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THE NOVEMBER MEETING OF THE IOWA SECTION.

The ninth regular meeting of the Iowa Section was held in the Chamber of Commerce library, Des Moines, on November 4, 1921, the Section, as is usual at the fall meeting, uniting with the Iowa Association of Mathematics Teachers. The vice-chairman, Professor C. W. EMMONS, presided.

One hundred and one persons were present, including the following thirteen members of the Association:

O. W. Albert, Julia T. Colpitts, I. S. Condit, C. W. Emmons, R. B. McClenon, F. M. McGaw, J. V. McKelvey, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, B. F. Simonson, E. R. Smith.

The portion of the program given by the Iowa Section consisted of a report on "The work of the National Committee on Mathematical Requirements" by Professor J. V. MCKELVEY, Iowa State College. Speaking as chairman of the Section's committee, of which Professors Emmons and Albert were the other members, Professor McKelvey discussed the suggestions and recommendations of the National Committee. His paper covered the general scope of the National Committee's report, lists of subjects to be included and those to be excluded, with some special mention of the functional concept in elementary mathematics, and of college entrance examinations.

Professors Rietz and Condit participated in the discussion.

J. F. REILLY, *Secretary.*

THE NOVEMBER MEETING OF THE MISSOURI SECTION.

The fifth regular meeting of the Missouri Section was held in St. Louis, on Friday and Saturday afternoons, November 25–26. The Friday session, held at Soldan High School, was presided over by Professor Louis Ingold, chairman of the Section. The Saturday session, held at Washington University, was a joint meeting with the Southwestern Section of the American Mathematical Society, and was presided over by Professor E. R. Hedrick, chairman of the Southwestern Section, and by Professor Ingold.

There were thirty-three in attendance, including the following twenty-eight members of the Association:

C. Ammerman, C. H. Ashton, A. Davis, O. Dunkel, W. W. Hart, E. R. Hedrick, Louise H. Huff, A. H. Huntington, L. Ingold, S. Lefschetz, W. A. Luby, E. B. Lytle, J. V. McKelvey, U. G. Mitchell, A. R. Nauer, P. R. Rider, P. Robertson, W. H. Roever, I. Roman, W. J. Ryan, A. J. Schwartz, H. E. Slaught,

E. R. Smith, E. Stephens, E. B. Stouffer, Eula A. Weeks, W. D. A. Westfall, Jessica M. Young.

On Friday evening at the American Hotel Annex a dinner was held jointly with the Southwestern Section of the American Mathematical Society and the Central Association of Science and Mathematics Teachers. Father W. J. Ryan, vice-president of St. Louis University and a member of the Association, acted as toastmaster at the dinner, and the following addresses were given: "The age of power" by Mr. A. S. Langsdorf, formerly dean of the Schools of Engineering and Architecture of Washington University; "Zoölogy in the secondary schools" by Dr. Caswell Grave, professor of zoölogy, Washington University.

At noon on Saturday the members of the Missouri Section of the Association and the Southwestern Section of the Society were the guests of Washington University at a luncheon which was served in the Tower Dormitory dining hall.

The following officers were elected for the ensuing year: Chairman, E. R. HEDRICK, University of Missouri; Vice-Chairman, W. A. LUBY, Kansas City Junior College; Secretary-Treasurer, P. R. RIDER, Washington University.

The 1922 meeting will be held in Kansas City in November, at the time of the meeting of the Missouri State Teachers' Association.

The following papers were read:

(1) "Mathematics clubs in junior high schools" by Mr. A. H. HUNTINGTON, Cleveland High School, St. Louis;

(2) "Some suggestions in regard to mathematics" by Father W. J. RYAN, vice-president of St. Louis University;

(3) "Correct methods of making drawings of space objects" by Professor W. H. ROEVER, Washington University;

(4) "The relation of mathematics to engineering" by Professor E. R. HEDRICK, University of Missouri;

(5) "Graphical methods of representing a function of a function and of solving allied problems" by Professors HEDRICK and ROEVER;

(6) "An elementary exposition of the theorem of Bernoulli with applications to statistics" by Professor H. L. RIETZ, University of Iowa;

(7) "Final report of the National Committee on Mathematical Requirements" by Dr. EULA A. WEEKS, Cleveland High School, St. Louis.

In addition to these papers, an informal talk was given by Professor H. E. SLAUGHT, of the University of Chicago, who told the Section of the recent grant to the Association by Mrs. Paul Carus of a sum of money to be used for the publication of expository monographs on mathematical subjects. In the absence of the author, the paper by Professor Rietz was read by Professor C. H. ASHTON of the University of Kansas. Several of the papers led to interesting discussions. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Mr. Huntington discussed mathematics clubs for pupils of junior high school age, maintaining that they offer opportunities not yet realized for enlisting the interest and effort of boys and girls in the study of mathematics.

2. Father Ryan offered some suggestions in regard to the present day status of mathematics in the high schools, which led up to a discussion of measures that should be taken to promote intensive work in mathematics.

3. In this paper Professor Roever first stated the criteria which must be satisfied by a drawing (on a plane or curved surface) in order that it should be an adequate and good looking representation of a space object. He then showed, as a consequence of these criteria, that a drawing must be a central or parallel (orthographic or oblique) projection of the space object to be represented, and gave the conditions under which one or two projections are required in order adequately to represent the object. Finally, rules of procedure were developed and exemplified.

4. Professor Hedrick called the attention of the Section to the organization of a joint committee under the auspices of the Society for the Promotion of Engineering Education and the Mathematical Association of America. He pointed out the close relationship between mathematics and engineering, and mentioned briefly both the history of previous investigations of mathematical teaching in schools of engineering and the plans of the present committee.

5. Professors Hedrick and Roever called attention to a recent paper (R. von Huhn, "A new graphic analytic method," *Science*, October 7, 1921) and pointed out that it is a very special case of the general method given by Professor Roever in this MONTHLY, *1916*, 330–333. They also showed how the method could be used in more general problems, including the elimination of one variable from two equations in three variables. [Published in *Science*, April 14, 1922.]

6. In this paper Professor Rietz showed by concrete illustrations the character of the statistical problems to which the theorem of Bernoulli is applicable, and presented an elementary demonstration of the theorem for the purpose of giving a clear view of a method of treatment of a fundamental problem in the fluctuations of results derived from random samples. [Published in *Mathematics Teacher*, December, 1921.]

7. The paper of Dr. Weeks, who is a member of the National Committee on Mathematical Requirements, gave a brief outline of the contents of the final report of the committee, with statements about the publication and distribution of this report.

PAUL R. RIDER, *Secretary-Treasurer.*

THE NOVEMBER MEETING OF THE TEXAS SECTION.

The first regular meeting of the Texas Section was held at the Bryan High School, Dallas, Texas, on November 25–26, in conjunction with a meeting of the Mathematics Section of the Texas State Teachers' Association. Three meetings were held. G. C. EVANS presided at the session of Friday morning, H. J. ETTLINGER, president of the Mathematics Section of the Texas State Teachers' Association, at the session of Friday afternoon, and J. O. MAHONEY at the closing session of Saturday morning.

One of the very enjoyable features of the occasion was the banquet for the members of the Texas Section of the Association held in the Violet Room of the Oriental Hotel. President Ettlinger presided as toastmaster.

There were sixty-five in attendance, including the following eighteen members of the Association:

W. D. Baten, H. Y. Benedict, A. A. Bennett, J. E. Burnam, P. J. Daniell, H. J. Ettlinger, G. C. Evans, L. R. Ford, A. J. Hargett, H. Hosford, E. H. Jones, J. O. Mahoney, R. L. Moore, H. Porter, J. L. Riley, P. C. Rockwell, E. R. Tucker, P. H. Underwood, N. Wunder.

The following officers were elected for the coming year: Chairman, G. C. EVANS; Vice-Chairman, A. A. BENNETT; Secretary-Treasurer, W. D. BATEN. The next meeting will be held at Houston, December 1–2, 1922, in conjunction with the annual meeting of the Mathematics Section of the Texas State Teachers' Association.

The following papers were read:

(1) "Dante and science" by Professor G. C. EVANS;

(2) "What the college or university expects of the freshman in the way of mathematics" by Professor H. J. ETTLINGER;

(3) "An actuary: his training, functions and service" by C. P. ROCKWELL;

(4) "The rôle of logic in the refinement of geometric intuition" by Professor R. L. MOORE;

(5) "What the state section of the Mathematical Association of America can do for mathematics in Texas" by Professor J. L. RILEY;

(6) "On teaching the fractional exponent" by Professor W. D. BATEN;

(7) "The number system of algebra" by Professor A. A. BENNETT.

Abstracts of most of the papers and discussions follow below, the numbers corresponding to numbers in the list of titles.

1. Professor Evans gave a résumé of his paper as published in *The Rice Institute Pamphlet*, volume 8, April, 1921, on the occasion of the Dante Sexcentenary. Before his exile Dante had demonstrated his extraordinary ability and his interest in both poetry and science. No reader can follow adequately a work like the *Divine Comedy* without some knowledge of the mechanical frame on which it is consciously built, and some understanding of the physical aspect of the mediæval universe. For here Dante himself was an expert. The astronomy of Dante's time, which occupied a secondary place to that of astrology, was merely the astronomy of Hipparchus. Interpreted in the light of modern knowledge it was an approximate representation of apparent angular motions of the stars and planets as seen from the earth, by means of a system of uniformly moving epicycles and eccentrics. It was thus an attempt to build up the variable apparent motions of the heavens out of a system of uniform circular motions about centers themselves also moving uniformly.

Knowing well the phenomena of the precession of the equinoxes discovered by Hipparchus, Dante was enabled to proceed accurately on his voyage of the heavens. According to Dante the earth occupies a central position at rest and

the orbits of the moon, planets, sun, and stars are fixed on concentric spheres, or heavens of various radii.

The mediæval, or Dantean, order of the spheres is this: (1) Moon, (2) Mercury, (3) Venus, (4) Sun, (5) Mars, (6) Jupiter, (7) Saturn, (8) Stars, (9) Crystalline or Primum Mobile, (10) the Empyrean Heaven.

2. Professor Ettlinger gave the conclusions drawn from the answers to a questionnaire concerning the knowledge of mathematics required in other departments of the University of Texas of students who are taking or have completed the work of the freshman year. Opinions were solicited from each department relative to the value of each topic listed. The plan of the questionnaire followed the fourth section of that issued by the National Committee on Mathematical Requirements. This paper will be published in the *Texas Mathematics Teachers' Bulletin*.

3. Mr. Rockwell, in the course of his remarks, said: "Mathematical training, with patient study and practice, may make a perfect mathematician, but still fail to give us an actuary. To the born mathematical genius, or to the person with acquired mathematical talent, must be added something that is absolutely essential to produce the actuary. That something includes a minute observation of the practical workings of life insurance, the ability to recognize and appreciate the weight and importance of business interests and business requirements."

Recent features of insurance practically new to American life, as group insurance, industrial insurance and annuities, have increased the responsibilities of the actuary. The generally accepted notion that the sole duty of the actuary is to furnish tables in keeping with the "policies" by the company agents has been dispelled. The actuary must be in close coöperation with the medical director so as to pass upon certain risks and with the general directors to advise on certain financial risks and policies.

4. Professor Moore spoke of the rôle of logic in the treatment of propositions in elementary geometry. The demonstration of some well-known elementary propositions led to rather fanciful results, or fallacies, when demonstrated without regard to logical sequence. The age-old problem of whether or not strictly logical treatment must be required was discussed.

6. Professor Baten spoke of the troubles that beset first year students when they take up the work of exponents in college algebra. He emphasized the very close relation existing between the study of exponents, the study of logarithms and the subject of radicals. The fact that the average freshman did not know that the value of $a^0 = 1$ was reasserted.

7. In this paper, Professor Bennett discussed the development of the number system of algebra through its principal historical and logical steps. He contrasted the usual method of elementary arithmetic, wherein the discussion of fractions precedes that of negative numbers, with the procedure in algebraic treatments, where this sequence is usually reversed, and spoke of the justification of each line of development. The principal feature of the paper was an analysis of the postulates of algebra into three groups, the first treating of rational operations

alone, the second, of order relations alone, the third, of order relations among the results of rational operations. This classification corresponds intimately to the characterizations of the theorems of algebra as falling into three analogous categories.

J. L. RILEY, *Secretary-Treasurer.*

CUSPIDAL ENVELOPE ROSETTES.

By WILLIAM F. RIGGE, Creighton University.

A point P moves in the line segment EG, Fig. 1, with simple harmonic motion of p cycles, while this segment makes q revolutions about A with uniform angular speed. Moritz[1] has exhaustively treated the case when the point A is in the line EG or in its prolongation. The writer[2] has shown that when the point A is out of the line EG and the rosette drawn is cuspidal, AL, the distance of EG from A, must be $n \sin \beta$ (in which $n = p/q$) and LR, the distance of R, the mid-point, or point of zero phase, of EG, from its point of tangency L on the *tangent* circle, must be $\cos \beta$. The point P remains on an ellipse whose conjugate semi-axis is unity ($= ER = RG$) and is always parallel to EG, whose major semi-axis $= n$, and whose center is the sine PR of the phase α distant from A, β being the eccentric angle of P.

When the point P is given a double rectilinear harmonic motion[3] with equal amplitudes but with periods in the ratio of m to 1, it may be conceived to move with simple harmonic motion of mn periods on the line segment DF, Fig. 2, while this line slides n times in a similar way along the line EG in one revolution of EG about A. For the sake of greater clearness these lines DF and EG are spaced a short distance apart in the figure. PS is then the sine of the phase of P on DF, while RS is the sine of the phase of S, the mid-point of DF, on EG. Hence the distance of the tracing point P from the *tangent* circle measured along the tangent line $GFED$ is

$$LP = - RL + RS + SP = - \cos \beta + \sin n\theta + \sin mn\theta,$$

in which θ is the phase of the circular motion about A. In the previous paper the point A was in the line EG. In the present A will be out of this line. The discussion will, as before, be confined to envelopes that are cuspidal.

Two Envelopes. As the points D and F are the limits of the excursions of P on this line segment DF, it is clear that these points themselves would trace the envelopes to all the lobes or loops or branches drawn by P, and that P must be on these envelopes when it is in phases 90° and 270°, respectively, on DF. The

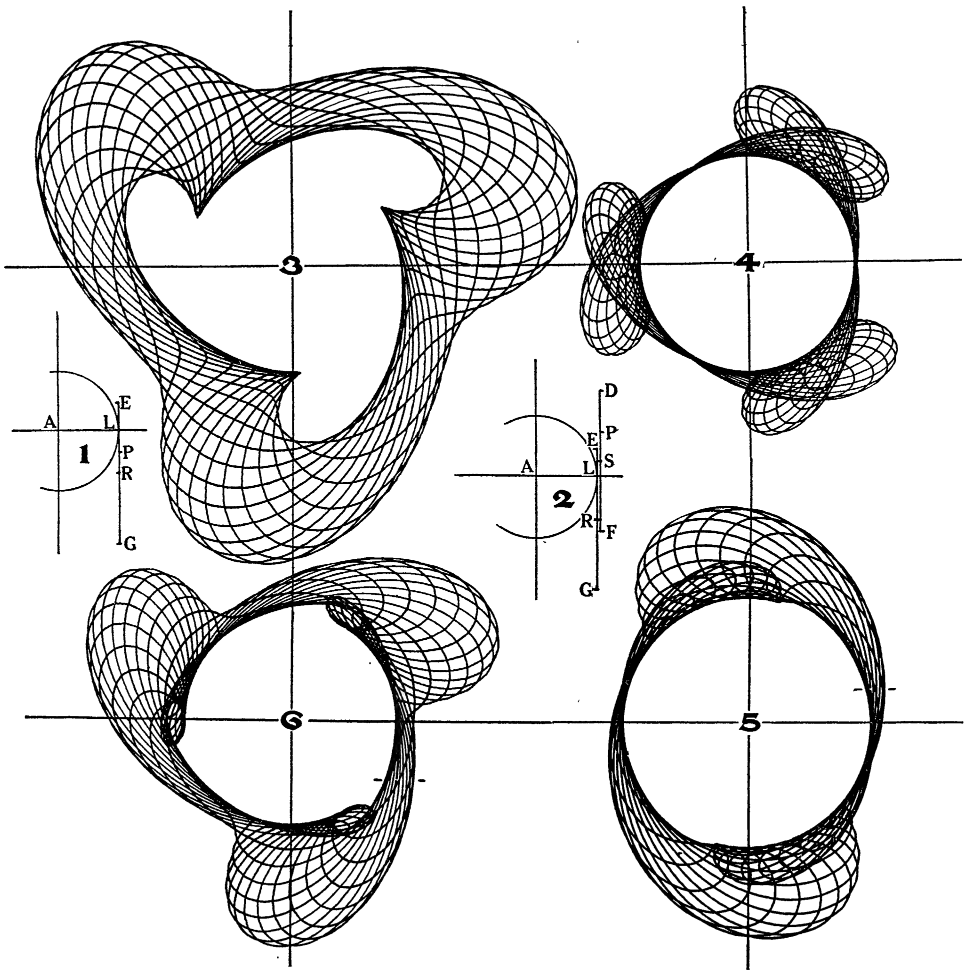
[1] "On the construction of certain curves in polar coördinates" by R. E. Moritz, in this MONTHLY, *1917*, 213–220.

[2] "Concerning a new method of tracing cardioids" by W. F. Rigge, in this MONTHLY, *1919*, 21–32. "Cuspidal rosettes," in this MONTHLY, *1919*, 332–340.

[3] "Envelope Rosettes," in this MONTHLY, *1920*, 151–157.

distance between the two envelopes measured in the line of motion of P is thus equal to 2, and the distance of P from them when in phase 0° or 180°, or multiples of them, must be -1 and $+1$.

The Starting Position of P for Cuspidal Envelopes. If D (or F) is to trace a cuspidal rosette, P must be set on a point of this rosette in phase 90° (or 270°) on DF, and then the phase of S on EG must also be set according to the nature of the curve. As the phase of S on EG, when $m-1$ is infinitesimal, may have



any value corresponding to any phase of P on DF, we select the convenient value of $\theta = 0°$, when both will be zero together, to start P moving. Then P, R, S are coincident in Fig. 2. But as the distance of P from the D (or F) envelope will then be $+1$ or -1, this starting position of P, R, S, in phase 0° must therefore be the distance unity on one side or other of the cuspidal rosette, that is, its starting ordinate $y_0 = -\cos\beta - 1$ or $-\cos\beta + 1$, while x_0 is always

$n \sin \beta$. From this it is evident that only one of the D and F envelopes can be cuspidal, except in a special case to be mentioned presently, when both may be so.

The Influence of β. When y_0 or $-\cos \beta \pm 1$ is small numerically, the D and F envelopes will as a rule intersect. Although unequal, they are always symmetrical on account of the positions of E and G on opposite sides of L. They are equal only when $y_0 = 0$, that is, when $\beta = 0°$ or $180°$, being coincident when n is odd, and symmetrically displaced when n is even. When y_0 or $-\cos \beta \pm 1$ is large numerically, the D and F envelopes cannot intersect, because E and G are then at very unequal distances from L and A. The inner one of the two will be cuspidal.

Illustrations. Fig. 3 with $n = 3$, $\beta = 30°$, $x_0 = n \sin \beta = 1.5$, $y_0 = -\cos \beta - 1 = -1.866$, shows the D or inner envelope cuspidal, while Fig. 4 with the same values of n, β, x_0, but with $y_0 = -\cos \beta + 1 = +0.134$, would seem at first sight to show two equal symmetrical and cuspidal envelopes. Only one however is in fact cuspidal and congruent to the inner envelope of Fig. 3. It is here the F envelope and is really smaller than the other or D envelope, because in this case $GL < EL$. The D envelope of Fig. 3 is in every way exactly equal to the F envelope of Fig. 4. When $\beta = 60°$ (but $n = 2$), as in Fig. 5, the inequality of the two envelopes is obvious, the smaller alone being cuspidal.

Transition Envelopes. Fig. 6 shows a transition envelope for $n = 3$. When the tracing point P is started in phases $0°$ with a smaller numerical value of y_0 than $-\cos \beta - 1$ as in Fig. 3, the cusp throws a lobe or a shoot like a growing bud, while the outer envelope is contracted. While y_0 is increasing in the positive direction, the two envelopes may become apparently equal or nearly so, although neither can be cuspidal. The cuspidal stage will be reached again when $y_0 = -\cos \beta + 1$ as in Fig. 4. After that the growth just mentioned will be reversed, until the value of $\beta' = 180° - \beta$ will again give cuspidal envelopes symmetrical however to those of β. For still larger positive values of y_0, the envelopes will tend to become nearly equal and circular.

Cuspidal envelope rosettes are best drawn when n is a small integer. When $n = 1$ one or both of the envelopes are cardioids for all values of β.

SEXAGESIMAL FRACTIONS AMONG THE BABYLONIANS.

BY FLORIAN CAJORI, University of California.

In view of the fact that certain writers have expressed the conviction[1] that, while the Babylonians operated with integers expressed in the sexagesimal system, they *did not use sexagesimal fractions*, it is worth while to refer briefly to a cuneiform tablet recently described by H. F. Lutz[2] of the University of Pennsylvania which unquestionably reveals the Babylonian use of sexagesimal frac-

[1] For example, see this MONTHLY, *1920*, 124.

[2] *American Journal of Semitic Languages and Literatures*, vol. 36, 1920, pp. 249–257. Tablet CBS 8536 in the University Museum, Philadelphia.

tions. According to Lutz, the tablet "cannot be placed later than the Cassite period, but it seems more probable that it goes back even to the First Dynasty period, *ca.* 2000 B.C."

To mathematicians the tablet is of interest because it reveals operations with sexagesimal fractions resembling modern operations with decimal fractions. For example, 60 is divided by 81 and the quotient expressed sexagesimally. Again, a sexagesimal number with two fractional places, 44(26)(40), is multiplied by itself, yielding a product in four fractional places, namely [32]55(18)(31)(6)(40). In this notation the [32] stands for 32 $\times$ 60 units, and to the (18), (31), (6), (40) must be assigned, respectively, the denominators 60, 60^2, 60^3, 60^4.

Numbers that are incorrect are marked by a *.

First column

. . . gal (?) -bi 40 -ám
šu *a- na* gal-bi 30 -ám

igi 2	30
igi 3	20
igi 4	15
igi 5	12
igi 6	10
igi 8	7(30)
igi 9	6(40)
igi 10	6
igi 12	5
igi 15	4
igi 16	3(45)
igi 18	3(20)
igi 20	3
igi 24	2(30)
igi 25	2(24)
igi 28*	2(13)(20)
igi 30	2
igi 35*	1(52)(30)
igi 36	1(40)
igi 40	1(30)
igi 45	1(20)
igi 48	1(15)
igi 50	1(12)
igi 54	1(6)(40)
igi 60	1
igi 64	(56)(15)
igi 72	(50)
igi 80	(45)
igi 81	(44)(26)(40)

Fifth column

	44(26)(40)
1	44(26)(40)
2	[1]28(53)(20)
3	[2]13(20)
4	[2]48(56)(40)*
5	[3]42(13)(20)
6	[4]26(40)
7	[5]11(6)(40)
9	[6]40
10	[7]24(26)(40)
11	[8]8(53)(20)
12	[8]53(20)
13	[9]27(46)(40)*
14	[10]22(13)(20)
15	[11]6(40)
16	[11]51(6)(40)
17	[12]35(33)(20)
18	[13]20
19	[14]4(26)(40)
20	[14]48(53)(20)
30	[22]13(20)
40	[29]37(46)(40)
50	[38]2(13)(20)*

44(26)(40)a-na 44(26)(40)
[32]55(18)(31)(6)(40)
44(26)(40) square

igi 44(26)(40)	81
igi 81	44(26)(40)

The tablet contains twelve columns of figures. The first column gives the results of dividing 60 in succession by twenty-nine different divisors from 2 to 81. The eleven other columns contain tables of multiplication; each of the numbers 50, 48, 45, 44(26)(40), 40, 36, 30, 25, 24, 22(30), 20 is multiplied by integers up to 20, then by the numbers 30, 40, 50, and finally by itself. Using our modern numerals, we reproduce the first and the fifth columns of the tablet, which exhibit a larger number of fractions than do the other columns. The Babylonians had

no mark separating the fractional from the integral parts of a number. Hence a number like 44(26)(40) might be interpreted in different ways; among the possible meanings are $44 \times 60^2 + 26 \times 60 + 40$, $44 \times 60 + 26 + 40 \times 60^{-1}$, and $44 + 26 \times 60^{-1} + 40 \times 60^{-2}$. Which interpretation is the correct one can be judged only by the context, if at all.

The exact meaning of the first two lines in the first column is uncertain. In this column 60 is divided by each of the integers written on the left. The respective quotients are placed on the right.

In the fifth column the multiplicand is 44(26)(40) or 44 4/9. The last two lines seem to mean "$60^2 \div 44(26)(40) = 81$, $60^2 \div 81 = 44(26)(40)$."

It is a source of gratification to find that scholars of several thousand years ago were fully as capable of committing errors in computation, as are arithmeticians of the present time.

The Babylonian use of sexagesimal fractions is shown also in a clay tablet described by A. Ungnad[1] (*Orient. Lit. Zeitung*, 19 Jahrg., 1916, p. 363-368). In it the diagonal of a rectangle whose sides are 40 and 10 is computed by the approximation $40 + 2 \times 40 \times 10^2 \div 60^2$, yielding 42(13)(20), and also by the approximation $40 + 10^2 \div \{2 \times 40\}$, yielding 41(15). Translated into the decimal scale, the first answer is 42.22 +, the second is 41.25, the true value being 41.23 +.

A BUDGET OF EXERCISES ON DETERMINANTS.

By THOMAS MUIR, Rondebosch, South Africa.

A collection of fresh exercises[2] on a mathematical subject, even if the plan and execution be far from perfect, can be of greater service to the student than any so-called "paper," covering the same amount of page space. It is in this belief that I have brought together the following budget of thirty. In a kind of way they range over the whole subject of determinants: at any rate they do not confine themselves to any special branch of it. They are of all degrees of difficulty, starting with commonplace instances of mere "evaluation." They naturally also differ in suggestiveness; one or two of them might in eager hands lead to the formulating of allied results, and thereby even to the evolution of material for a "paper." None of them, so far as I can at present recall, has been printed before, and certainly, the number of them that may so turn out must be comparatively trifling.

1. If $\begin{vmatrix} a_1 - x & a_2 - x & a_3 - x \\ b_1 - x & b_2 - x & b_3 - x \\ c_1 - x & c_2 - x & c_3 - x \end{vmatrix} = 0$, then $x = -\,|a_1b_2c_3| \div \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a_1 & a_2 & a_3 \\ 1 & b_1 & b_2 & b_3 \\ 1 & c_1 & c_2 & c_3 \end{vmatrix}$.

[1] Our information is drawn from *Mitteilungen zur Geschichte der Medizin und der Naturwissenschaften*, Leipzig, vol. 17, 1918, p. 203.

[2] For any technical terms, in the following, which the student may have doubt about he is referred to the text-books of Weld and Hanus.

2. $$\begin{vmatrix} 1 & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ 1 & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \\ 1 & \sin(\alpha+\beta) & \cos(\alpha-\beta) \end{vmatrix} = \begin{vmatrix} \cos 2\alpha & \cos\alpha & \sin\alpha \\ \cos 2\beta & \cos\beta & \sin\beta \\ \cos 2\gamma & \cos\gamma & \sin\gamma \end{vmatrix}$$

3. $$\begin{vmatrix} \cos(\beta+\gamma) & \sin(\beta+\gamma) & \cos(\beta-\gamma) \\ \cos(\gamma+\alpha) & \sin(\gamma+\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha+\beta) & \sin(\alpha+\beta) & \cos(\alpha-\beta) \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} \sin 2\alpha & \cos 2\alpha & 1 \\ \sin 2\beta & \cos 2\beta & 1 \\ \sin 2\gamma & \cos 2\gamma & 1 \end{vmatrix}$$

4. Any axisymmetric determinant is expressible as the product of its principal term and an axisymmetric determinant with a diagonal of units.

5. In an n-line zero-axial determinant the number of positive terms is $n-1$ greater or less than the number of negative terms according as n is odd or even.

6. The reversible continuant $K(x^a x^b x^c x^c x^b x^a x)$ of odd order is resolvable into two continuants.

7. Express the square of the persymmetric determinant

$$\begin{vmatrix} a & b & c & d \\ b & c & d & am \\ c & d & am & bm \\ d & am & bm & cm \end{vmatrix}$$

as a persymmetric.

8. $$|(a_r+s-1)^m|_{m+1} = 1^1 2^2 \cdots m^m \zeta^{1/2}(a_1, a_2, \cdots, a_{m+1}).$$

What is written here between the two vertical lines is the (r, s)th element of the determinant, and the suffix outside denotes the order: the first row is thus

$$a_1{}^m, \quad (a_1+1)^m, \quad \cdots, \quad (a_1+m)^m.$$

When instead of the suffix $m+1$ we put n, we are faced with a more complicated problem. $\zeta^{1/2}(a_1, a_2, \cdots, a_{m+1})$ denotes the difference product of the a's.

9. If $A = (b+c)(b+d)(c+d)$, $B = (a+c)(a+d)(c+d)$, $\cdots$, then

$$\zeta^{1/2}(A, B, C, D) = (a+b+c+d)^6 \cdot \zeta^{1/2}(a^2, b^2, c^2, d^2).$$

10. If Δ stand for $|a_1 b_2 c_3 d_4|$, then

$$\begin{vmatrix} A_1 - \dfrac{\Delta}{a_1} & A_2 & A_3 & A_4 \\ B_1 & B_2 - \dfrac{\Delta}{b_2} & B_3 & B_4 \\ C_1 & C_2 & C_3 - \dfrac{\Delta}{c_3} & C_4 \\ D_1 & D_2 & D_3 & D_4 - \dfrac{\Delta}{d_4} \end{vmatrix} = \frac{\Delta^3}{a_1 b_2 c_3 d_4} \begin{vmatrix} 0 & a_2 & a_3 & a_4 \\ b_1 & 0 & b_3 & b_4 \\ c_1 & c_2 & 0 & c_4 \\ d_1 & d_2 & d_3 & 0 \end{vmatrix}$$

11. $$\begin{vmatrix} abcd & abd & acd & bcd \\ abc & abcd & acd & bcd \\ abc & abd & abcd & bcd \\ abc & abd & acd & abcd \end{vmatrix} = a^3b^3c^3d^3(abcd - \Sigma ab + 2\Sigma a - 3)$$

and the cofactor of $a^3b^3c^3d^3$ on the right is expressible as a continuant and also as a recurrent. What does this cofactor become when the number of variables and the order of the determinant are each made n?

12. In a 3-by-5 array whose row-sums vanish there are three independent linear relations between the determinants of the array.

13.
$$\begin{vmatrix} x_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & x_2 & a_3 & a_4 & a_5 \\ -a_1 & -a_2 & x_3 & a_4 & a_5 \\ -a_1 & -a_2 & -a_3 & x_4 & a_5 \\ -a_1 & -a_2 & -a_3 & -a_4 & x_5 \end{vmatrix} = \begin{vmatrix} x_1 & -1 & 0 & 0 & 0 \\ a_1a_2 & x_2 & -1 & 0 & 0 \\ a_1x_2a_3 & a_2a_3 & x_3 & -1 & 0 \\ a_1x_2x_3a_4 & a_2x_3a_4 & a_3a_4 & x_4 & -1 \\ a_1x_2x_3x_4a_5 & a_2x_3x_4a_5 & a_3x_4a_5 & a_4a_5 & x_5 \end{vmatrix}$$
$$= x_1x_2x_3x_4x_5 + \Sigma x_1x_2x_3a_4a_5 + \Sigma x_1a_2a_3a_4a_5.$$

14. Simplify the determinant whose first three columns are taken from those of any 4-by-5 array and whose other column is a column taken from the square of the array.

15. Show that the cofactor of c in
$$\begin{vmatrix} 0 & a & b & c \\ l & 1 & x & yz - x^2 \\ m & 1 & y & zx - y^2 \\ n & 1 & z & xy - z^2 \end{vmatrix}$$
is a factor of the determinant, and find the other factor.

16. Any positive unit orthogonant $|\alpha_1\beta_2\gamma_3|$ is axisymmetric if $1+\alpha_1+\beta_2+\gamma_3$ vanishes or if $\alpha_1 = \beta_2 = \gamma_3 = 1$.

17. If Δ stand for $ad - bc$, then
$$\frac{\partial(a\Delta, b\Delta, c\Delta, d\Delta)}{\partial(a, b, c, d)} = 3\Delta^4.$$

18. The bordered axisymmetric determinant
$$\begin{vmatrix} 0 & x & y & z \\ u & -d+e+f & f & e \\ v & f & d-e+f & d \\ w & e & d & d+e-f \end{vmatrix} = \begin{vmatrix} x & y & z \\ & d-e & d-f \\ & & e-f \end{vmatrix} \cdot \begin{vmatrix} u & v & w \\ & d-e & d-f \\ & & e-f \end{vmatrix}$$
and therefore equals the bordered skew
$$\begin{vmatrix} 0 & x & y & z \\ -u & 0 & d-e & d-f \\ -v & e-d & 0 & e-f \\ -w & f-d & f-e & 0 \end{vmatrix}.$$

19. Eliminate x, y, z, w from the equations got by putting $r = 1, 2, 3, 4$ in
$$a_rxy + b_rxz + c_rxw + d_ryz + e_ryw + f_rzw = 0.$$

20. If

$$\begin{aligned} ax + by + cz &= 0, \\ by + du + ev &= 0, \\ cz + ev + fw &= 0, \end{aligned} \qquad \text{and} \qquad \begin{aligned} mx + ny + pz &= 0, \\ ny + qu + rv &= 0, \\ pz + rv + sw &= 0, \end{aligned}$$

then

$$(mc - ap)(bq - dn)(es - rf) = (dr - qe)(pf - cs)(an - bm).$$

21. $$\left\Vert \begin{matrix} |a_2b_3| & |a_1b_3| & |a_1b_2| \\ |c_2x_3| & |c_1x_3| & |c_1x_2| \\ |y_2c_3| & |y_1c_3| & |y_1c_2| \end{matrix} \right\Vert = \left\Vert \begin{matrix} |b_2c_3| & |b_1c_3| & |b_1c_2| \\ |a_2c_3| & |a_1c_3| & |a_1c_2| \\ |x_2y_3| & |x_1y_3| & |x_1y_2| \end{matrix} \right\Vert = |a_1b_2c_3|\,|c_1x_2y_3|.$$

22. If P be the product of n linear homogeneous functions of n variables, then (H standing for Hessian)

$$\frac{H(P)}{H(\log P)} = -(n-1)P^n.$$

23. If the determinant whose columns are the lth, mth, nth, rth columns of any 4-by-8 array be denoted by $|lmnr|$, then

$$\left\Vert \begin{matrix} |1567| & |1568| & |1578| & |1678| \\ |2567| & |2568| & |2578| & |2678| \\ |3567| & |3568| & |3578| & |3678| \\ |4567| & |4568| & |4578| & |4678| \end{matrix} \right\Vert = |1234| \cdot |5678|^3.$$

24. $$\begin{vmatrix} ax & bx+\beta & cx+\gamma \\ bx-\beta & ex & dx+\delta \\ cx-\gamma & dx-\delta & fx \end{vmatrix} = \begin{vmatrix} ax & a\beta & M \\ -a\beta & aPx & N \\ -M & -N & PQx \end{vmatrix} \div a^2P^2,$$

where

$$P = \begin{vmatrix} a & b \\ b & e \end{vmatrix}, \qquad Q = \begin{vmatrix} a & b & c \\ b & e & d \\ c & d & f \end{vmatrix}, \qquad M = \begin{vmatrix} a & b & c \\ b & e & d \\ 0 & \beta & \gamma \end{vmatrix},$$

$$N = \begin{vmatrix} 0 & a & b & c \\ 0 & b & e & d \\ a & 0 & \beta & \gamma \\ b & -\beta & 0 & \delta \end{vmatrix}.$$

25. $$\begin{vmatrix} 0 & a_1b_2+a_2b_1 & a_1b_3+a_3b_1 & a_1b_4+a_4b_1 & \cdots \\ a_1b_2+a_2b_1 & 0 & a_2b_3+a_3b_2 & a_2b_4+a_4b_2 & \cdots \\ a_1b_3+a_3b_1 & a_2b_3+a_3b_2 & 0 & a_3b_4+a_4b_3 & \cdots \\ a_1b_4+a_4b_1 & a_2b_4+a_4b_2 & a_3b_4+a_4b_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n$$

$$= \left\{ -n + 1 - \sum \frac{|a_1b_2|^2}{(-2a_1b_1)(-2a_2b_2)} \right\} \prod(-2a_1b_1).$$

26. The sum of a quadric and its discriminant is expressible as a Pfaffian.

27. The determinant of the sum of an axisymmetric matrix and a zero-axial skew matrix is expressible as a Pfaffian.

28. If D_1, D_2 be n-line determinants, the one axisymmetric and the other zero-axial skew, and Δ_1, Δ_2, $\cdots$ be the series of determinants each of which is identical in m columns with D_2 and in the remaining columns with D_1, then when m is odd

$$\sum \Delta = 0.$$

29. If from the n-line determinants D_1, D_2, $\cdots$, D_p, there be formed a new n-line determinant Δ consisting of α_1 rows from D_1, α_2 rows from D_2, and so on, the rows in each case occupying the same places in Δ as in their own determinant, then

$$\sum \Delta = \sum \Delta',$$

where Δ' is the determinant formed like Δ save that columns are used instead of rows.

30. If A and O be n-line determinants, A axisymmetric and O orthogonant, the sum of the r-line coaxial minors of OAO' is equal to the corresponding sum in A.

Δ' is used to denote the conjugate of Δ, *i.e.*, the determinant got by changing the rows of Δ in order into columns.

February 15, 1921.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

18. Sylvester as a Poet.

The natural relation of the mathematical to the poetical mind has been observed so often as to make it hardly worth while to comment upon it. It is probably not too much to say that every mathematician is a poet at heart, and it is possible that every poet has more of the mathematical in his soul than he thinks. However this may be, it would not be without interest to study the tangible evidence of the relation of poetry to mathematics, and, indeed, of mathematics to all the various arts in which rhythm and symmetry and imagination enter. In such a study the productions of that interesting character, Professor Sylvester, would naturally be considered—not because they represent a high type of imaginative literature, but because they illustrate an interesting type of mind.

Among a large number of autograph letters of Professor Sylvester which I possess is one written on November 9, 1884, to W. J. C. Miller, Esq., mathematical editor of the *Educational Times*. This contains a poem entitled "Retrospect," which was published a little later. It shows the alterations which Professor Sylvester made and which, having never before appeared in print, may be welcomed by the few who have already seen the verses. The poem as a whole may be of interest to those who have attempted to solve what may be designated as Professor Sylvester's "other" problems. The verses and accompanying note are as follows:

RETROSPECT.

'Ere told it's message, spent
The warning flash at sea,
Stars through dim firmament
Making mad revelry;

Before earth's[1] sovereign eye
Night's silver mantle drawn,
With sobering alchemy
In pale day quenched the dawn;

Strange pause in merriment,
The hush of minstrelsy,
Silence more eloquent
Than tuneful threnody,

Life's bootless endless chain
Things born to cease to be
Pleasure subserving[2] pain
Thought bitter[3] reverie,

Loved lost Eurydice
Snatched to redeemless death,
Her spouse distraught to see
And suck once more her breath,—

What tales of eld rehearse,
Whisper these signs to me,
"Hope lights to swift reverse
Death is faith's destiny."

Dr Mr Miller

These are the verses referred to in my previous note in their more perfect form. The *third* stanza is new[4] the 18th and 19th lines altered and the 22nd retouched.

Yours truly,
J. J. S.

9th Novr.[5] 1884.

19. LEWIS CARROLL AS A CRITIC.

To those who have read *Alice in Wonderland* (and who has not?) or Mr. Stuart Collingwood's *Life of Lewis Carroll* (which everyone should do), personal letters of Mr. Dodgson can hardly fail to be of interest. Mr. Collingwood has given many letters in his biography, but in general they are such as were written to the child friends of Dodgson,—or, to use his literary name, of Lewis Carroll. Among those of his letters which are now in my collection is one which may have some interest because it shows Dodgson as the crusty bachelor that he was, and hence in a somewhat different light. It was written when he was fifty-nine years old, while he was still engaged in his tutoring at Christ Church, Oxford (where he had matriculated forty-one years before), and nearly thirty years after he began his various adventures of Alice, culminating in the children's

[1] In the manuscript "night's" is replaced by "earth's."
[2] "Subserving" replaces "the mask to."
[3] "Bitter" replaces "thrall to."
[4] "and and" erased after "new."
[5] "Novr." replaced "Oct.," showing his mental lapse.

well-known classic which appeared in 1865. It is not improbable that his remarks relative to editorial revision will strike a responsive note in the soul of other writers, and possibly of certain editors as well. At any rate, my friend, the late W. J. C. Miller, to whom it was written, laughed over it when he gave it to me, many years ago.

The letter is as follows:

Ch. Ch. Oxford.
Feb. 21/91

Dear Sir,

I have carefully examined "The Foundations of Geometry" by E. T. Dixon, published by Deighton, Bell, & Co., Cambridge, and would be glad to send you a short notice of it, if you would be at all likely to print it. You would be free to reject it, of course; & also to abridge it, if you thought fit: the only condition I make is that no word shall be *altered*, or *added*. Also you would be free to put my name to it, or not, as you thought best—The latter would be the best, in my own opinion. I think the notice would have more weight, & get more attention, as an unsigned 'notice' than as a signed letter.

Kindly tell me whether to send it or not. I enclose a card for your reply.

Truly yours,
C. L. Dodgson.

The Mathematical Editor of
"The Educational Times."

P.S. The book is on the "direction" theory, and involves, I believe, the same logical fallacy as lies at the root of Wilson's treatise.

RECENT PUBLICATIONS.

REVIEWS.

Geschichte der Elementar-Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. Von J. Tropfke. Erster Band, Rechnen. Zweite, verbesserte und sehr vermehrte Auflage. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. 8vo. 7 + 177 pages. Price, to Germans, bound, 46 marks.

The first edition of Tropfke's history, in two volumes, 1902–1903, made a notable contribution to the subject by its clarity of statement, and wealth of exact references to sources. The two parts of the first volume were devoted to "Das Rechnen" (pp. 1–122) and "Die Algebra" (pp. 123–332). The twelve parts of the second volume (496 pages) dealt with geometry, logarithms, plane and spherical trigonometry, spherical geometry, stereometry, series, compound interest, partial fractions, analytic geometry, conic sections, maxima and minima, combinatory analysis and theory of probability.

But the numerous discoveries of the past twenty years have rendered many of the statements of this work inaccurate and otherwise inadequate. A thorough revision was therefore very much to be desired. The previous two-volume edition, costing 22 marks, is to be expanded somewhat and published in seven volumes, costing 322 marks—if the cost of the volumes is to be uniform: I, Das Rechnen; II, General arithmetic; III, Proportion and equations; IV, Geometry; V, Trigonometry and spherics; VI, Analysis; VII, Name and subject indexes.

The first volume of the new edition covers the first 120 pages of the old edition; there is thus an increase in the number of pages of more than 45 per cent. The expansions are mainly in connection with footnotes, more than doubled in number, and with the sections on "Numbers in general" and "Integers." The treatment of perfect numbers in this latter-named section would have been vastly improved if due account had been taken of the comprehensive survey in Dickson's *History of the Theory of Numbers*, volume 1, 1919, (see *1921*, 140–141). Tropfke gives no reference to results obtained regarding odd perfect numbers, and the twelve known perfect numbers are, therefore, without indicated warrant called the *first* twelve numbers of this type.

It is incorrectly stated, on page 26, that the first printed work to have its pages numbered with Arabic numerals was a certain work of Petrarch published in 1471. In the Annmary Brown Memorial of Providence, R. I., is a copy of a work by Werner Rolewinck published in 1470, and with pages so numbered (see *1921*, 423).

As a whole, this first volume of the new edition offers, at present, the most satisfactory treatment of its kind in the field discussed, and is a great improvement on the corresponding part of the first edition. There is no change in the general plan of the work. That the author has had the benefit of coöperative assistance on the part of G. Eneström and H. Wieleitner tends to fortify the feeling of the reader that down to the smallest detail great care has been exercised to make exact statements.

The Vereinigung wissenschaftlicher Verleger has, for some time, pursued a campaign of extortion so far as America is concerned, by charging for their books more than five times the published prices. For a bound copy of the book under review $2.30 are demanded. It is to be hoped that such action will be suitably resented.

August, 1921.

R. C. ARCHIBALD.

Technical and Scientific Serials in the Libraries of Providence, 1920. Compiled by the various libraries and edited by F. K. W. DRURY. Providence, R. I., 1921. Royal 8vo. 9 + 63 pp. Price 60 cents.

This publication lists about 2500 titles of serials (as defined by the American Library Association) in mathematics, chemistry, physics, biology, botany, and geology, as well as in the mechanic and useful arts, engineering, trades, medicine and the fine arts. "The list looks forward not simply to a complete list of *all* serials in the libraries of Providence or of the State, but to a complete sectional union list for New England and the East. Such a list is under way in the North Central states; this list is a contribution toward one on the Atlantic seaboard. With this in mind the national abbreviations for the libraries have been used rather than symbols purely local."

The publication is not above criticism in connection with the mathematical titles. A few selected indications of this must suffice. The two entries for this

MONTHLY on page 4 indicate confusion in the mind of the compiler and "Funkel" is given for "Finkel." On page 13, for "Férpussac" read "Ferussac." The entry "Notes in mathematics, 1905, 06, 12" under Johns Hopkins University is very misleading since these are only three of many such numbers in the *Johns Hopkins University Circulars* of which Brown University has practically a complete set. On page 19 it is stated that the first volume of the *Proceedings of the Edinburgh Mathematical Society* was published in 1883, when the correct date is 1894; volume two was published in 1884. *Revista Matematica hispano-americana* was not "continuing *Revista de matemáticas*, in Buenos Aires" (pages 49, 53).

In spite of its imperfections, however, the bibliographer and research student will be grateful for this valuable new aid in the promotion of their inquiries.

R. C. ARCHIBALD.

Mathematics of Finance. By H. L. RIETZ, A. R. CRATHORNE, and J. C. RIETZ. New York, Henry Holt and Company, 1921. 12mo. 13 + 280 pp. Price $3.00.

Preface: "It is the main purpose of this book to present a teachable elementary course in the application of mathematics to a broad class of financial problems.

"The material for the book has been obtained from many sources and tested in the teaching of such courses at three large universities. Experience has shown that the material is especially adapted to the needs of the students in schools and colleges of commerce and business administration, although general liberal arts and engineering students also find the course of much value.

"The man with a liberal business education should surely be thoroughly and accurately trained in the operation of interest in relation to finance. This course is designed to supply such training. In particular, this book treats of the relation of interest to the amortization of debts, to the creation of sinking funds, to the treatment of depreciation, to the valuation of bonds, to the accumulation of funds in building and loan associations, and to the elements of life insurance.

"Three chapters are devoted to an introduction to the elements of the mathematics of insurance. This is not a technical actuarial treatment of insurance, but simply a sufficient introduction to insurance so that the general business man who studies the book may obtain proper quantitive knowledge about the first principles of life insurance; and, as a student, may come to appreciate the beautiful system of long time finance involved in legal reserve life insurance.

"For the study of the book, no mathematical preparation, except that usually included in the high school course, is absolutely necessary, but courses in freshman and even in sophomore college mathematics will be found very useful, especially if only a short time is devoted to the work on this book.

"The plan of the book is such as to afford much elasticity in the time required to cover the work with a class. This is accomplished in part by the inclusion of many applied problems that should be solved by students when this work is given as a full course of, say three hours per week for a year. When the work is given as a briefer course, many problems may be omitted. Indeed, the entire work beginning with the chapter on probability, or with the previous chapter on building and loan associations may be omitted, without destroying the continuity of the course. These omissions would make possible a very brief course. Answers are given to some of the problems to meet the needs of those teachers who find them useful.

"The necessary subjects in pure mathematics beyond the elements of algebra are given in the body of the text, in foot-notes or in special chapters at the end of the text. These final chapters are on logarithms and progressions. Students who have not studied these subjects will do well to take portions of these chapters first.

"Some rather complicated formulas are developed in the book. It is important to guard the student against the tendency to substitute half blindly in the formulas. In fact, it is one of the features of our treatment to stress the formulas which the student should think out from first principles rather than the formulas most convenient for substitution without thinking.

"From experience in teaching the subject, we feel justified in the view that the careful study of the course presented in this book will do much to create in the business student an appreciation of exact science in business."

Contents—Chapter I: Interest, 1–30; II: Annuities certain, 31–61; III: The sinking fund method of paying a debt by periodical instalments, 62–76; IV: Valuation of bonds and other securities, 77–106; V: Mathematics of depreciation, 107–134; VI: The operation of funds in building and loan associations, 135–152; VII: Theory of probability with special reference to its application in insurance, 153–171; VIII: Life annuities, 172–189; IX: Net premium for some simple forms of life insurance, 190–202; X: Valuation of life insurance policies, 203–217; XI: Logarithms, 218–242; XII: Progressions, 243–248; Tables, 251–275; Index, 277–280.

NOTES.

The last number of *Proceedings of the Royal Society,* series A, volume 99, published September 1, 1921, contains biographical notices of five deceased members. These include sketches of Robert Bellamy Clifton (1836–1921; compare *1921,* 237) by R. T. G., vi–ix; of Srinivasa Iyengar Ramanuja Iyengar (1887-1920) by G. H. Hardy, xiii–xxix; and of Woldemar Voigt (1850–1919; compare *1921,* 32) by H. L., xxix–xxx. Hardy's sketch of Ramanujan appears to be identical with the one which he published earlier in *Proceedings of the London Mathematical Society* (see *1921,* 458).

Holland now publishes six mathematical journals, five of which are well known: *Nieuw Archief voor Wiskund,*[1] *Nieuw Tijdschrift voor Wiskunde, Revue Semestrielle des Publications Mathématiques, Wiskundige Opgaven met de Oplossingen,* and *Wiskundig Tijdschrift.* The sixth journal is a bimonthly periodical entitled *Christiaan Huygens, International Mathematisch Tijdschrift,* and the first number (64 pages) was published in October, 1921, by P. Noordhoff of Groningen who is also the publisher of the *Nieuw Tijdschrift.* It is planned that these periodicals shall be issued in alternate months (12 florins a year for the two; 8 florins for *Christiaan Huygens* alone). *Christiaan Huygens* is edited by Dr. F. Schuh, professor at the Technical University in Delft, with the collaboration of several others. The editors will accept articles (in English, Dutch, South-African, French, and German), which deal with topics of pure mathematics, mechanics, and mathematical physics. An honorarium of 20 florins per 16 pages is given for accepted articles. A few problems are also to be published in each number, the solutions appearing in subsequent issues. The standard of the publication is summed up in the announcement as follows: "In order to attract a wide field of readers and in that way to secure the largest useful result, the editors propose not to carry the contents to too high a level; it will have to remain largely within the field of study of doctors, doctorandi, engineers, and those who hold the diploma K_5." (The K_5-diploma is a diploma held by those who are entitled to teach in the highest type of secondary schools.)

Mathematical Philosophy, a Study of Fate and Freedom is the title of a new series of lectures by C. J. KEYSER, crown 8vo, cloth, \$4.20, published by E. P.

[1] This is the leading Dutch mathematical journal. It began as *Archief, uitgegeven door het Wiskundig Genootschap* in 1856; the third, and last, volume of this series was completed in 1875. The first volume of *Nieuw Archief voor Wiskunde* was started in 1875.

Dutton & Co., New York.—The Department of Commerce, Bureau of the Census, announces *United States Life Tables, 1890, 1901, 1910 and 1901–1910* [496 pp., 52 full page graphs, 22 diagrams, and 185 tables. Price $1.25].

The twenty-seventh volume of the *Bulletin of the American Mathematical Society* contains a portrait of Professor F. N. Cole and the following dedication:

By order of the Council of the Society, this volume is dedicated to

FRANK NELSON COLE

in appreciation of his devotion to the Society during his twenty-six years as Secretary and in recognition of his efficient leadership in the editorial work of the Bulletin for the past twenty-four years.

These are published with the concluding number (June–July, 1921).

ARTICLES IN CURRENT PERIODICALS.

Annales Scientifiques de l'École Normale Supérieure (3d series), volume 38, January, 1921: "Aplatissement suivant l'axe polaire . . . d'une goutte liquide de révolution" . . . by J. Boussinesq, 1–12 [First paragraph: "Parmi les analogies physiques auxquelles pensèrent les théologiens du XIII^e siècle pour s'expliquer la sphéricité de la Terre, il y a celle de gouttes de pluie ou de rosée que l'on voit pendre aux feuilles des arbres, gouttes si bien arrondies surtout après s'être détachées pour tomber en chute libre. Ces théologiens sembleraient donc avoir admis, au moins implicitement, la fluidité primitive de notre globe, comme le firent d'une manière explicite, cinq cent ans plus tard, Newton et ses disciples en recourant à la pesanteur. Or il peut y avoir un certain intérêt théorique à poursuivre la même analogie des gouttes d'eau, mais d'une manière plus précise que ne l'a fait Plateau" . . .]—February to June: "Sur certaines fonctions automorphes de deux variables" by Georges Giraud, 43–164 [Quotation from page 55: "Voici donc le tableau d'ensemble de notre classification des substitutions linéaires: Substitutions hyperboliques; Substitutions elliptiques, à trois points doubles, à plan double—pénétrant a l'hypersphère principale, —extérieure a cette hypersphère; Substitutions paraboliques, à deux points doubles, à plan double, à point double unique; Substitution identique."]

The Electrician, volume 87, August 19, 1921: "The thermionic tube, a return to simplicity" by L. C. Pocock, 232–234 ["It is of course important to remember that the practical formulæ are only approximately true under finite conditions, and it is important to understand the differential equations associated with the action of the tube, but since the final formulæ in many cases depend upon the simplifying assumptions rather than on the differential equations, it may be of use to show how a simple explanation can be given"]—September 23: "Notes of the week: Progress in harmonic analysis," 373 ["Seeing however that in actual electromotive force waves, there are an infinite number of harmonics present and only a limited number of ordinates are drawn, though Mr. Clayton's method may work out well in practice, it is hardly likely to be held in very high favor among mathematicians who cling closely to Cambridge traditions"]; "The Heaviside unit and unit impulse functions" by A. Press, 376–377 ["Consider the functions, $y = x^m$, $U = x^{dm}$. We may define the Heaviside unity function, U, as the limiting form of the prior function as m approaches zero value from positive values of m only. However, the matter is not as simple as might appear due to the prevailing laxity in the use of the word 'limit' in mathematical literature. (Compare in this regard, *Fundamental Conceptions of Modern Mathematics* by Richardson and Landis, Open Court Publishing Company)" . . . "$x^0 = x^{dm} = U$. The index, zero, implies here that m approaches this value from plus values whence the index may be written dm if preferred."—A mathematical (?) exposition.]

L'Enseignement Mathématique, volume 21, 1920, nos. 5–6 (published July, 1921): "La notion d'équivalence dans la théorie des groupes" by G. A. Miller, 251–254; "Quelques

remarques sur un théorème relatif à la série hypergéométrique et sur la série de Kummer" by C. Cailler, 255–259; "Méthode à suivre pour déterminer la totalité des nombres premiers comprise entre 1 et le nombre $N = 1.2.3.5.7.11. \cdots p$, produit des $(\alpha+1)$ nombres premiers consécutifs pris a partir de 1" by E. Barbette, 260–265; "Sur l'équation fonctionelle $f[\varphi_1(t)] = f[\varphi_2(t)]$" by R. Wavre, 265–277; "Sur l'enseignement de calcul différentiel et intégral en Grèce" by P. Zervos, 278–281; "La préparation théorique et pratique des professeurs de mathématiques de l'enseignement secondaire en Argentine" by N. B. Moreno, 281–304; "La Commission Internationale de l'Enseignement Mathématique de 1908 à 1920.—Compte rendu sommaire suivi de la liste complète des travaux publiés par la Commission et les Sous-commissions nationales" by H. Fehr, 305–342; "Chronique," 343–349; "Bibliographie," 349–353; "Bulletin bibliographique," 354–366; Index, 367–370.

Giornale di Matematiche di Battaglini, volume 58, July–December, 1920: "Dimostrazione vettoriale di alcuni teoremi di Kasner," Nota di Clara Abbondati, 193–208 [Foot-note: "Edw. Kasner, 'The trajectories of Dynamics,' *Trans. American Mathem. Society*, v. 7, pp. 401–424, 1906; 'Dynamical trajectories: the motion of a particle in an arbitrary Field of Force,' *Ibid.*, v. 8, pp. 135–158, 1907."]

The International Studio, volume 74, November, 1921: "Dynamic symmetry and its practical value today" by Maxwell Armfield, 76–85 [See *1920*, 314].

Journal of the Indian Mathematical Society, volume 13, June, 1921: "Third conference of the Indian Mathematical Society," 81–89; "An extension of Feuerbach's theorem" by M. B. Rao, 109–116; Questions and Solutions, 117–120.

Mathematical Gazette, volume 10, October, 1921: "Mathematicians and their work" by L. J. Mordell, 321–323 [Review of F. Cajori, *A History of Mathematics*, 2d edition. "Prof. Cajori has taken the opportunity of adding a few new chapters, and of completely revising, in the light of modern knowledge, that part of his *History* dealing with the older work. The greatest attraction, however, of the volume is the account of the nineteenth and twentieth centuries, which now forms some two hundred of the present five hundred pages, making it almost a new history. There are two obvious difficulties in writing this section; not only are many of those mentioned still living, but of far more importance is the difficulty of keeping in touch with, and being conversant with, the relative importance of developments along so many different and unconnected lines. Prof. Cajori has solved the second difficulty by quoting extracts from reports and addresses by numerous experts. These, moreover, he has blended so well with his other material that he has produced a most useful and fascinating volume"]; "Mathematical Notes: A proof of the formula for the volume of a tetrahedron, in terms of the rectangular coördinates of its vertices" [not involving the square root which appears in most proofs but disappears in the result] by R. F. Muirhead, 324–325.

The National Marine, New York, volume 18, July, 1921: "The evolution of the patent log. How early mariners measured distances through the water and the improvements that have been wrought" by Bradley Jones, 1–14.

Nature, volume 108, October 6, 1921: "L'espace dans la chemie" (editorial), 171–172 [Review of D. A. Clibbens, *The Principles of the Phase Theory*. "The present work of Dr. Clibbens is the first book in English to attack seriously the real geometry of the subject. He carries us from the delightful simplicity of the binary systems right to the real thing—quaternary and quinary systems"]; Review of J. Cusack, *The Arithmetic of the Decimal System* by S. Brodetsky, 174–175 [The author does not adopt the metric system, but uses the decimal system with the present English units, the yard, the hundredweight, the gallon, the pound sterling, etc.]—October 13: "An algebraical identity" by R. F. Whitehead, 212 [Reply to an inquiry as to the primitive character of a solution of Gauss's cyclotomic equation]; "Notes," 222–226 ["In his address on October 3 to the newly constituted Glasgow Society for Psychical Research Sir Oliver Lodge gave an interesting exposition of his view of the æther of space as a region of possibilities in contrast with matter as a region of facts . . . he argued that if mind when dissociated from matter continues to exist, it can only be that there is something else which can perform the function of matter and serve as its instrument. For himself he has told us he is convinced that disembodied spirit personalities do exist in fact, and therefore for him it would seem the æther is a necessary postulate. His acceptance of the principle of relativity does not apparently in the least affect his belief in the real physical existence of the æther; it seems only to have added a few more negative qualities to that exceedingly elusive stuff and made its residual positive reality more than ever difficult to imagine. Still, perhaps the new society may succeed where Michelson and Morley failed, for psychical research, as Sir Oliver conceives it, is purely and essentially physical research, however

suspect to some of us its methods may appear."]—October 20: "The tendency of elongated bodies to set in the north and south direction" by A. Schuster, 240 [. . . "it may be pointed out that the earth's centrifugal force would act in a manner tending in the direction of the alleged effect, though the resulting couple is so minute that it would be extremely difficult to verify it experimentally. If a horizontal rod be placed in the north and south direction, its southern end is—in the northern hemisphere—further away from the earth's axis. The centrifugal force is therefore greater at the southern end, and if the rod be slightly displaced, the horizontal component of that force will tend to bring the rod back into the meridian plane."]; "Metaphysics and materialism" by H. W. Carr, 247–248 ["Materialism does not stand for any particular theory of the nature of matter, but for the general world-view that matter . . . exists and constitutes the reality of the universe, including reason and will, which as qualities or properties of some of its forms give rise to knowledge of it. This materialism reached the zenith of its expression in the Darwinian theory of natural selection, not in that theory itself . . . but in the implications which were generally accepted as contained in it, and especially in the application which was made of it to rationalize a world-view. It seemed to point a way by which it was possible to conceive, and to some extent to follow in its history, an evolution which had produced mind from an original matter. It may not be obvious at once that the mere rejection of the Newtonian concept of absolute space and time and the substitution of Einstein's space-time is the death-knell of materialism, but reflection will show that it must be so."]; "Notes" [Quotation from page 255: "*The Dictionary of Applied Physics* which Messrs. Macmillan and Co. Ltd. propose to issue under the editorship of Sir Richard Glazebrook is now in an advanced stage of preparation. The work will appear in five volumes of 600–700 pages each. . . . Dr. Horace Lamb has provided several articles on related mathematical questions."]; "Our astronomical column," 256 ["The Astronomical Union will meet next year in Rome, and among the Committee meetings that will be held there is one on calendar reform. . . . The main outlines of the reforms to be discussed include a more uniform arrangement of the lengths of the months, alteration in the position of the leap-day (the end of the year would be far more convenient from the point of view of astronomical tables), and making the incidence of the week-days the same every year by placing one day a year (two in leap-year) outside the weekly reckoning. . . . It is much easier to recognize the inconvenience of the present system than to agrée on an alternative one."]

La Nature, volume 49_2, August 27, 1921: "Le problème de la suspension des voitures automobiles" by E. Weiss, 129–132—September 3: "Calculateur circulaire à disque mobile Arnault-Paineau" by J. Boyer, 70 supplément.

Periodico di Matematiche (See *1921*, 317) series 4, volume 1, no. 4, July, 1921: "Noterelle di logica matematica" by F. Enriques, 233–244; "I moti planetari e le leggi di Keplero" by E. Daniele, 245–262; "Gli inviluppi di linee curve ed i primordi del metodo inverso delle tangenti" by E. Bortolotti, 263–276; "Una interpretazione geometrica ed una estensione della divisibilità dei polinomî" by S. Pincherle, 276–282 [Suggests the representation of a polynomial of degree n by the point in space of $n + 1$ dimensions whose coördinates are the coefficients of the polynomial].

Philosophical Magazine, volume 42, September, 1921: "On certain fundamental principles of scientific inquiry" by Dorothy Wrinch and H. Jeffreys, 369–390; "The mental multiplication and division of large numbers" by V. A. Bailey, 390–397—October: "The convection coefficient in a dispersive medium" by A. Anderson, 509–511; "The forms of planetary orbits on the theory of relativity" by W. B. Morton, 511–522.

["Summary" (of article on certain fundamental principles of scientific inquiry): "It is shown that intensive theories of the structure of Nature which involve the use of infinite classes of entities of kinds known only by observation are not capable of yielding satisfactory accounts of the method by which we have acquired our knowledge of physics, and that the same applies to the theory of universal consent as a basis of scientific knowledge. The whole of a single person's knowledge is based upon a finite number of observations and his individual judgments, and the problem of a theory of scientific knowledge is to show how this can be carried out.

"It is clear that such a theory must depend on the theory of probability, and the question of the probability of physical laws and of inferences based on them is discussed. It is shown that it will never be possible to attach appreciable probability to an inference if it is assumed that all laws of an infinite class, such as all relations involving only analytic functions, are equally probable *a priori*. If inference is possible, the admissible laws must not be all equally probable *a priori*. It is suggested that all admissible laws can be arranged in a well-ordered sequence, each

having a finite prior probability, and such that each is more probable than any that follows it in the sequence. The probabilities of the laws must form a convergent series. On this basis it is shown that with sufficient empirical verification of a law the probability of further inferences from it will approach certainty.

"There is reason to believe that all admissible laws must form an enumerable aggregate, and this condition, and apparently all others which are necessary, are satisfied if we suppose that all laws admissible in physics are expressible as differential equations of finite order and degree, with rational coefficients. The most natural ways of well-ordering these are such that those of low order and degree, and involving no numerical constants other than small integers and fractions with small numerators and denominators, come earliest in the sequence. Accordingly, the practice of adopting the simplest law that fits the available observations appears to be closely related to the possibility of a satisfactory theory of inference."]

Proceedings of the London Mathematical Society, series 2, volume 20, part 3, August, 1921: "The classification of rational approximations" by P. J. Heawood, 233–240 [The construction of irrational numbers θ for which there are at most only a finite number of approximations x/y such that $|x/y - \theta| < 1/ky^2$ for a given value of k, x/y being a fraction in its lowest terms].

Revista de Matematicas y Fisicas Elementales (See *1920*, 475; Spanish), volume 3, August, 1921: "Notes on analytical geometry" by E. Rebuelto, 73–78; "On projective coördinates" (conclusion) by F. D. Jaime, 78–84; "Note: On the sums of certain series" by J. S. Corti, 84–85—September: "The vectorial product" by T. Lalescu, 97–99; "Construction of conics" by J. S. Cortie, 99–101; "Notes: Rapid elimination of x from two equations of the third degree" by B. I. Baidaff, 106; "A relation of the theory of numbers" by T. I. G. Lintes, 106–107; Problems solved, Exercises, Problems proposed, 112–120.

Revue Générale des Sciences, volume 32, August, 1921: "Développement mathématique élémentaire de la relativité restreinte" by M. Sauger, 453–459 [A very simple derivation of the Lorentz equations and some mechanical and physical consequences]; Bibliographie: H. Broggi, *Análisis matemático*, 480 ["L'apparition de ce volume est un nouvel indice du mouvement qui tend à substituer, dans l'Amérique latine, aux ouvrages didactiques en langues étrangères ou aux traductions d'ouvrages étrangers, des œuvres dues aux professeurs et savants locaux."]—September: Bibliographie, 515, "Du Pasquier—le développement de la notion du nombre" [review] by Dr. Sorel.

Revue Scientifique, volume 59, August 27, 1921: "La description géométrique des Alpes françaises" by L. R., 465 [Comments on the geodesic work of M. Helbronner as described in *La Montagne*, January, 1921]—September 10: "Notes et actualites: La Société des Amis des Sciences," 505–507 [Address of its President, M. Emile Picard. Quotation: "Mesdames, Messieurs, nous nous efforçons d'employer le mieux possible notre petite fortune, en suivant les intentions des fondateurs de la Société. Vous savez qu'ils s'étaient proposé de mettre à l'abri du besoin les savants et inventeurs peu fortunés, ainsi que leur famille après leur mort. Il nous était déjà impossible de remplir ce programme à la lettre avant la guerre, dans ces temps à la fois si proches et si lointains, où nous n'appréciions pas suffisamment, semble-t-il, la douceur de vivre à bon marché. . . . Quand ces temps heureux seront revenus, notre Société, recevant de toutes parts des dons généreux, deviendra une sorte de Ministère de la reconnaissance publique envers les savants et inventeurs malheureux. Je veux croire que ce n'est pas là un rêve; en attendant sa réalisation, continuons, mes chers collègues, notre propagande en faveur de l'œuvre qui nous est chère, et cherchons à lui amener de nouvelles recrues pour son action bienfaisante."]

School Science and Mathematics, volume 21, October, 1921: "A dream come true" [an account of an experiment in teaching geometry inductively] by W. A. Austin, 621–627; "An extension of a process in factorization" by S. M. Karmalkar, 628–630; "Falling bodies in ancient and modern times" by F. Cajori, 639–648 [discussion of the question whether the story of Galileo's experiments at the tower of Pisa is mythical. See *Science*, n.s., volume 51, pp. 615–616, June 18, 1920; volume 52, pp. 272–273, September 17, 1920, and p. 409, October 29, 1920. "Recent criticism endeavors to sift the truth from a conglomerate mass of facts and fiction in the history of science. The movement is to be welcomed, whenever the work is carefully done. But an iconoclast may brush aside both facts and fiction, and leave unto the reader of science only a barren waste. The rebuilder of the historic structure may be an untrained architect with no sense of proportion or comprehension of the values of historic materials. The subject of falling bodies has had its trained historians and its amateurish historians. Comparatively simple questions have received different answers. Has Aristotle been misrepresented by Galileo to modern

writers? Did Aristotle actually claim that a heavier body falls faster in proportion to its weight? Some years ago an English writer, J. H. Hardcastle, denied this. Quoting a brief passage from Aristotle he argued that historians had grievously erred, that the great Peripatetic had been stupidly misunderstood. . . . Other of England's noted scientists made brief comments, basing their arguments on the one passage taken from Aristotle. Thus G. Greenhill, William Ramsay, and Oliver Lodge arrive at the conclusion that Aristotle has been misinterpreted, that he did not mean a body ten times heavier would fall from rest ten times faster, that he was thinking of a body moving through a resisting medium, like the air, which greatly modifies the motion, and was considering, not its initial, but its terminal velocity." They "did not themselves go back to search in Aristotle's writings; their deductions were based on the one quotation which had been brought to their attention. As a matter of fact, that one quotation gave only a partial exposition of Aristotle's theory."]; "Gerbert's letter to Adelbold" by G. A. Miller, 649–653; "The mathematics needed in freshman chemistry" by L. W. Williams, 654–665 [A classified list of over three hundred "mathematical" terms is given. The list includes such terms as "subdivisions," "double," "year," "1/3."]; "Do high school pupils dislike mathematics" by W. E. Gingery, 674–675; Problems and Solutions, 676–679; "Reception to the members of the National Committee of Mathematics Requirements for Secondary Schools" by W. J. Ryan, 696, 698—November: "An analysis of an experiment in teaching first year mathematics" by Ina E. Holroyd, 757–764; "Some plane geometry problems" by Lida C. Martin, 765–769; Problems and Solutions, 787–790; "The National Committee on Mathematical Requirements," 798, 800.

Tôhoku Mathematical Journal, volume 19, nos. 3–4, July, 1921: "The late Prof. Dr. Tetsugo Kojima" (Japanese), insert; "Sur les surfaces sphériformes" by G. Tiercy, 149–163; "Über Maxima und Minima von quadratischen Formen mit unendlichvielen Variabeln" by T. Kubota, 164–168; "Sur les équations différentielles homogènes" by F. Sibirani, 169–172; "On the roots of an algebraic equation" by M. Tajima, 173–174; "On the solutions of Mathieu's equations of the second kind" by S. C. Dhar, 175–182; "Cyclotomic sexe-section" by P. O. Upadhyaya, 183–186; "On the twisted cubic of constant curvature" by S. Narumi, 187–195; "On the characteristic property of the conic section" by S. Narumi, 196–204; "On the osculating conics of a plane curve" by T. Hayashi, 205–209; "Ueber gewisse Infinitesimal-Operationen der höheren Operationsstufen" by R. E. Moritz, 210–237; "Bemerkungen zu der Arbeit von Herrn Ogura: 'On the theory of the tides'" by O. Perron, 238–240; "Bemerkung über die Mittag-Lefflerschen Funktionen $E_a(z)$" by G. Pólya, 241–248; "Pentaspherical geometries in noneuclidean space, I" by T. Ota, 249–270; Shorter notices and reviews, 271–275; Miscellaneous notes, 276–279.

Zeitschrift für angewandte Mathematik und Mechanik, volume 1, August, 1921: "Über algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen" by J. Schur, 307–311.

UNDERGRADUATE. MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

The Mathematical Club of Adelphi College, Brooklyn, N. Y.

The Mathematical Club of Adelphi College was founded in the fall of 1898; and is apparently one of the oldest mathematical clubs in the country. Membership is open not only to students but to their friends. Meetings are held the second Wednesday of each month, and are well attended. The club has also a social meeting once a year.

During the year 1920–21 talks have been given upon the topics: "Lewis Carroll as a mathematician;" "Women as mathematicians;" "The trisection of an angle;" and certain topics suggested in The American Mathematical Monthly.

(Report by Evelyn Brisbane, secretary.)

The Mathematics Club of Cooper Union, New York City.

The Mathematics Club of Cooper Union was organized October 11, 1920. At this meeting it was decided to limit active membership for the current year to first-year and second-year

THE EUCLID CLUB OF THE UNIVERSITY OF WASHINGTON, Seattle, Wash.

[*1919*, 170.]

In November, 1920, the Mathematics Club of the University of Washington was reorganized under the name of the Euclid Club of the University of Washington, and the following officers were elected: Rubin Raport '23, president; Howard Robertson '23, vice-president; Lillie Siler '21, secretary.

The following programs were given:

December 16, 1920: "Einstein's theory of relativity" by Professor E. T. Bell.
January 20, 1921: "Greek numbers" by Helen Dunn '22.
February 3: "The Cyclo-harmonograph, a machine for drawing curves" by Professor R. E. Moritz (the inventor).
February 15: "Who's who in modern mathematics" by Gustene Rupe '23.
March 10: "Modern mathematical machines and famous mathematicians" by Lillie Siler '21.
April 7: "The use of mathematics in science" by Howard Robertson '23.
May 26: "The proof and use of the planimeter" by Rubin Raport '23.

(Report by Mr. Raport.)

NOTES.

Through the courtesy of one of our contributors, Mr. F. V. MORLEY, a Rhodes scholar from the United States at New College, Oxford, we are permitted to inspect the first two numbers of the manuscript *Proceedings of the Oxford University Undergraduate Mathematical Club*. The Club was started in October, 1920, and seven meetings were held during the year. One paper is read at each meeting and the *Proceedings* contain the complete papers. For the Michaelmas Term the papers were: "John Wallis" by J. S. HUGHES, of New College, and "Some circles connected with the triangle" by H. O. NEWBOLT, of Balliol. For the Hilary Term: "Introduction to inversive geometry" (2 parts) by F. V. MORLEY and "The twisted cubic" by Mr. TITCHMARCH, of Balliol. During the Summer Term two meetings were held in May, and Professor FRANK MORLEY, of Johns Hopkins University, and W. R. BURWELL, of Brown University, were the speakers.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2941. Proposed by W. D. CAIRNS, Oberlin College.

$1^2 + 2^2 + 3^2 + \cdots + (n-1)^2$ is a function of n. Find its derivative with respect to n.

2942. Proposed by L. E. DICKSON, University of Chicago.

I am dealt 13 cards at whist. What is the chance that all my cards will be diamonds?

2943. Proposed by L. E. DICKSON, University of Chicago.

In the game of bridge, what is my chance: (*a*) that my hand will contain 4 aces? (*b*) that some hand will contain 4 aces? (*c*) that my hand will be a Yarborough, *i.e.*, contain no honor?

2944. Proposed by S. A. COREY, Des Moines, Iowa.

A particle of mass m, starting from rest, is drawn by a string over a smooth horizontal plane, the other end of the string moving in the plane with uniform acceleration n along a line perpendicular to the initial position of the string. Prove that the tension of the string is $3mn \cos \theta$, where θ is the angle which the string makes with the given line. Also prove that the motion of the particle is vibratory.

2945. Proposed by T. M. BLAKSLEE, Ames, Iowa.

A point P in the plane of the triangle ABC rotates in a given direction around the vertices taken in either cyclical order, in each case through an angle equal to the corresponding angle of the triangle. That is, for example, AP rotates around A through an angle equal to the angle A of the triangle; then BP around B through an angle equal to the angle B, and so on. Prove that P coincides with its original position at the end of six of these rotations. (See problem 2899, *1921*, 228.)

2946. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

Cut a regular hexagon into the smallest number of pieces that can be fitted together to form an equilateral triangle: (*a*) no piece to be turned over; (*b*) some pieces may be turned over.

2947. Proposed by D. H. MENZEL, Princeton University.

An oil tank has the shape of a cylinder with ends which are segments of a sphere and with horizontal axis. The diameter of the cylinder being given, and the radius of the spherical segments, derive a formula that will express the volume of the liquid contained in the tank in terms of its depth.

2948. Proposed by J. B. REYNOLDS, Lehigh University.

Find the envelope of the normal planes to the curve,

$$x = a \cos t, \qquad y = a(1 - \cos t), \qquad \text{and} \qquad z = a \sin t.$$

2949. Proposed by J. B. REYNOLDS, Lehigh University.

Find the lateral area of the cone with vertex at (0, 0, h) and whose base is the epicycloid,

$$x = \tfrac{3}{2}a \cos \theta - \tfrac{1}{2}a \cos 3\theta, \qquad y = \tfrac{3}{2}a \sin \theta - \tfrac{1}{2}a \sin 3\theta.$$

2950. Proposed by T. M. SIMPSON, JR., Randolph-Macon College, Ashland, Va.

Determine the curve which cuts the radius vector at an angle proportional to the radius vector.

NOTES.

25. The Area of a Quadrilateral.—The first expression for the area of an inscribed convex quadrilateral, in terms of its sides, was given by Brahmagupta (born 598 A.D.), without proof, in the following form:[1] "The product of half the sides and countersides is the gross area of a triangle and tetragone. Half the sum of the sides set down four times, and severally lessened by the sides, being multiplied together, the square root of the product is the exact area." The latter result appeared in a treatise written by Mahāvīrācārya, about 850 A.D., who gives[2] "The rule for arriving at the minutely accurate measurement of the

[1] *Algebra with Arithmetic and Mensuration from the Sanscrit of Brahmegupta and Bháscara.* Translated and edited by H. T. Colebrooke, London, 1817, pp. 295–296. See H. Weissenborn "Das Trapez bei Euklid" *Abhandlung zur Geschichte der Mathematik*, Heft 2, Suppl. historisch-literarischen Abtheilung, *Zeitschrift für Math. u. Physik*, vol. 4, 1879, pp. 181–184.

[2] *The Ganita-Sāra-Sangraha of Mahāvīrācārya* with English translations and notes. By M. Raṅgācārya, Madras, 1912, p. 198. Neither Brahmagupta nor Mahāvīrācārya knew that their rule for the exact area of a quadrilateral was only true for cyclic figures. The inexactness of the rule for all quadrilaterals was first pointed out by Bháskara (*l.c.*, § 167) who was born 1114 A.D.

area (of trilateral and quadrilateral figures):—. . . Four quantities represented (respectively) by half the sum of the sides as diminished by (each of) the sides (taken in order) are multiplied together; and the square root (of the product so obtained) gives the minutely accurate measure (of the area of the figure)." From this the area of a trilateral figure may be represented algebraically as[1] $\sqrt{[s(s-a)(s-b)(s-c)]}$, where s is half the sum of the sides whose lengths are a, b, and c; while the area of the quadrilateral figure[2] is

$$A = \sqrt{[(s-a)(s-b)(s-c)(s-d)]}, \tag{1}$$

where s is half the sum of the sides whose lengths are a, b, c, d. Moreover both Brahmagupta (*l.c.*, pp. 300–301) and Mahāvirācārya (*l.c.*, p. 199) state what is equivalent to the following expressions for the lengths of the diagonals of the quadrilateral:

$$m = \sqrt{\left[\frac{(ac+bd)(ab+cd)}{ad+bc}\right]}, \qquad n = \sqrt{\left[\frac{(ac+bd)(ad+bc)}{ab+cd}\right]}. \tag{2}$$

Formula (1) was rediscovered by Snellius who gives it in his commentary on the first book of Ludolphe van Ceulen's *De problematibus miscellaneis*.[3] The statement of the rule is given by Ozanam, without proof or reference to any other writer, in his *Cours de Mathematique* (Paris, vol. 3, 1690, p. 212; nouvelle édition, 1699, p. 147). Noticing this result Philip Naudé gave two demonstrations[4] of extraordinary complexity[5] to which Euler pays his respects introductory to furnishing a geometrical proof.[6]

Taking A, B, C, D as the vertices, in order, of an inscribed quadrilateral, and supposing AB, DC to meet in E, Euler first finds the following expression for the area, Q, of the quadrilateral,

$$QQ = \frac{1}{16}\cdot\frac{(AD-BC)(BE+CE+BC)}{BC}\cdot\frac{(AD-BC)(BE+CE-BC)}{BC}$$
$$\cdot\frac{(AD+BC)(BC+BE-CE)}{BC}\cdot\frac{(AD+BC)(BC-BE+CE)}{BC}.$$

He then shows that, apart from the numerical factor, the successive factors

[1] A result for which Heron of Alexandria gave an elegant proof hundreds of years earlier. The result was discovered by Archimedes who flourished still earlier.

[2] In connection with the following formulae of this note it will be supposed, unless otherwise stated, that a convex quadrilateral only is considered. The modifications for other cases are not difficult; some of them are later noted.

[3] *Oeuvres mathématiques de Ludolphe Van Ceulen*, traduites du hollandais en latin et enrichies de notes, par Snellius. Leyden, 1619. This was about 200 years before the first publication of Brahmagupta's results.—See Chasles, *Aperçu historique sur l'origine et le développement des méthodes en géométrie* . . . 2e édition, Paris, 1875, pp. 292, 432.

[4] "Demonstratio trium theorematum," *Miscellanea Berolinensia* . . . tome 3, 1727, pp. 259–269.

[5] C. L. A. Kunze comments on the proof which "zwar streng und bündig, aber mit einer *unerträglichen Weitschweifigkeit* abgefasst ist" (*Ueber einige theils bekannte, teils neue Sätze vom Dreieck und Viereck*, zweite vermehrte Ausgabe, Halle, 1848, p. 4).

[6] L. Euler, "Variae demonstrationes geometriae," *Nova acta acad. sc. Petrop.*, vol. 1 (1747–1748), 1750, pp. 57–63.

of the right-hand member are, respectively, $2S - 2BC$, $2S - 2DA$, $2S - 2AB$, $2S - 2CD$, where $2S = BC + DA + AB + CD$; whence he arrives at the form

$$Q = \sqrt{[(S - AB)(S - BC)(S - CD)(S - DA)]}.$$

A neat trigonometrical derivation of this result was given by Fuss[1] in a paper of considerable interest for the history of Poncelet polygons. Several other results here given will be noted in what follows.

The area was given in determinant form by Dostor:[2]

$$16A^2 = - \begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix}.$$

Such a form would be suggested naturally to anyone familiar with formula (1), and noticing the factors of this determinant, with a substituted for $-a$, given by Ferrers[3] in 1861.

In 1782 Lhuilier gave,[4] in effect, the following expression for the radius, R, of the circle circumscribing the quadrilateral:

$$R = \frac{1}{4}\left[\frac{(ab + cd)(ac + bd)(ad + bc)}{(s - a)(s - b)(s - c)(s - d)}\right]^{\frac{1}{2}}. \tag{3}$$

For the numerator, within brackets, of the right-hand member, Klügel-Mollweide-Grunert substituted:[5] $abcd\Sigma a^2 + (abcd)^2\Sigma(1/a)^2$.

It was early recognized that, in general, there are three inscribed convex quadrilaterals with sides of given lengths; that these were equivalent in area but not superposable; that they have only three diagonals different in length. In 1626 Albert Girard stated[6] that the product of the lengths of the three different diagonals divided by twice the diameter of the circumcircle, is equal to the area of each of the quadrilaterals.

Hence, from (3), it appears that the product of the lengths of the three diagonals is $[(ab + cd)(ac + bd)(ad + bc)]^{1/2}$ which would be expected, since, from "Ptolemy's theorem," or (2), the product of the diagonals (m, n) of the quadrilateral whose sides are consecutively a, b, c, d, is given by $mn = ac + bd$.

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[1] N. Fuss, "De quadrilateris quibus circulum tam inscribere quam circumscribere licet," Conuentui exhib. die 14. Aug 1794, *Nova acta acad. sc. Petrop.*, vol. 10 (1792), 1797, pp. 103–125.

[2] G. Dostor, *Eléments de la théorie des déterminants*, Paris, 1877, p. 198.

[3] N. M. Ferrers, *An Elementary Treatise on Trilinear Coördinates*, Cambridge, 1861, p. 70.

[4] S. Lhuilier, *De Relatione Mutua . . . seu de Maximis et Minimis*, Warsaw, 1782, p. 21. Formula (3), in just this form, was given by Fuss (*l.c.*), p. 105.

[5] Klügel-Mollweide-Grunert, *Mathematisches Wörterbuch*, fünfter Theil, zweiter Band, Leipsic, 1831, p. 875.

[6] A. Girard, *Tables de sinus, tangentes et sécantes selon le raid de 10000 parties . . .*, 1626; Dutch translation, 1629. Compare Kästner, *Geschichte der Mathematik*, Göttingen, vol. 3, 1799, pp. 107–110; and S. Günther, *Vermischte Untersuchungen zur Geschichte der mathematischen Wissenschaften*, Leipsic, 1876, pp. 17–21.

The area, A', of the diagonal triangle of an inscribed quadrilateral was given by Dostor[1] in 1868 as

$$A' = \frac{4a^2b^2c^2d^2A}{(a^2b^2 - c^2d^2)(a^2d^2 - b^2c^2)};$$

whence Dostor found

$$\frac{A}{A'} = \frac{1}{4}\left(\frac{a^2}{c^2} - \frac{b^2}{d^2} + \frac{c^2}{a^2} - \frac{d^2}{b^2}\right) = \frac{1}{4}\left(\frac{ab}{cd} - \frac{cd}{ab}\right)\left(\frac{ad}{bc} - \frac{bc}{ad}\right).$$

With the above notation Dostor showed,[2] in 1848, that in the case of any quadrilateral,

$$16A^2 = 4m^2n^2 - (a^2 - b^2 + c^2 - d^2)^2. \tag{4}$$

He showed also that if the quadrilateral is circumscribed about a circle

$$\begin{aligned} A &= \tfrac{1}{2}\sqrt{[(mn + ac - bd)(mn - ac + bd)]}, \\ &= \tfrac{1}{2}\sqrt{[(mn + a'a''-b'b'' + c'c''-d'd'')(mn-a'a'' + b'b''-c'c'' + d'd'')]}, \end{aligned} \tag{5}$$

where a', a''; b', b''; c', c''; d', d'' are the segments of the respective sides formed by the points of contact such that $a'' = b'$, $b'' = c'$, $c'' = d'$, $d'' = a'$.

Dostor found (*l.c.*, 1848, p. 73) that if p and q are the lengths of the line segments joining the middle points of pairs of opposite sides of any quadrilateral,

$$A = \tfrac{1}{2}\sqrt{[(p + q + m)(p + q - m)(p + q + n)(p + q - n)]}.$$

In 1874, he gave[3] yet another expression in terms of the coördinates of the four vertices:

$$2A = \begin{vmatrix} 1 & 0 & x_1 & y_1 \\ 0 & 1 & x_2 & y_2 \\ 1 & 0 & x_3 & y_3 \\ 0 & 1 & x_4 & y_4 \end{vmatrix}.$$

If a, b, c, d are the consecutive sides of a convex quadrilateral, and $\delta(\neq 0)$ is the length of the line joining the middle points of the principal diagonals, Catalan found[4] that

$$16\delta^2A = (d^2 - b^2)\sqrt{[4a^2c^2 - (a^2 + c^2 - 4\delta^2)^2]} + (c^2-a^2)\sqrt{[4b^2d^2 - (b^2 + d^2-4\delta^2)^2]}.$$

Strehlke showed[5] that if A and C are a pair of opposite angles of the quadrilateral

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[1] G. Dostor, *Propriétés Nouvelles des Quadrilatères en Général* . . . Greifswald, [1868], p. 28; also in *Archiv der Mathematik und Physik*, vol. 48. In connection with this result E. W. Hobson erroneously omits the factor 4 (*A Treatise on Plane Trigonometry*, second edition, 1897, p. 205; fourth edition, 1918, p. 208).

[2] G. Dostor, *Nouvelles Annales de Mathématiques*, vol. 7, 1848, pp. 70 and 230; also vol. 33, 1874, p. 563.

[3] *Nouvelles Annales de Mathématiques*, vol. 33, 1874, p. 562; also *Archiv der Mathematik und Physik*, vol. 56, 1874, p. 240.

[4] E. Catalan, *Nouvelle Correspondance Mathématique*, vol. 6, 1880, pp. 52–53.

[5] Strehlke, "Zwei neue Sätze vom ebenen und sphärischen Viereck" . . ., *Archiv der Mathematik und Physik*, vol. 2, 1842, p. 324. J. F. König found (*Archiv* . . ., vol. 34, 1860, p. 14), for the area of a spherical quadrilateral the following expression which, in part, reminds one of Strehlke's formula:

$$A^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos^2\tfrac{1}{2}(A+C). \tag{6}$$

Hence[1] from formula (4)

$$\left.\begin{aligned}16A^2 &= 4(ac+bd)^2 - (a^2-b^2+c^2-d^2)^2 - 16abcd\cos^2\tfrac{1}{2}(A+C),\\ &= 4(ac-bd)^2 - (a^2-b^2+c^2-d^2)^2 + 16abcd\sin^2\tfrac{1}{2}(A+C).\end{aligned}\right\} \tag{7}$$

If a circle can be inscribed in the quadrilateral, $a+c=b+d$, and this formula becomes[2]

$$A^2 = abcd\sin^2\tfrac{1}{2}(A+C).$$

If the quadrilateral is also inscribable[3] (Fuss, *l.c.*, p. 114),

$$A^2 = abcd; \tag{8}$$

under these conditions the radius of the inscribed circle is[4] $r = A/s = A/(a+c)$; for R, Klügel-Mollweide-Grunert wrote (*l.c.*)

$$R = \tfrac{1}{4}\sqrt{[\Sigma a^2 + abcd\Sigma(1/a)^2]}.$$

$$\sin\frac{E}{2} = \frac{\sqrt{[\sin(s-a)\sin(s-b)\sin(s-c)\sin(s-d) - \sin a\sin b\sin c\sin d\cos^2\frac{1}{2}(B+D)]}}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}$$
$$+\frac{\sin a\sin b\sin B\cos\frac{1}{2}(c+d)\cos\frac{1}{2}(c-d) + \sin c\sin d\sin D\cos\frac{1}{2}(a+b)\cos\frac{1}{2}(a-b)}{8\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d\cos^2\frac{1}{2}m},$$

where m is the diagonal AC, and E is the spherical excess of the quadrilateral $ABCD$. See also *Archiv. . . .*, vol. 4, 1843, p. 448.

[1] R. Baltzer, *Die Elemente der Mathematik*, Leipsic, vol. 2, 1862, p. 305.

[2] S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, p. 253.

[3] Corresponding to formulæ (4), (5), and (8), P. Serret gave for the spherical quadrilateral, (*Des Méthodes en Géométrie*, Paris, 1855, p. 43):

$$\sin^2\frac{E}{2} = \frac{(\sin\frac{1}{2}m\sin\frac{1}{2}n + \cos\frac{1}{2}a\cos\frac{1}{2}c - \cos\frac{1}{2}b\cos\frac{1}{2}d)(\sin\frac{1}{2}m\sin\frac{1}{2}n + \cos\frac{1}{2}b\cos\frac{1}{2}d - \cos\frac{1}{2}a\cos\frac{1}{2}c)}{4\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}; \tag{4'}$$

$$\sin^2\frac{E}{2} = \frac{\sin\frac{1}{2}(s-a)\sin\frac{1}{2}(s-b)\sin\frac{1}{2}(s-c)\sin\frac{1}{2}(s-d)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}; \tag{5'}$$

$$\sin^2\frac{E}{2} = \tan\tfrac{1}{2}a\tan\tfrac{1}{2}b\tan\tfrac{1}{2}c\tan\tfrac{1}{2}d; \tag{8'}$$

where E is the spherical excess. From formula (5′) we get (Grunert, *Nouvelles Annales de Mathématiques*, vol. 22, 1863, p. 336)

$$\cos^2\frac{E}{2} = \frac{\cos\frac{1}{2}s\cos\frac{1}{2}(s-a-b)\cos\frac{1}{2}(s-a-c)\cos\frac{1}{2}(s-a-d)}{\cos\frac{1}{2}a\cos\frac{1}{2}b\cos\frac{1}{2}c\cos\frac{1}{2}d}.$$

[4] Fuss found this result (*l.c.*, p. 112) by considering the question: What shall be the radius of a circle to which a quadrilateral with given sides is circumscribed, in order that the area of the quadrilateral shall be a maximum? He showed that it was when the quadrilateral was cyclic, *i.e.*, inscribed also in a circle. He found also an expression for the distance between the centers of these circles in terms of their radii. In this connection one may consult "The in-and-circumscribed quadrilateral" by W. E. Byerly (*Annals of Mathematics*, vol. 10, pp. 123–128, 1909); it is there shown that the necessary and sufficient condition for two circles (radii r, R) the distance between whose centers is x, to have a quadrilateral circumscribed to the first and inscribed to the second is that

$$\frac{1}{r^2} = \frac{1}{(x+R)^2} + \frac{1}{(x-R)^2}.$$

Déprez gave the following relation between the diagonals and the radii: $mn/2r = r + \sqrt{r^2+4R^2}$ (*Mathesis*, vol. 8, 1888, pp. 80, 104, 237–239). See also O. Kolberg, Program Braunsberg, 1853–1856.

It is well known that when there is one quadrilateral at the same time circumscribed about one circle and inscribed in another, there are an infinite number of quadrilaterals so related to these circles. The question naturally arises: Which one of these quadrilaterals has the greatest area? Or the least area? Welsch showed (*L'Intermédiaire des Mathématiciens*, vol. 16, 1909, p. 37) that the former occurred when the diagonals of the quadrilateral were at right angles to one another; that is, when two opposite vertices of the quadrilateral were at the extremities of the diameter of the circumcircle, center O, passing through the center, I, of the in-circle. He showed also that the quadrilateral is a minimum when it is an isosceles trapezium whose parallel sides touch the in-circle where it is intersected by OI.

Dostor showed [1] that in every quadrilateral:

(a) $$A=\tfrac{1}{4}(a^2-b^2+c^2-d^2)\cdot|\tan(m, n)|,$$

when (m, n) is not 90°; [2]

(b) $$A = \tfrac{1}{4}(m^2 - n^2)\cdot|\tan(p, q)|,$$

when (p, q) is not 90°; and

(c) $$A = \tfrac{1}{2}mn \sin(m, n).$$

If g and h, k and l, are the segments of the interior diagonals,

$$A = \tfrac{1}{2}(gl + lh + hk + kg)\sin(m, n),$$

a form which Kummer found useful in his discussion of a rational quadrilateral, that is, one whose sides, diagonals and area are rational numbers[3] (*Journal für die reine und angewandte Mathematik*, vol. 37, 1848, p. 11).

If the exterior angles of an inscribed quadrilateral are bisected, the area of the quadrilateral formed by these bisecting lines is (E. W. Hobson, *Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, p. 222)

$$\frac{1}{2}\frac{s^2(ab + cd)(ad + bc)}{(a + c)(b + d)\sqrt{[(s - a)(s - b)(s - c)(s - d)]}}.$$

Meier Hirsch gave the following formula (*Sammlung geometrischer Aufgaben*, Erster Teil, Berlin, 1805, p. 36), for the area of a quadrilateral whose sides are known and in which a pair of opposite angles, (a, d) and (b, c) are equal:

$$A = \frac{1}{4}\frac{ad + bc}{ad - bc}\sqrt{[(a + b + c + d)(a + b - c - d)(a + d - b - c)(b + d - a - c)]}.$$

When the opposite angles are right angles P. F. Verhulst showed that we have

[1] G. Dostor, *Propriétés Nouvelles des Quadrilatères en Général* . . ., Greifswald, [1868], pp. 3, 6, 7.

[2] If the quadrilateral is circumscribable about a circle this formula becomes

$$A = \tfrac{1}{2}(bd - ac)\cdot\tan(m, n)$$

(C. Davison, *Subjects for Mathematical Essays*, London, 1915, p. 33).

[3] The problem of the rational quadrilateral has an extensive history; see the discussion by L. E. Dickson in this MONTHLY, *1921*, 244–250; and in his *History of the Theory of Numbers*, vol. 2, 1920, pp. 216–221.

(*Correspondance Mathématique et Physique* (Quetelet), Brussels, vol. 6, 1830, p. 121) $A = (s - a)(s - d) = (ad + bc)/2$.

Other expressions for the area are given in E. Heis and T. J. Eschweiler, *Lehrbuch der Geometrie, Dritter Teil, Ebene und sphärische Trigonometrie*, zweite Auflage, Köln, 1875, pp. 86–90, and in M. Hirsch, *l.c.*, pp. 33–41.

In 1782 Lhuilier considered the question of when the polygon with given sides should have a maximum area, and found that this occurred when it was inscribed in a circle.[1] The result for the quadrilateral follows at once on setting $A + C = 180°$, $\cos C = -\cos A$, in formula (6) or the first part of formula (7). Lhuilier considered also the question of when the area is a minimum and showed [2] that this arose when the quadrilateral was no longer convex but when the sides cut one another. From the second part of formula (7) it appears that the area is a minimum when $A + C = 0$, that is, when[3] $\cos C = \cos A$, the case of the inscribed concave quadrilateral. In this case

$$16A_1^2 = 4(ac - bd)^2 - (a^2 - b^2 + c^2 - d^2)^2,$$

$$= (a+b+c+d)(-a-b+c+d)(-a+b-c+d)(-a+b+c-d),$$

$$= -\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \qquad (9)$$

which is Ferrers's determinant referred to above. From this, it follows at once that for any convex quadrilateral

$$\sqrt{[s(s-a-d)(s-b-d)(s-c-d)]} \not> A,$$

a result given by Wolstenholme[4] in one of his problems. Among a number of other maxima and minima results connected with the quadrilateral, R. Sturm proved [5] that for given sides, in a given order, the inscribed convex quadrilateral has the greatest diagonal product and hence the greatest area (compare (4)).

In *Astronomische Nachrichten*, no. 42, November, 1823, Shumacher reports that on page 61 of the description by Möbius of the observatory at Leipzig the following problem occurs: Let $ABCDE$ be any five points in the plane joined so as to form the five triangles ABC, BCD, CDE, DEA, EAB whose areas α, β, γ, δ, ϵ, respectively, are known; required the area of the pentagon $ABCDE$.

[1] S. Lhuilier, *l.c.*, p. 5; see also Enneper "Ueber das Maximum eines Vierecks von gegebenen Seiten," *Gött. Nachrichten*, 1885, pp. 175–180. The same result holds true for a spherical quadrilateral (W. J. McClelland and T. Preston, *Treatise on Spherical Trigonometry*, part 2, London, 1903, p. 50.)

[2] S. Lhuilier, *l.c.*, pp. 23–24; formula (9) was given, in effect, by Lhuilier. Details of this question are discussed by R. Sturm and E. Lampe in *Journal für die reine und angewandte Mathematik*, vol. 96, 1884, pp. 78–80.

[3] R. Baltzer, *Die Elemente der Mathematik*, vol. 1, 1860, pp. 129–130; vol. 2, 1862, p. 306. For inscribed quadrilaterals $A^2 - A_1^2 = abcd$ (Möbius, *Journal für die reine und angewandte Mathematik*, vol. 3, 1828, pp. 17–18).

[4] J. Wolstenholme, *A Book of Mathematical Problems*, London, 1867, problem 377; third edition, 1878, problem 596. It is clear that Wolstenholme might have written "$< A$" instead of "$\not> A$."

[5] R. Sturm, *Maxima and Minima in der elementaren Geometrie*, Leipsic, 1910, p. 25.

Shumacher appends a solution handed to him by Gauss (*Carl Friedrich Gauss Werke*, vol. 4, 2ter Abdruck, 1880, pp. 406–407); this leads to the equation

$$A^2 - (\alpha + \beta + \gamma + \delta + \epsilon)A + (\alpha\beta + \beta\gamma + \gamma\delta + \delta\epsilon + \epsilon\alpha) = 0.$$

P. Serret noted (*Nouvelles Annales de Mathématiques*, vol. 7, 1848, p. 28) that if the pentagon becomes the quadrilateral $ABCD$, α, β, γ being the areas of the triangles ABC, BCD, CDA, the above equation for the area of the quadrilateral becomes $A^2 - (\alpha + \beta + \gamma)A + \alpha\beta + \beta\gamma = 0$,—which can, on solving, be at once verified.

The problem of the construction with ruler and compasses of an inscribed quadrilateral, being given its sides, has an interesting history extending over five hundred years. This is, in general, only a particular case of the problem discussed by Ozanam (*Dictionaire Mathematique*, Amsterdam, 1691, pages 461–464): "Construct the quadrilateral, of given area, given the lengths of its four sides."

R. C. ARCHIBALD.

SOLUTIONS

2832 [1920, 227]. Proposed by S. A. COREY, Des Moines, Iowa.

Prove that the square of the sum of four squares is the sum of four squares, that the square of the sum of eight squares is the sum of six squares, and that the square of the sum of sixteen squares is the sum of ten squares.

I. SOLUTION BY THE PROPOSER.

Given the identity,

$$\begin{aligned}(Pp&+Pq+Qp-Qq-Rr+Rs-Sr-Ss-Tt-Tu+Ut-Uu-Vv-Vx+Xv-Xx)^2\\ &+(Pr+Ps-Qr+Qs-Rp+Rq+Sp+Sq-Tv+Tx+Uv+Ux+Vt-Vu-Xt-Xu)^2\\ &+(Pp-Pq-Qp-Qq+Rr+Rs-Sr+Ss+Tt-Tu+Ut+Uu+Vv-Vx+Xv+Xx)^2\\ &+(Pr-Ps+Qr+Qs+Rp+Rq+Sp-Sq+Tv+Tx+Uv-Ux-Vt-Vu-Xt+Xu)^2\\ &+(Pt+Pu+Qt-Qu-Rv-Rx+Sv-Sx+Tp+Tq-Up+Uq+Vr-Vs+Xr+Xs)^2\\ &+(Pv-Px-Qv-Qx-Rt+Ru-St-Su+Tr+Ts-Ur+Us-Vp+Vq-Xp-Xq)^2\\ &+(Pt-Pu-Qt-Qu+Rv-Rx+Sv+Sx-Tp+Tq-Up-Uq-Vr-Vs+Xr-Xs)^2\\ &+(Pv+Px+Qv-Qx+Rt+Ru-St+Su-Tr+Ts-Ur-Us+Vp+Vq-Xp+Xq)^2\\ &\equiv 2(P^2+Q^2+R^2+S^2+T^2+U^2+V^2+X^2)(p^2+q^2+r^2+s^2+t^2+u^2+v^2+x^2). \quad (1)\end{aligned}$$

Add $(P^2 + Q^2 + R^2 + S^2 + T^2 + U^2 + V^2 + X^2)^2$ and $(p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + x^2)^2$ to each member of (1). The second member then becomes the square of the sum of 16 squares, and the first member becomes the sum of 10 squares, as required. If $P = Q = R = S = t = u = v = x =$ zero, it follows that the square of the sum of eight squares is the sum of six squares. That the square of the sum of four squares is the sum of four squares is a direct consequence of Euler's well known theorem: (sum of four squares) (sum of four squares) = (sum of four squares).

II. Mr. NORMAN ANNING, of the University of Michigan, contributes formulæ exhibiting these squares as the sum of 3, 5 and 9 squares, instead of 4, 6, and 10. The results are given for record.

Corresponding to $(x_1^2 + x_2^2)^2 = (x_1^2 - x_2^2)^2 + 4x_1^2x_2^2$, we have

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 = (x_1^2 - x_2^2 + x_3^2 - x_4^2)^2 + 4(x_1^2 + x_3^2)(x_2^2 + x_4^2)$$
$$= \text{the sum of three squares.}$$

$$(x_1^2 + x_2^2 + \cdots + x_8^2)^2$$
$$= (x_1^2 - x_2^2 + - + \cdots - x_8^2)^2 + 4(x_1^2 + x_3^2 + x_5^2 + x_7^2)(x_2^2 + x_4^2 + x_6^2 + x_8^2)$$
$=$ the sum of five squares, since by Euler's Identity the last term can be expressed as the sum of four squares.

Similarly,

$$(x_1^2 + x_2^2 + \cdots + x_{16}^2)^2$$
$$= (x_1^2 - x_2^2 + - + \cdots - x_{16}^2)^2 + 4(x_1^2 + x_3^2 + \cdots + x_{15}^2)(x_2^2 + x_4^2 + \cdots + x_{16}^2)$$
$$= \text{the sum of nine squares.}$$

The fact that the product of two numbers each of which is the sum of eight squares is itself the sum of eight squares is stated, with a reference to "Thomson, 1877," on page 115 of Carmichael's *Diophantine Analysis*, 1915.

An identity for the eight-square case is as follows:

$$\begin{aligned}&(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\ &\equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cv - du + ew + fs - gr - ht)^2\\ &+ (ar - bv - cp - dt + es - fw + gq + hu)^2 + (as + bu + ct - dp - er - fq - gw + hv)^2\\ &+ (at - bw - cs + dr - ep + fv - gu + hq)^2 + (au - bs + cw + dq - ev - fp + gt - hr)^2\\ &+ (av + br - cq + dw + eu - ft - gp - hs)^2 + (aw + bt - cu - dv - eq + fr + gs - hp)^2.\end{aligned}$$

NOTE (written after the above was in type)—Euler's identity given in a letter to Goldbach dated May 4, 1748 (*Correspondance Mathématique et Physique* ed. by Fuss, vol. 1, 1843, p. 452) was as follows: "Si $m = aa + bb + cc + dd$ et $n = pp + qq + rr + ss$ erit $mn = A^2 + B^2 + C^2 + D^2$ existente $A = ap + bq + cr + ds$, $B = aq - bp - cs + dr$, $C = ar + bs - cp - dq$, $D = as - br + cq - dp$." This was generalized by C. F. Degen in 1822 (*Mém. Acad. Sc. St. Pétersbourg*, vol. 8 [1817–18], pp. 207–219) as follows:

$$\begin{aligned}&(a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \times (p^2 + q^2 + r^2 + s^2 + t^2 + u^2 + v^2 + w^2)\\ &\equiv (ap + bq + cr + ds + et + fu + gv + hw)^2 + (aq - bp + cs - dr + eu - ft + gw - hv)^2\\ &+ (ar - bs - cp + dq \mp ev \pm fw \pm gt \mp hu)^2 + (as + br - cq - dp \pm ew \pm fv \mp gu \mp ht)^2\\ &+ (at - bu \pm cv \mp dw - ep + fq \mp gr \pm hs)^2 + (au + bt \mp cw \mp dv - eq - fp \pm gs \pm hr)^2\\ &+ (av - bw \mp ct \pm du \pm er \mp fs - gp + hq)^2 + (aw + bv \pm cu \pm dt \mp es \mp fr - gq - hp)^2.\end{aligned}$$

Cf. L. E. Dickson, "On quaternions and their generalization and the history of the eight square theorem," *Annals of Mathematics*, second series, volume 20, p. 164, 1919.

2840 [1920, 274]. Proposed by NORMAN ANNING, University of Michigan.

It is observed in a table of values of

$$\log_{10} (\text{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

SOLUTION BY C. C. WYLIE, University of Illinois.

Let $y = \log_{10} (\text{colog}_{10} x)$ and $M = \log_{10} e$. Then

$$y = M \log \left(\log \frac{1}{x}\right) + M \log M.$$

$$\frac{dy}{dx} = -\frac{M}{x \log \frac{1}{x}}, \quad \frac{d^2y}{dx^2} = \frac{M\left(\log \frac{1}{x} - 1\right)}{\left(x \log \frac{1}{x}\right)^2} = 0,$$

if $x = 1/e = 0.368$ approx. Therefore, the second differences in a table of values of this function must be practically zero for values of x near 0.368.[1]

Also solved by H. L. Olson, H. S. Uhler, A. H. Wilson, and the Proposer.

2842 [1920, 274]. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6 - 6k_1x^5 + (7k_1^2 - 9k_2^2)x^4 - 2(2k_1^3 - 4k_1k_2^2 - 3k_3^3)x^3 + [(k_1^2 - k_2^2)^2 - 9k_1k_3^3]x^2 - (k_1^2 - 2k_2^2)(k_1k_2^2 - 9k_3^3)x + (k_1^2 - 3k_2^2)(k_2^4 - 3k_1k_3^3).$$

I. Solution by W. D. Cairns, Oberlin College.

If $k_1 = k_3 = 0$, the expression can be factored thus:

$$(3x^4 + k_2^4)(x^2 - 3k_2^2).$$

If $k_1 = k_2 = 0$, it can be factored thus:

$$3x^3(x^3 + 2k_3^3) \text{ or } (3x^4 + 6k_3^3x)x^2.$$

If $k_2 = k_3 = 0$, it can be factored thus:

$$x^2(3x^2 - 3k_1x + k_1^2)(x^2 - k_1x + k_1^2).$$

These will be consistent only if one factor contains, at least, the terms $3x^4 - 3k_1x^3 + k_1^2x^2 + 6k_3^3x + k_2^4$, and the second $x^2 - k_1x + k_1^2 - 3k_2^2$. The last term of the given expression shows that the first factor must contain also the term $-3k_1k_3^3$, and multiplication of these two tentative factors shows that the term $-k_1k_2^2x$ completes the first factor. Thus we have as the two factors: $3x^4 - 3k_1x^3 + k_1^2x^2 - (k_1k_2^2 - 6k_3^3)x + (k_2^4 - 3k_1k_3^3)$ and $x^2 - k_1x + (k_1^2 - 3k_2^2)$.

II. Remarks by H. P. Manning, Providence, R. I.

The polynomial can be written $u_3k_3^3 + v_6$, where u_3 and v_6 are polynomials of degrees 3 and 6 in x, k_1 and k_2. If there are rational factors k_3^3 must go entirely with one factor, since u_3 and v_6 are not the cubes of rational expressions. The factor which does not contain k_3 will be the highest common factor of u_3 and v_6. We can also factor u_3, noting that it contains only the second power of k_2.

2845 [1920, 326]. Proposed by E. L. POST, Princeton University.

Prove that if y_x is a solution of the functional equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1}$$

for positive integral values of x with $y_x > 0$, then

$$\lim_{x \to \infty} y_x \log x = 1.$$

Solution by the Proposer, and Otto Dunkel, Washington University.

If x and y_x are both positive, the equation

$$y_x = \frac{y^2_{x+1}}{x} + y_{x+1} \tag{1}$$

has one positive root,

$$y_{x+1} = \frac{\sqrt{x^2 + 4xy_x} - x}{2}.$$

From (1), we clearly have then that

$$y_x > y_{x+1} > 0. \tag{2}$$

[1] Since the second difference $y(a + 2h) - 2y(a + h) + y(a) = h^2y''(\theta)$, where $a < \theta < a + 2h$.—Editors.

The equation may be written

$$\frac{1}{y_{x+1}} - \frac{1}{y_x} = \frac{y_{x+1}}{xy_x}.$$

Hence

$$\frac{1}{y_x} - \frac{1}{y_1} = \sum_1^{x-1} \frac{1}{x} \cdot \frac{y_{x+1}}{y_x} = \sum_1^{x-1} \frac{1}{x}\left[1 - \frac{y_x - y_{x+1}}{y_x}\right] = \sum_1^{x-1} \frac{1}{x} - \sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2_{x+1}}{y_x}.$$

Now

$$\sum_1^{x-1} \frac{1}{x} = \log x + r,$$

where r approaches Euler's constant C when x becomes infinite (Goursat-Hedrick, vol. 1, p. 103, Ex. 1); so that, if we substitute and divide by $\log x$, we have

$$\frac{1}{y_x \log x} - 1 = \frac{\dfrac{1}{y_1} + r - \displaystyle\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2_{x+1}}{y_x}}{\log x}.$$

But from (2) $y_{x+1} < y_x < y_1$, $y^2_{x+1} < y_1 y_x$. Therefore

$$\sum_1^{x-1} \frac{1}{x^2} \cdot \frac{y^2_{x+1}}{y_x} < y_1 \sum_1^{x-1} \frac{1}{x^2} < \frac{\pi^2 y_1}{6}.$$

The numerator of the fraction remains numerically less than a fixed number when x becomes infinite and we have

$$\lim y_x \log x = 1.$$

It is to be noticed that the above method can be immediately extended to

$$y_x = f(x) y^2_{x+1} + y_{x+1},$$

for certain functions $f(x)$, where $\sum_1^{\infty} [f(x)]^2$ is a convergent series.

NOTE. From (1) we obtain

$$\frac{y_x}{y_{x+1}} - 1 = \frac{y_{x+1}}{x}, \qquad \text{and} \qquad x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = \frac{y_{x+1}}{y_x}. \tag{3}$$

When x becomes infinite, the first equation in (3) together with (2) shows that y_x/y_{x+1} approaches unity. Hence the second equation gives

$$\lim_{x \to \infty} x\left(\frac{1}{y_{x+1}} - \frac{1}{y_x}\right) = 1.$$

It is easily seen that the limit is the same if we replace x by $x + 1$ and when this is done it follows that

$$\lim_{x \to \infty} \frac{1}{\log (x+1) y_{x+1}} = 1,$$

by use of the theorem on page 108, § 162, E. Cesàro, *Elementares Lehrbuch der algebraischen Analysis* . . ., Leipzig, 1904. The desired result easily follows from the above.

2846 [1920, 326].

Find the entire volume within the surface $x^{1/2} + y^{1/2} + z^{1/2} = a^{1/2}$. (W. A. Granville, *Elements of Differential and Integral Calculus*, revised ed., 1911, p. 420.)

This equation, rationalized, is the equation of Steiner's quartic surface, every tangent plane to which cuts it in two conics. (Cf. Salmon-Rogers, *Analytic Geometry of Three Dimensions*, 5th ed., vol. 2, 1915, pp. 171, 201, 207, 213f. Also C. M. Jessop, *Quartic Surfaces* 1916, chapter 7.)

I. SOLUTION BY L. A. EASTBURN, North Arizona Normal School, Flagstaff, Ariz.

The required volume inclosed by the surface is

$$v = \int_0^a \int_0^{(a^{1/2} - x^{1/2})^2} \int_0^{(a^{1/2} - x^{1/2} - y^{1/2})^2} dz\,dy\,dx,$$

$$= \int_0^a \int_0^{(a^{1/2}-x^{1/2})^2} [(a^{1/2} - x^{1/2})^2 - 2(a^{1/2} - x^{1/2})y^{1/2} + y]dydx,$$

$$= 1/6 \int_0^a [a^2 - 4a^{3/2}x^{1/2} + 6ax - 4a^{1/2}x^{3/2} + x^2]dx,$$

$$= 1/6 \left[a^3 - 8/3a^3 + 3a^3 - 8/5a^3 + \frac{a^3}{3} \right] = \frac{a^3}{90}.$$

II. Notes by R. C. Archibald, Brown University.

The integral here arising is a particular case of one considered by Lejeune Dirichlet in 1839.[1] If $V = \int x^{a-1}y^{b-1}z^{c-1} \cdots dxdydz \cdots$, considering all positive values of $x, y, z, \cdots$ such that

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \cdots < 1$$

the constants $a, b, c, \cdots, p, q, r, \cdots, \alpha, \beta, \gamma, \cdots$ being also positive, then

$$V = \frac{\alpha^a\beta^b\gamma^c \cdots}{pqr \cdots} \frac{\Gamma\left(\frac{a}{p}\right)\Gamma\left(\frac{b}{q}\right)\Gamma\left(\frac{c}{r}\right)\cdots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} + \cdots\right)}.$$

For the surface $(x/a)^{1/n} + (y/b)^{1/n} + (z/c)^{1/n} = 1$, the volume in the first octant would be $abc(n!)^3/(3n)!$ When $n = 7/2$ we have a result given[2] in 1883

$$V = \frac{abc}{(2/7)^3} \cdot \frac{[\Gamma(7/2)]^3}{\Gamma(23/2)}.$$

The values, for $n = 3/2$ and $n = 1/2$ were given in Williamson, *Elementary Treatise on the Integral Calculus*, 6 ed., 1891, pp. 289–290. The result for the surface $(x/a)^2 + (y/b)^2 + (z/c)^4 = 1$, given in Todhunter, *A Treatise on the Integral Calculus*, 4 ed., 1874, p. 186, follows at once from the Dirichlet formula above.

2852 [1920, 377]. Proposed by D. H. Richert, Bethel College, Newton, Kans.

What is the radius of a cylinder inscribed in a right cone, radius of base $R = 5$ inches, and altitude $h = 18$ inches, the volume of the cylinder to be $1/n$ $(= 3/4)$ that of the cone?

Solution by H. S. Uhler, Yale University.

Let V denote the volume of the cone, and let r, v, and z denote, respectively, the radius, the volume, and the altitude of the cylinder. Then

$$V = \tfrac{1}{3}\pi hR^2, \qquad v = \pi zr^2, \qquad \text{and} \qquad v = \frac{1}{n}V;$$

hence,

$$zr^2 = \frac{hR^2}{3n}. \tag{1}$$

From the similar right triangles obtained by passing a plane through the common axis of the cone and cylinder we find

$$z = \frac{h(R - r)}{R}. \tag{2}$$

These two equations are homogeneous in h and z, and also in R and r; therefore they determine the ratio of the altitudes and the ratio of the radii independently as functions of n alone.

Substituting from (2) for z in equation (1) we obtain

$$r^3 - Rr^2 + R^3/(3n) = 0.$$

[1] *Comptes Rendus . . . de l'Académie des Sciences*, vol. 8, p. 156; also in *Journal de Mathématiques Pures et Appliquées*, vol. 4, p. 168.

[2] *Mathematical Questions with their solutions from the "Educational Times,"* vol. 38, p. 104.

Let $\rho \equiv r/R$ so that the cubic becomes

$$\rho^3 - \rho^2 + 1/(3n) = 0. \tag{3}$$

Since the constant term of this equation is positive there is always one real negative root, and this is inadmissible. Therefore the remaining roots must be real in order that a positive root may exist. The derivative of the left-hand member of (3) is $\rho(3\rho - 2)$ which vanishes for $\rho = 0$ and $\rho = 2/3$. The expression for the derivative shows that there is a maximum for $\rho = 0$, and a minimum for $\rho = 2/3$; the value of the minimum is $1/3(1/n - 4/9)$ and hence if $1/n > 4/9$ there is only one real root, the negative root already mentioned.

Consequently the given value $1/n = 3/4$ conflicts with the preceding condition and makes the problem impossible.

Also solved by T. M. Blakslee, L. A. Eastburn, R. A. Johnson, R. H. Marshall, J. Q. McNatt, Arthur Pelletier, and A. V. Richardson.

2855 [1920, 377]. Proposed by J. L. Riley, Stephenville, Texas.

Show that the circle of curvature at any point of the ellipse cannot pass through the centre unless the eccentricity be greater than $1/\sqrt{2}$.

I. Solution by A. V. Richardson, Bishop's College, Lennoxville, Quebec.

In the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b),$$

the normal at the point $(a \cos \theta, b \sin \theta)$ is

$$\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2. \tag{1}$$

To find the coördinates of the center of curvature, we solve (1) and its derivative with respect to θ,

$$\frac{ax \sin \theta}{\cos^2 \theta} + \frac{by \cos \theta}{\sin^2 \theta} = 0,$$

giving

$$x = \frac{a^2 - b^2}{a} \cos^3 \theta; \qquad y = -\frac{a^2 - b^2}{b} \sin^3 \theta. \tag{2}$$

The (radius of curvature)^2 is, of course, the (distance)^2 between this point and $(a \cos \theta, b \sin \theta)$ *i.e.*,

$$\frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^3}{a^2b^2}.$$

If we put the radius of curvature equal to the distance of the center of curvature from the origin we shall get after some reductions

$$(a^2 - 2b^2) \cos^2\theta = (2a^2 - b^2) \sin^2\theta. \tag{3}$$

Hence, for real values of θ, we must have $a^2 > 2b^2$, *i.e.*,

$$a^2 > 2a^2 (1 - e^2) \qquad \text{or} \qquad e > \frac{1}{\sqrt{2}}.$$

II. Remarks by Otto Dunkel, Washington University.

If we determine the condition that the perpendicular bisector of the segment from the origin to the point $(a \cos \theta, b \sin \theta)$ shall pass through the center of curvature we shall get equation (3) more easily.

Or we may express this by saying that the center of curvature projects into the middle point of the segment. This condition takes the form $2(x_1x_c + y_1y_c) - (x_1^2 + y_1^2) = 0$, which reduces to the same equation (3).

We find also that e may equal $1/\sqrt{2}$. In the ellipse for which $e = 1/\sqrt{2}$ the radius of curvature at the extremity of the major axis is equal to $a/2$.

Also solved by A. M. Harding, William Hoover, and H. S. Uhler.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Mr. J. W. BLINCOE has been appointed instructor of mathematics in Randolph-Macon College, Ashland, Va.

Mr. F. J. BURKETT, of the Pennsylvania State College, was appointed instructor of mathematics in the University of Pittsburgh in September, 1920.

At Columbia University, the following are temporarily associated with the department of mathematics as visiting lecturers: Professor M. A. NORDGAARD, of the University of Iowa, Professor J. A. NORTHCOTT, of Syracuse University, and Professor R. R. HITCHCOCK, of the University of North Dakota. Professor H. B. MITCHELL is absent on leave for the first half of the current academic year, and Professors D. E. SMITH and EDWARD KASNER will be absent during the second half of the year.

Mr. A. O. HICKSON, of Acadia University, has been appointed instructor of mathematics at Brown University for the second semester of the present academic year, Professor R. C. ARCHIBALD being absent on sabbatic leave (cf. *1921*, 336).

Mr. C. J. CARDIN, mechanical engineer and industrial instructor of the Bethlehem Shipbuilding Company, has been appointed instructor of Mathematics at Washington and Jefferson College.

Professor T. O. WALTON, of Michigan Agricultural College, has been appointed assistant professor of mathematics at Kalamazoo College.

Dr. L. H. HILL, of the University of Montana, has been appointed associate professor of mathematics at the University of Maine.

W. H. HILL, who during the past year was instructor of mathematics at the University of Colorado, is now assistant professor of mathematics at the Manual Training Normal School at Pittsburg, Kansas.

Mr. F. W. JOHN, of Cornell University, has been appointed instructor of mathematics in Washington Square College of New York University.

Mr. A. S. PRATT, who was instructor of mathematics at the University of Maine 1919–1921 (*1920*, 141), is teacher of mathematics at the Moses Brown School, Providence, R. I.

Miss HELEN BARTON, for the past two years instructor of mathematics at Wellesley College (*1919*, 219), has been appointed dean of women at Albion College, Albion, Mich.

Dr. H. R. KINGSTON, formerly assistant professor of mathematics and astronomy, University of Manitoba, has been appointed professor and head of the department of mathematics at Western University, London, Ontario.

Last June Lecturer JOSEPH DE PEROTT, of Clark University, retired from active service in the department of mathematics.

Instructor H. L. SMITH, of the University of Wisconsin, and a charter member of the Mathematical Association, has been appointed professor of mathematics at the University of the Philippines.

At Dartmouth College, Assistant Professor R. D. BEETLE has been promoted to a full professorship. Professor J. W. YOUNG has returned to the department after a leave of absence of two years during which he has served as chairman of the National Committee on Mathematical Requirements. Assistant Professor F. M. MORGAN has been granted leave of absence for the second semester of the current academic year.

At Cornell University, Professor JAMES MCMAHON retired from active service last June. The previously announced Heckscher research grant to Mr. H. S. VANDIVER (*1921*, 287) for the continuation of his investigations on the theory of algebraic numbers has been extended to cover the first semester of the year 1921–1922, and Mr. VANDIVER has been granted leave of absence for that period. Mr. JESSE OSBORNE, of Pennsylvania State College, and Mr. R. L. JEFFERY, of Acadia College, have been appointed instructors.

Mr. G. I. HOPKINS, head of the mathematics department and for the last three years principal of the Manchester, N. H., high school, resigned from this position in June. He had reached the age of 72 and had given faithful service to this school for a continuous period of forty-one years. He is the author of *Manual of Plane Geometry on the Heuristic Plan,* Boston, 1891.

Miss HENRIETTA S. LEAVITT, member of the staff of the Harvard Astronomical Observatory, died in Cambridge, Mass., December 12, 1921. She was born in Lancaster, Mass., July 4, 1868. She graduated at Radcliffe College with the degree of B.A. in 1892, took graduate work in astronomy in 1892–1893, and research work at the Harvard Observatory in 1895–1896. In 1902 she became engaged in photographic researches, which she continued till her death, making a special study of the photographic brightness of the stars and of the relation between the brightness and periods of variable stars. The scientific results of her work are found in volumes 60, 71, 84 and 85 of the *Annals of the Harvard Observatory.*

The following reports of Summer Sessions in 1922 have been received.

University of California, June 26–August 5. In addition to the usual courses in Plane trigonometry, Graphic algebra, Differential calculus, Integral calculus, and Plane analytic geometry, the following courses are offered. By Professor G. E. F. SHERWOOD: Solid analytic geometry and theory of infinite series. By Dr. P. H. DAUS: Synthetic projective geometry. By Professor B. A. BERNSTEIN: Elementary algebra for advanced students. By Professor L. E. DICKSON: Applied trigonometry, Selected topics in the theory of equations. By Professor G. D. BIRKHOFF: Differential equations, Theory of relativity. Each course is equivalent to two units.—Southern Branch, July 1–August 12. In addition to courses in Plane trigonometry, Plane analytic geometry, and Differential calculus, the following course is offered. By Professor D. N. LEHMER: Introduction to synthetic projective geometry, 2 units.

University of Chicago, First Term, June 19–July 26; Second Term, July 27–September 1. By Professor E. H. MOORE: Fundamental number systems of

analysis, four hours, first term only; Limits and series, four hours, first term only. By Professor H. E. SLAUGHT: Elliptic integrals, four hours; Differential equations, four hours. By Professor G. A. BLISS: Functions of a complex variable, four hours; Calculus of variations, four hours. By Professor E. J. WILCZYNSKI: Metric differential geometry, four hours; College algebra, five hours. By Professor J. W. A. YOUNG: Theory of equations, four hours; Differential calculus, five hours. By Professor F. R. MOULTON: Celestial mechanics, four hours. By Professor A. C. LUNN: Statistics and probability, four hours; Vector analysis, five hours. By Professor R. L. MOORE: Foundations of analysis situs, four hours; Integral calculus, five hours. By Mr. J. H. VANDIVER: Theory of numbers, four hours; Plane analytic geometry, five hours. By Professor C. H. GINGRICH: Plane trigonometry, five hours.

Columbia University, July 10–August 18. In addition to the usual courses in Logarithms and trigonometry, Solid geometry, College algebra, Analytic geometry, and Calculus, the following courses are offered. By Professor J. F. RITT: Theory of functions of a complex variable. By Professor DUNHAM JACKSON: Differential equations, Fourier's series. By Professor W. B. FITE: Projective geometry. By Dr. G. A. PFEIFFER: Introduction to higher algebra. Each of these courses meets five times per week.

Cornell University, July 8–August 18. In addition to the courses in Solid geometry, Advanced algebra, Trigonometry, Analytic geometry, and Calculus, the following advanced courses are offered. By Professor VIRGIL SNYDER: Projective geometry. By Professor D. C. GILLESPIE: Analysis. The following reading and research courses are also offered. By Professor SNYDER: Algebraic curves and surfaces. By Professor F. R. SHARPE: Applied mathematics. By Professor W. B. CARVER and Professor F. W. OWENS: Foundations of geometry and problems in synthetic geometry. By Professor GILLESPIE: Functions of a real variable, Point-sets, Calculus of variations. By Professor W. A. HURWITZ: Advanced analysis. By Professor C. F. CRAIG: Functions of a complex variable.

Harvard University, July 10–August 19. By Mr. R. E. LANGER: Trigonometry, Analytic geometry. By Professor W. C. GRAUSTEIN: Differential and integral calculus. By Professor O. D. KELLOGG: Differential and integral calculus. Each course meets five times per week.

University of Illinois, June 19–August 12. In addition to the usual courses in College algebra, Plane trigonometry, Analytic geometry, Differential and integral calculus, the following courses are offered. Each course meets five times per week, except as otherwise stated. By Professor G. A. MILLER: Advanced algebra; Critical study of Cajori's *History of Mathematics,* 3 hours; Seminar in group theory, 2 hours. By Dr. C. C. CAMP: Differential equations. By Dr. C. F. GREEN: Advanced analytic geometry. By Professor E. B. LYTLE: Teachers' course. By Professor ARNOLD EMCH: Geometric transformations. By Professor J. B. SHAW: Vector methods.

University of Iowa, First Term, June 7–July 22. By Professor J. F. REILLY: Subject matter and teaching of mathematics, Actuarial theory. By Professor

E. W. CHITTENDEN: Integral calculus, Differential equations, Theory of functions of a complex variable. By Mr. R. E. GLEASON: Differential calculus, Theory of equations, Projective geometry. By Mr. F. S. HARPER: Algebra, Trigonometry. By Mr. O. E. BROWN: Trigonometry, Analytic geometry. By Mr. H. W. CHANDLER: Solid geometry. Second Term, July 24–August 25. By Professor R. P. BAKER: Calculus, Projective geometry, Geometric magnitudes. By Dr. W. H. WILSON: Algebra, Determinants. By Mr. R. E. GLEASON: Analytic geometry. The courses are five hours per week in class through the sessions. When transformed into hours of the ordinary academic year, the credit for each course taken during the first session is two semester hours and for the second session five sixths of this amount.

University of Kansas, First Session, June 12–July 21. By Professor C. H. ASHTON: Complex numbers, College algebra. By Professor U. G. MITCHELL: Differential equations, Teachers' course. By Professor J. J. WHEELER: Analytic geometry, Calculus. Second Session, July 24–August 18. By Professor S. LEFSCHETZ: Theory of equations, Trigonometry.

University of Michigan, June 26–August 18. By Professor W. B. FORD: Advanced algebra, Advanced calculus. By Professor Peter FIELD: Theory of potential. By Professor T. R. RUNNING: Graphical methods. By Professor L. C. KARPINSKI: History of mathematics. By Professor T. H. HILDEBRANDT: Differential equations. By Professor W. B. CARVER: Theory of statistics, Theory of probability. By Mr. N. H. ANNING: Advanced geometry. By Mr. C. H. RICHARDSON: Finite differences. Each course meets for four hours a week.

University of Minnesota, June 19–July 29. In addition to the usual courses in College algebra, Plane trigonometry, Analytic geometry, and Differential and integral calculus, the following course is offered: By Professor R. W. BRINK, W. L. HART, and A. L. UNDERHILL: Selected topics in advanced mathematics.

University of Oklahoma, June 7–August 1. In addition to courses in College algebra, Trigonometry, Analytic geometry, Differential calculus, and Integral calculus, the following courses are offered. By Professor S. W. REAVES: Analytic geometry of space, 3 hours. By Professor J. O. HASSLER: Teachers' course, 2 hours. By Professor N. ALTSHILLER-COURT: History of mathematics, 2 hours; College geometry, 3 hours.

University of Wisconsin, June 19–August 4. In addition to courses in Solid geometry, College algebra, Trigonometry, Analytic geometry, Differential and integral calculus, and Mathematical theory of investment, the following courses are given. By Professor E. B. SKINNER: Elements of the group theory. By Professor H. W. MARCH: Mathematical formulation of scientific problems, Fourier series. By Professor ARNOLD DRESDEN: Advanced integral calculus, Theory of point sets. By Professor W. W. HART: The teaching of mathematics, Higher Euclidean geometry for teachers. By Professor E. P. LANE: Differential geometry. By Dr. F. E. ALLEN: Theory of equations. By Mr. H. T. DAVIS: Differential equations.

National Committee on Mathematical Requirements.

The complete report of the National Committee on Mathematical Requirements is in the press and will, it is hoped, be ready for distribution in April. [1] It is published under the title "The Reorganization of Mathematics in Secondary Education" and will constitute a volume of about 500 pages. The table of contents given below indicates its general character.

Through the generosity of the General Education Board the National Committee is in a position to distribute large numbers of this report free of charge. It is hoped that the funds available will be sufficient to place a copy of this report in every regularly maintained high school library, and also to furnish every individual with a copy free of charge who is sufficiently interested to ask for it. Requests from individuals for this report are now being received. They should be sent as early as possible to J. W. Young, Chairman, Hanover, New Hampshire. The receipt of these requests is not in general acknowledged, but applicants may rest assured that their requests will be filled when the report is ready for distribution. If the number of requests received exceeds the number the Committee is able to distribute, the earlier requests will receive the preference.

The table of contents of the report is as follows: Part I: General Principles and Recommendations—Chapter I: A brief outline of the report; II: Aims of mathematical instruction, general principles; III: Mathematics for years seven, eight and nine; IV: Mathematics for years ten, eleven and twelve; V: College entrance requirements; VI: Lists of propositions in plane and solid geometry; VII: The function concept in secondary school mathematics; VIII: Terms and symbols in elementary mathematics. Part II: Investigations Conducted for the Committee—Chapter IX: The present status of disciplinary values in education, by Vevia Blair; X: The theory of correlation applied to school grades, by A. R. Crathorne; XI: Mathematical curricula in foreign countries, by J. C. Brown; XII: Experimental courses in mathematics, by Raleigh Schorling; XIII: Standardized tests in mathematics for secondary schools, by C. B. Upton; XIV: The training of teachers of mathematics, by R. C. Archibald; XV: Certain questionnaire investigations; XVI: Bibliography on the teaching of mathematics, by D. E. Smith and J. A. Foberg.

[1] The Committee has since announced that the report will not be ready before May.

Published June 15, 1922.

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BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

Eighth Summer Meeting of the Association, Vassar College, September 5–6, 1923

Eighth Annual Meeting, University of Cincinnati, December, 1923

The following are dates of Section meetings of the Association in 1923 (unless otherwise specified):

Illinois, Knox College, Galesburg, May 4–5

Iowa, Cornell College, Mount Vernon, April 27–28

Kansas, Topeka, January 20

Kentucky, University of Kentucky, Lexington, April

Maryland-Virginia-District of Columbia, Baltimore, May 12

Minnesota, St. Paul, May 27

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Ohio, Ohio State University, Columbus, March 30–31

Rocky Mountain, University of Colorado, Boulder, April

Southeastern, Agnes Scott College, Decatur, March 10

Texas, Houston, December 1–2

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE PATH OF LIGHT IN A GRAVITATIONAL FIELD.

By H. S. UHLER, Yale University.

While reading the recent books on the generalized principle of relativity, Einstein's theory, etc., I failed to find any numerical data which would give, even approximately, a quantitative idea of the angular deviation at various points on the curved ray which represents the course taken by a light wave in coming from a remote star to the earth after having passed close to the limb of the sun. All of the books examined give the total deviation,[1] $4m/R = 1.745''$, from infinity to infinity (that is, the angle between the asymptotes to the ray) but nothing more about numerical angles. Since my investigation of this problem may contain something of interest to the readers of the MONTHLY, it seems appropriate to present the details of the analysis in this place.

As a purely mathematical problem, we shall first find the polar equation of a ray of light in an unlimited medium in which the index of refraction is a linear function of the reciprocal of the distance of the wave from the pole or center of symmetry. We shall then simplify the equation by adapting it to the special case of a gravitational field. Finally, we shall obtain so much of a series expansion as will be necessary for the calculation of the very small angular deviations involved.

In order to avoid undue distraction in the midst of the subsequent analysis we shall now state a theorem,[2] borrowed from geometrical optics, which will constitute the keynote to the plan of attack to be followed in this paper. For a transparent medium having the property that the index of refraction possesses spherical symmetry with respect to a given center, the product of the index of refraction at any point on a ray of light in the medium by the length of the perpendicular dropped from the center of symmetry upon the tangent to the ray at the point in question is constant. Let n_1, n_2, n_3, $\cdots$ denote the indices of refraction at points along the same ray and let p_1, p_2, p_3, $\cdots$ symbolize the lengths of the corresponding perpendiculars, then

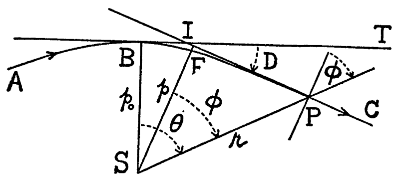
$$n_1p_1 = n_2p_2 = n_3p_3 = \cdots = \mathit{constant}. \tag{1}$$

This fact is independent of the functional relation between the index of refraction and the distance from the center of symmetry.

In the figure let S denote the center of symmetry and let ABC represent the curved ray of light. B is the point on the ray nearest to S. Take the line SB as the axis of polar coördinates (r, θ) of any point P on the ray. $SP = r$, $\angle BSP = \theta$. Draw tangents BT and FP to the curved ray at the points B

[1] R denotes the radius of the sun and m a quantity which Eddington calls "the mass of the sun in astronomical units," while Weyl styles it the "gravitational radius." m is always expressed in terms of a unit of length, usually the kilometer.

[2] *Cf.* R. S. Heath, *Geometrical Optics*, 1887, p. 328.



and P, respectively. Let the lengths of the perpendiculars (SB and SF) dropped from S upon these tangent lines be denoted by p_0 and p. Let ϕ represent the acute angle which the radius vector to the point P makes with the normal to the ray at P. Then $\angle\, FSP = \phi$. The deviation of the ray, D, at the point P will be defined as the angle which the tangent FP makes with the apsidal tangent BT. $D = \angle\, TIP = \angle\, BSF = \theta - \phi$.

In general

$$\tan\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}. \tag{2}$$

From the figure

$$\tan\phi = \frac{\overline{FP}}{\overline{SF}} = \frac{\sqrt{r^2 - p^2}}{p}. \tag{3}$$

Elimination of ϕ from relations (2) and (3) gives

$$\frac{dr}{d\theta} = \frac{r}{p}\sqrt{r^2 - p^2}. \tag{4}$$

In order to integrate equation (4), p must be given as a function of r. As stated in the second paragraph, let

$$n = h + \frac{l}{r}. \tag{5}$$

For the point B we may write $n_0 = h + l/p_0$. Hence, according to lemma (1) we have $p = r(hp_0 + l)/(hr + l)$.

Substituting the last expression for p in equation (4) we obtain

$$d\theta = \frac{dr}{r\sqrt{\left(\frac{hr + l}{hp_0 + l}\right)^2 - 1}}. \tag{6}$$

Putting $u \equiv 1/r$ and $u_0 \equiv 1/p_0$ in equation (6) we find

$$\frac{(\sqrt{h})u_0 d\theta}{\sqrt{h + 2lu_0}} = \frac{-du}{\sqrt{1 - \left[\frac{(h + 2lu_0)u - lu_0^2}{u_0(h + lu_0)}\right]^2}}.$$

Substituting z for the expression between the brackets, the last equation reduces to the simple form

$$\frac{\sqrt{h(h + 2lu_0)}}{h + lu_0}\, d\theta = \frac{-dz}{\sqrt{1 - z^2}};$$

whence

$$\frac{\sqrt{h(h + 2lu_0)}}{h + lu_0}\,\theta = \cos^{-1} z + c.$$

When $\theta = 0$; $r = p_0$, $u = u_0$, and $z = 1$, therefore $c = 0$. Consequently the equation of the curved ray may be written as

$$\frac{1}{r} = \frac{1}{p_0(hp_0 + 2l)} \left\{ l + (hp_0 + l) \cos \left[\frac{\sqrt{hp_0(hp_0 + 2l)}}{hp_0 + l} \theta \right] \right\}. \tag{7}$$

We shall next apply equation (7) to the special case of the gravitational field associated with the sun. The point S of the figure may now be considered as situated at the center of this star.

According to Einstein's theory of gravitation the index of refraction at the point P is given by $n = 1 + 2m/r$ to the first order of approximation in m/r. As a matter of fact, m^2/r^2 would be negligible in the case of the sun since, for this mass, $m = 1.473$ km., $p_0 = 6964 \times 10^2$ km. (corresponding to the sun's limb) and $m^2/p_0^2 = 4.474 \times 10^{-12}$. Consequently we may obtain an approximate equation of the path of a wave of light in the sun's gravitational field by neglecting all powers of m/r higher than the first in equation (7), and by putting $h = 1$ and $l = 2m$. [*Vide* relation (5).]

Under these conditions equation (7) reduces to

$$\frac{1}{r} = \frac{2m}{p_0^2} + \frac{p_0 - 2m}{p_0^2} \cos \theta. \tag{8}$$

This is the familiar equation of one branch of an hyperbola referred to the nearer focus as pole. The eccentricity of the hyperbola is given by $e = p_0/2m - 1$ which, in the practical case under consideration, has the very large value 2364×10^2. The transverse semi-axis $= (2mp_0)/(p_0 - 4m) = 2.946$ km., approximately.

For an infinite value of r equation (8) shows that θ is obtuse and that the approximate value of the total angular deviation, $(2D_\infty)$, from asymptote to asymptote, equals $4m/p_0$. (*Vide supra.*)

In order to calculate the deviations at various points along the curved ray it will be necessary to derive a special formula, since the numerical values of the deviations are so extremely small as to preclude the use of equation (8) in conjunction with ordinary logarithmic-trigonometric tables.

This may be accomplished in the following manner. Let

$$a \equiv \frac{p}{r} = \frac{p_0 + 2m}{r + 2m}, \qquad b \equiv \frac{p_0^2 - 2mr}{r(p_0 - 2m)}.$$

By referring to the figure and equation (8) we readily see that

$$D = \theta - \phi = \cos^{-1} b - \cos^{-1} a,$$

or

$$D = \sin^{-1} a - \sin^{-1} b,$$

or

$$D = \sin^{-1} [a \sqrt{1 - b^2} - b \sqrt{1 - a^2}]. \tag{9}$$

To the first order in m/p_0 and m/r we obtain

$$a \sqrt{1 - b^2} = c \sqrt{1 - \frac{p_0^2}{r^2}}, \qquad b \sqrt{1 - a^2} = \left(c - \frac{2m}{p_0}\right) \sqrt{1 - \frac{p_0^2}{r^2}},$$

where

$$c \equiv \frac{p_0 + 2m}{r} - \frac{2mp_0^2}{r^2(r + p_0)} .$$

Hence, relation (9) reduces to

$$D = \frac{2m}{p_0}\sqrt{1 - \left(\frac{p_0}{r}\right)^2} = \frac{1296 \times 10^3\, m}{\pi p_0 r}\sqrt{(r - p_0)(r + p_0)}'' . \qquad (10)$$

The first form of relation (10) shows that the deviation very rapidly approaches the asymptotic limit $2m/p_0$ as r increases, and the second form is especially convenient for logarithmic computation.

In calculating the data collected in the table given below the values 1.473 km. and 6964×10^2 km. were taken for m and p_0, respectively. Thus the table pertains to a ray that has grazed the sun's limb. For integral values, θ was taken as the independent variable and r was calculated from equation (8). The numbers given for r in the table have been rounded off. The angles tabulated in the right hand column were obtained by the aid of formula (10).

θ°	r km.	D''
0	6964×10^2	0
5	6991×10^2	0.07605
10	7071×10^2	0.15152
15	7210×10^2	0.22584
20	7411×10^2	$0.2984\bar{4}$
25	7684×10^2	0.36876
30	8041×10^2	0.43628
35	8501×10^2	0.50048
40	9091×10^2	0.56087
45	9849×10^2	0.61700
50	1083×10^3	0.66842
55	1214×10^3	0.71476
60	1393×10^3	0.75566
65	1648×10^3	0.79081
70	2036×10^3	0.81994
75	2691×10^3	0.84283
80	4010×10^3	0.85931
85	7990×10^3	0.86925
θ_M	5794×10^4	0.87250
θ_E	1495×10^5	0.87256
90	1646×10^8	0.87257
θ_∞	∞	0.87257

The fourth and third rows from the bottom of the table pertain respectively to the orbits of the planets Mercury and Earth. $\theta_M = 89^\circ\ 18'\ 41.5''$, $\theta_E = 89^\circ$

$44' 0''$, and $\theta_\infty = 90° + D_\infty$. The data in the last four rows fulfill the expectation that the curved ray has sensibly reached its asymptote not only for the earth's orbit but even for an observer at the distance of the inferior planet.[1]

GRAPHICAL DISCUSSION OF THE ROOTS OF A QUARTIC EQUATION.

By E. L. REES, University of Kentucky.

It is the purpose of this note to give a graphical study of the conditions which determine the nature of the roots of a quartic equation. Using the reduced form $f(x) = x^4 + qx^2 + rx + s = 0$, with q, r and s real and with discriminant Δ, we have types of the quartic for which the following are criteria[2] regarding the nature of the roots:

$\Delta < 0$, roots distinct, two real, two imaginary;

$\Delta > 0$, roots distinct, all real or all imaginary;

$$q < 0,\ s > \frac{q^2}{4}, \text{ roots imaginary;}$$

$$s < \frac{q^2}{4}, \text{ roots real;}$$

$$q \geqq 0, \text{ roots imaginary;}$$

$\Delta = 0$, at least two equal roots;

$$q < 0,\ s > \frac{q^2}{4}, \text{ two equal real roots, two imaginary;}$$

$$-\frac{q^2}{12} < s < \frac{q^2}{4}, \text{ roots real, two and only two equal;}$$

$$s = \frac{q^2}{4}, \text{ two pairs of equal real roots;}$$

$$s = -\frac{q^2}{12}, \text{ roots real, three equal;}$$

$$q > 0,\ s > 0,\ r \neq 0, \text{ two equal real roots, two imaginary;}$$

$$s = \frac{q^2}{4},\ r = 0, \text{ two pairs of equal imaginary roots;}$$

$$s = 0, \text{ two equal real roots, two imaginary;}$$

$$q = 0,\ s > 0, \text{ two equal real roots, two imaginary;}$$

$$s = 0, \text{ four equal real roots.}$$

The discriminant is the product of the squares of the differences of the roots

---

[1] The deviation of a ray from its original direction at the star is obtained by adding $2m/p_0$ to the value of D given in the table, and the total deviation, 1.745″, from asymptote to asymptote is twice the value of D given in the last two lines of the table.—EDITORS.

[2] Compare L. E. Dickson, *Elementary Theory of Equations*, 1914, p. 45.

of $f(x) = 0$. We prove first the following theorem[1] which gives a geometric interpretation of the discriminant that will be used later in the discussion.

The discriminant of a real quartic equation $f(x) = 0$ with leading coefficient unity equals 256 times the product of the ordinates of the turning points[2] of the graph of $y = f(x)$.

Denoting by α_i or α_j the roots of $f(x) = 0$, and by α_k' those of $f'(x) = 0$, we have

$$\begin{aligned}\Delta &= \prod_i \prod_{j>i} (\alpha_i - \alpha_j)^2 = \prod_i \prod_j{}_i (\alpha_i - \alpha_j) = \prod_i f'(\alpha_i)\\ &= 4^4 \prod_i \prod_k (\alpha_i - \alpha_k') = 4^4 \prod_k \prod_i (\alpha_k' - \alpha_i) = 4^4 \prod_k f(\alpha_k').\end{aligned}$$

Since $f(\alpha_k')$ are the ordinates of the turning points of $y = f(x)$, the proof is complete. We may now state as a corollary the following results:

$\Delta < 0$, *ordinates all negative;*
one ordinate negative, two positive;
one ordinate negative, two imaginary;
$\Delta > 0$, *ordinates all positive;*
one ordinate positive, two negative;
one ordinate positive, two imaginary;
$\Delta = 0$, *at least one ordinate zero.*

Assume the graph of $y = f(x)$ drawn. Let a line start in the position of the x-axis and revolve about the origin into the position of the line $y = -rx$, and let each point of the graph move in a vertical line at such a rate that its distance measured vertically from the revolving line remains constant.[3] The resulting curve is symmetric with respect to the y-axis and is the graph of the equation $y = x^4 + qx^2 + s$, which we shall call the auxiliary quartic. The roots of $f(x) = 0$ are the abscissas of the points of intersection of the line and this curve. The turning points of the original curve correspond to those points of the transformed curve at which the tangent is parallel to the line $y = -rx$. We shall call these points of the graph of the auxiliary quartic *transformed turning points*. The inflection points of one curve correspond to the inflection points of the other curve. The proofs of these statements are quite simple and are left to the reader.

By the usual calculus method we easily deduce the following facts concerning the auxiliary quartic curve.

---

[1] The corresponding theorem for the n-ic $x^n + c_1x^{n-1} + \cdots + c_n = 0$ (c's real) is

$$\Delta = (-1)^{\frac{n(n-1)}{2}} n^n \cdot \textit{(product of ordinates of turning points)}.$$

[2] In this discussion we shall understand that a point of inflection with horizontal tangent is to be considered a multiple turning point.

[3] This of course is the same as adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$. The effect of this transformation in our discussion is to replace the original quartic by the auxiliary quartic curve while the line $y = -rx$ in its relation to the graph of the auxiliary quartic in a certain sense replaces the x-axis.

The three turning points[1] are $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}}, s - \frac{q^2}{4}\right)$ and the two inflection points are $\left(\pm\sqrt{-\frac{q}{6}}, s - \frac{5}{36}q^2\right)$.

If $q < 0$, these turning points and inflection points are all real and distinct.

The difference between the ordinate of the middle turning point and the common ordinate of the other two is $q^2/4$.

The distance between the y-intercept point of the curve and that of the inflection tangents $= q^2/12$.

The y-intercept of the inflection tangents is $s + q^2/12 = I$.

If $q = 0$, the y-intercept point of the curve is a triple turning point (double inflection point).

If $q > 0$, there is only one real turning point; the inflection tangents are imaginary and meet inside the curve at the point $(0, I)$; and there is a real double tangent $y = s - q^2/4$, with conjugate imaginary points of contact.

We shall study the various cases corresponding to the different forms and positions of the auxiliary quartic curve according to the sign of q and the value of s.

The corollary of the theorem on discriminants proved above may now be restated in a slightly different and more useful form, namely:

At least one and possibly three real transformed turning points lie above or below the line $y = -rx$ according as Δ is greater than or less than 0; and at least one transformed turning point (real with one exception[2]) lies on the line if $\Delta = 0$.

Bearing this corollary in mind, an examination of a figure in each case will make clear the following classification of quartics, equivalent to that given at the beginning of this paper.

Case I. $q < 0$,

$s > \frac{q^2}{4}$,	$\Delta < 0$, two roots real and distinct;
	$\Delta = 0$, two roots real and equal;
	$\Delta > 0$, no real root;
$s = \frac{q^2}{4}$,	$\Delta < 0$, two roots real and distinct;
	$\Delta = 0$, two pairs of equal real roots;
$0 < s < \frac{q^2}{4}$,	$\Delta < 0$, two roots real and distinct;
$s = 0$,	$\Delta = 0$, all roots real, two equal;
$-\frac{q^2}{12} < s < 0$,	$\Delta > 0$, all roots real and distinct;

[1] Not the "transformed turning points" mentioned in the preceding paragraph, but the turning points of the auxiliary quartic itself.

[2] It will be seen (next foot-note) that there is one case when $\Delta = 0$ in which a real turning point (the other two being imaginary) is not on the line.

$s = -\dfrac{q^2}{12}$, $\Delta < 0$, two roots real and distinct;

$\Delta = 0$, all roots real, three equal;

$s < -\dfrac{q^2}{12}$, $\Delta < 0$, two roots real and distinct.

Case II. $q = 0$,

$s > 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;
$\Delta > 0$, no real roots;

$s = 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, four equal real roots;

$s < 0$, $\Delta < 0$, two roots real and distinct.

Case III. $q > 0$,

$s > 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;[1]
$\Delta > 0$, no real root;

$s = 0$, $\Delta < 0$, two roots real and distinct;
$\Delta = 0$, two roots real and equal;

$s < 0$, $\Delta < 0$, two roots real and distinct.

If $I < 0$, then $s < -q^2/12$ and the quartic has two real and distinct and two imaginary roots.

For a triple root it is necessary and sufficient that the line $y = -rx$ be an inflection tangent. I being the y-intercept of the inflection tangents, it follows that $I = 0$ is a necessary condition for a triple root; and since the slopes of the inflection tangents are $\pm \dfrac{4q}{3}\sqrt{-\dfrac{q}{6}}$ one of which must equal $-r$, it results that $8q^3 + 27r^2 = 0$ is also a necessary condition.

Conversely, if $I = 0$ and $8q^3 + 27r^2 = 0$, the quartic has a triple root.

Noting that

$$J\left(= \frac{qs}{6} - \frac{r^2}{16} - \frac{q^3}{216}\right) = \frac{q}{6}I - \frac{8q^3 + 27r^2}{432}$$

we see that these results are equivalent to the following familiar theorem:

A necessary and sufficient condition that the quartic equation have three or more roots equal is $I = J = 0$.[2]

The geometric arguments for most of the cases of the theorem may be made without the use of the derivative. For this purpose we apply the transformation

[1] Except when $s = q^2/4$ and $r = 0$, in which case there are two conjugate imaginary double roots.

[2] The method of this paper enables us also to recognize the order of succession of the simple and multiple roots. Thus by noting the double root cases in which the inflexions of the transformed quartic both lie below the line $y = -rx$ we find as a necessary and sufficient condition for a double root separating two simple real roots $\Delta = 0$, $8q^3 + 27r^2 < 0$.

In the other cases where all roots are real and there is one double or triple root, this will be the greatest or least root according as r is positive or negative.—EDITOR.

$x' = x^2$, $y = y$ to the parabola $y = x'^2 + qx' + s$, thus obtaining the auxiliary quartic curve. Note that the vertex of the parabola is $(-q/2, s - q^2/4)$ and that the parabola and quartic have in common the y-intercept s.

The different forms of the quartic curve depend on the position of the parabola relative to the y-axis. The three cases follow.

$q < 0$, the vertex of the parabola is to right of the y-axis; the quartic has three real and distinct turning points, $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}}, s - \frac{q^2}{4}\right)$.

$q = 0$, the vertex of the parabola is on the y-axis; the quartic has a triple turning point $(0, s)$.

$q > 0$, the vertex of the parabola is to left of the y-axis; the quartic has only one real turning point $(0, s)$.

The arguments for the various cases of the theorem with few exceptions are identical with those sketched above.[1]

TWO NEW CONSTRUCTIONS OF THE STROPHOID.

By R. M. MATHEWS, Wesleyan University.

(Read before the American Mathematical Society December 28, 1920.)

1. The classic construction for the strophoid uses a pencil of circles each of which has its center on a "medial" line g and passes through a fixed point, the node O, on g (Fig. 1). Let each circle be cut by that one of its diameters which passes through a fixed point, the singular focus F. The curve is the locus of these intersections.[2] The object of this note is to make this construction more general for the same curve: first, by using any line through the node as locus for the centers of the circles; and second, by using a pencil of circles through any two conjugate points of the curve. In preparation for this we describe certain well known features of the curve.[3]

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[1] Instead of adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$, the author might have started with the curve $y = x^4 + qx^2 + s$ and regarded the roots of the given quartic as the abscissas of the intersections of this curve and the line $y = -rx$. The form of this curve depends only on q; its position, or the position of the origin with respect to it, depends on s, while the character of the roots of the equation, when q and s are given, depends on r. Thus the classification, based first on q, and then on s, would finally be based on r.

The range of values of r for any type of equation, when q and s are given, depends on those values which correspond to the real tangents from the origin. These values of r are the roots of the equation $\Delta = 0$, and for any particular type of equation Δ will have a particular sign or be zero. Conversely, the sign or vanishing of Δ, with the given values of q and s, will usually determine the type of the equation. These considerations would enable us to dispense with the author's theorem on discriminants. Results obtained as depending on r could be interpreted at once as depending on Δ, and so when the classification has been obtained, the various classes could be grouped and arranged with respect to Δ, q and s if such an arrangement is more convenient for use.—EDITOR.

[2] Gino Loria, *Spezielle algebraische und transcendente ebene Kurven*, volume 1, Leipzig, 1910, p. 60. The strophoid of our text-books is the *right* strophoid, the form this curve takes when the node is at the foot of the perpendicular from the focus to the median.

[3] Loria, *loc. cit.*, chapter 8.

2. The strophoid is *the* orthotomic circular cubic. With the tangents at the node as axes, its equation may be written

$$(x^2 + y^2)(y + cx) - axy = 0;$$

or in parametric form

$$x = \frac{am}{(1 + m^2)(m + c)}, \qquad y = mx. \tag{1}$$

The real asymptote is parallel to the medial line g: $y + cx = 0$; while the two imaginary asymptotes meet at the singular focus F which is on the line $y - cx = 0$. the *axis* of the curve.

The nodal tangents bisect the angles formed by the axis and the median. Two points, P and P_1, whose parameters are m and $-m$, are "conjugate" points; that is, the tangents at these points meet the curve at the same "tangential point" T. The medial line bisects the join of every pair of conjugate points. Evidently, the nodal tangents bisect the angles determined at the node by each pair of conjugate points.

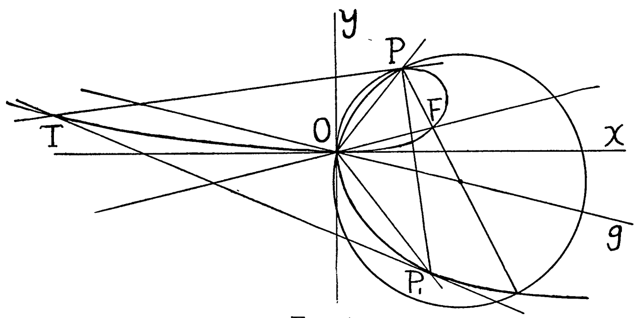


FIG. 1.

On substituting the parametric values of x and y in the equation $ux + vy + 1 = 0$ we find that the necessary and sufficient condition that three points m_1, m_2, m_3 of the strophoid be collinear is $m_1m_2m_3 = - c$.

3. Let us consider the pencil of circles of parameter h which are specified by the equation

$$x^2 + y^2 - 2hx - 2lhy = 0. \tag{2}$$

Each circle passes through the node O (Fig. 2), and has its center (h, lh) on the line $y = lx$, which we may suppose to be the line OP. When this equation is solved simultaneously with that of the strophoid, we obtain, besides the node counted twice and the circular points at infinity, the points whose parameters are roots of the equation

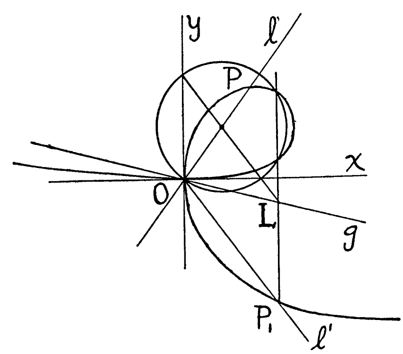


FIG. 2.

$$m^2 + \frac{1}{2lh}(2h + 2lhc - a)m + \frac{c}{l} = 0.$$

Hence $m_1m_2 = c/l$, a relation which implies $m_1m_2(-l) = -c$, and shows that

the points of intersection are collinear with the point P_1 which is conjugate to P. Thus the curve appears as the intersection of a pencil of lines with a projective pencil of circles. To correlate the two forms, we find that the join of m_1 and m_2 cuts the medial line $y + cx = 0$ in the point L

$$L: \qquad x = \frac{2lh}{l - c}, \qquad y = \frac{-2clh}{l - c}.$$

This result justifies the following construction of the curve, given the node O and two conjugate points P, P_1. Construct the tangents at the node and the medial line. Take the line OP for the line $y = lx$. The circle (2) through O with its center on this line cuts the axes again in $(2h, 0)$ and $(0, 2lh)$. The join of these points cuts the medial line in the point L, and the line P_1L cuts the circle in the desired points of the strophoid.

If we take l as the medial line, then P_1 is the singular focus and we have the classical construction.

4. Another construction can be obtained by considering the pencil of circles through two distinct conjugate points P and P_1 of parameters m and $-m$. A circle

$$x^2 + y^2 - 2hx - 2ky + d = 0 \tag{3}$$

cuts the strophoid in the circular points at infinity and at the points whose parameters are roots of the equation

$$dm^4 + 2(dc - ak)m^3 + (d + dc^2 + a^2 - 2ah - 2ack)m^2 + 2c(d - ah)m + dc^2 = 0.$$

Thus the necessary and sufficient condition that four points be concyclic is

$$m_1m_2m_3m_4 = c^2.$$

Taking m and $-m$ for m_1 and m_2 we have the pencil of circles on PP_1 such that

$$m_3m_4(m^2/c) = -c.$$

Therefore, every circle through the conjugate points of parameters m, $-m$ cuts the strophoid in two points Q, Q_1 which are collinear with a certain point K of parameter m^2/c. (Fig. 3.)

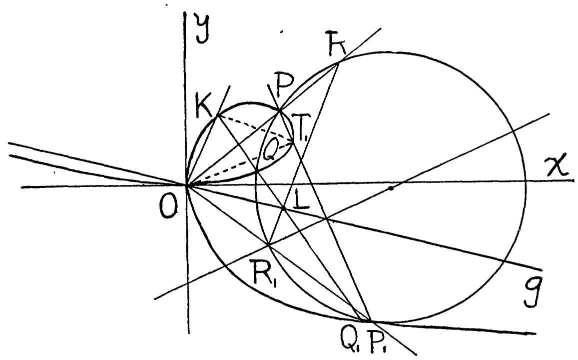


FIG. 3.

From the parametric equations it is easy to show that the slope of PP_1 is $-m^2/c$. The perpendicular from the origin upon this line meets it at the point c/m^2 of the strophoid, say T_1 (for it is the conjugate to the point T, the common tangential of P and P_1). The point at infinity of the strophoid has for parameter $-c$ and is collinear

with K and T_1, since $(m^2/c)(c/m^2)(-c) = -c$. Therefore to construct K we draw OT_1 perpendicular to PP_1, OK the reflection of OT_1 around the bisector of the angle between the nodal tangents, and T_1K parallel to the medial line g.

It remains to put the pencil of circles on PP_1 into graphical correspondence with the pencil of lines at K. Each circle through P and P_1 cuts the nodal radii OP and OP_1 again in two points R and R_1. To determine R, substitute $y = mx$ in (3). The result is a quadratic in x the product of whose roots is $d/(1 + m^2)$. But one of these is the x in (1); therefore the other root is $d(m + c)/am$. In this way we find the coördinates of R and R_1 to be:

$$\left[\frac{d}{am}(m+c), \quad \frac{d}{a}(m+c)\right] \quad \text{and} \quad \left[\frac{d}{am}(m-c), \quad \frac{-d}{a}(m-c)\right].$$

The equation of the line RR_1 is then

$$am^2x - acy - d(m^2 - c^2) = 0,$$

and it requires only some algebraic drudgery to show that this line meets KQQ_1 on the medial line g.

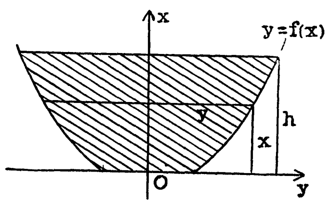
Accordingly, the strophoid may be constructed as follows, given the node and two conjugate points P and P_1. Construct the nodal tangents, the medial line, the line OK and the point K where it will cut the curve. Each circle of the pencil through PP_1 cuts the nodal radii OP, OP_1 in two points R, R_1; the line RR_1 cuts the medial line in a point L, and the line LK cuts the circle in its remaining real intersections with the strophoid.

5. While these constructions are not superior to the classical one in case of actual use on the drawing board, they are of importance as bases for the study of new properties of the curve.

AN APPLICATION OF ABEL'S INTEGRAL EQUATION.

By W. C. BRENKE, University of Nebraska.

Let the shaded area in the figure represent the cross section of a weir notch, the cross section being symmetrical with respect to the x-axis. The quantity of flow through the notch per unit time will be given by



$$Q = C\int_0^h \sqrt{h-x}\, f(x)dx,$$

where the form of the notch is determined by $y = f(x)$; $x \geqq 0$.

Consider the problem of determining $f(x)$ so that the quantity of flow per unit of time shall be proportional to a given power of the depth of stream; *i.e.*, $Q = k'h^m$, $m > 0$. Hence we must find $f(x)$ from an integral equation of the form

$$\int_0^h \sqrt{h-x}\, f(x)dx = kh^m. \tag{1}$$

Differentiation with respect to h gives

$$\int_0^h \frac{f(x)}{\sqrt{h-x}}\,dx = 2kmh^{m-1}, \tag{2}$$

and a solution of (2) will be a solution of (1) also. But (2) comes under the form of Abel's integral equation,[1]

$$\int_a^x \frac{f(y)dy}{(x-y)^s} = g(x), \qquad (0 < s < 1),$$

which has the continuous solution

$$f(x) = \frac{\sin s\pi}{\pi}\int_a^x \frac{g'(y)dy}{(x-y)^{1-s}},$$

provided that $g(x)$ is continuous and has a finite derivative $g'(x)$ with at most a finite number of discontinuities in the range of integration, and that $g(a) = 0$. These conditions are satisfied in the problem under consideration if $m \geqq 2$, and hence we have

$$f(x) = \frac{2km(m-1)}{\pi}\int_0^x \frac{y^{m-2}dy}{\sqrt{x-y}}; \qquad m \geqq 2.$$

Evaluation of the last integral leads to a simple closed form for $f(x)$. By making the change of variable $y = xt$ we obtain

$$\int_0^x \frac{y^{m-2}dy}{\sqrt{x-y}} = x^{m-3/2}\int_0^1 \frac{t^{m-2}dt}{\sqrt{1-t}}. \tag{3}$$

But from the theory of the gamma-function[2] we have

$$\int_0^1 \frac{t^{p-1}dt}{(1-t)^{1-q}} = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Applying this to (3) and substituting in the equation for $f(x)$, we get

$$f(x) = \frac{2km(m-1)}{\pi}\cdot\frac{\Gamma(m-1)\Gamma(\frac{1}{2})}{\Gamma(m-\frac{1}{2})}\,x^{m-\frac{3}{2}},$$

or, since $k\Gamma(k) = \Gamma(k+1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$f(x) = \frac{2k\Gamma(m+1)}{\sqrt{\pi}\,\Gamma(m-\frac{1}{2})}\,x^{m-\frac{3}{2}}; \qquad m \geqq 2. \tag{4}$$

When $m = n$, where n is a positive integer $\geqq 2$, we have

$$f(x) = \frac{k}{\pi}\cdot\frac{2^n\, n!}{1\cdot 3\cdot 5\cdot\,\cdots\,(2n-3)}\,x^{n-\frac{3}{2}}. \tag{5}$$

[1] M. Bôcher, *An Introduction to the Study of Integral Equations*, Cambridge, University Press, 1909, pp. 8–9.

[2] Nielsen, *Handbuch der Theorie der Gamma-Funktion*, 1906, p. 133.

When $m = n + 1/2$, n as above, we have

$$f(x) = k\,\frac{1\cdot 3\cdot 5\cdot\;\cdots\;(2n+1)}{2^n(n-1)!}\,x^{n-1}. \tag{6}$$

When $n = 2$, (5) gives the parabola, in which the quantity of flow is proportional to the square of the depth of stream, and (6) gives the triangular weir notch, with flow proportional to $h^{5/2}$.

It is easy now to show that (4) is still a solution of (1), though not necessarily continuous, for values of $m > 1/2$. To do this we put $x = ht$ in (1), which gives

$$\int_0^1 \sqrt{1-t}\,f(ht)dt = kh^{m-\frac{3}{2}}.$$

Substituting $f(ht) = ch^n t^n$ gives

$$c\int_0^1 \sqrt{1-t}\,t^n dt = kh^{m-n-\frac{3}{2}}.$$

The left member of this equation is independent of h, hence we must have

$$n = m - 3/2, \qquad \text{and} \qquad c\int_0^1 \sqrt{1-t}\,t^n dt = k,$$

which gives, since $\Gamma(3/2) = \sqrt{\pi}/2$,

$$c = \frac{2k}{\sqrt{\pi}}\cdot\frac{\Gamma(m+1)}{\Gamma(m-\frac{1}{2})}, \quad \text{provided } m > 1/2.$$

Hence (4) is a solution if $m > 1/2$.

When $m = 3/2$ we get the rectangular notch, $y =$ constant, and when $m = 1$ we get the curve $y = 1/\sqrt{x}$, such that the flow is directly proportional to the depth of stream.

DEPRECIATION BY A CONSTANT PERCENTAGE PLUS A CONSTANT.

By C. R. FORSYTH, Dartmouth College.

There are two ways in which a piece of property may depreciate which are usually considered in any complete treatise on the mathematical theory of depreciation, treatments in which possible interest accumulations are given no consideration. The methods employed to compute the annual or periodic allowance for depreciation corresponding to these two ways are known familiarly as the "straight line" method and the "constant percentage of book value" method. The annual allowance corresponding to the first case is the constant

$$k = \frac{C - S}{n}, \tag{1}$$

where C denotes the original cost, S the scrap value and n the estimated lifetime

of the property. The constant percentage in the second case is

$$x = 1 - \sqrt[n]{\frac{S}{C}}\cdot \tag{2}$$

It is not the purpose of this paper to dwell upon the practical virtues of the application of the first method or the practical defects of the application of the second method but rather, since both methods occupy a fairly important place in the mathematical theory of depreciation from the point of view of the mathematical theory itself, to present results analogous to those given above when a piece of property depreciates by a constant percentage plus a constant.

If we denote the original cost by C, the constant percentage by x and the constant by k, then the amount to be charged off the first year is $Cx + k$ leaving a residue of $C(1 - x) - k$. Continuing this reasoning it is easily shown that the residue R_n at the end of the nth year will be

$$R_n = (1 - x)^n\left(C + \frac{k}{x}\right) - \frac{k}{x}\cdot \tag{3}$$

Suppose that the valuation engineer is able to give at least three estimates R_n, R_{2n}, R_{3n}, etc., of the value of the property corresponding to the ends of at least three intervals of n years. Assuming first R_n to be known, equation (3) is easily solved for k to give

$$k = x\frac{R_n - C(1 - x)^n}{(1 - x)^n - 1}\cdot \tag{4}$$

It remains then to determine x. Assuming now that R_{2n} and R_{3n} are also known, we may write

$$R_n = (1-x)^n\left(C + \frac{k}{x}\right) - \frac{k}{x}, \qquad R_{2n} = (1-x)^{2n}\left(C + \frac{k}{x}\right) - \frac{k}{x},$$

$$R_{3n} = (1-x)^{3n}\left(C + \frac{k}{x}\right) - \frac{k}{x}\cdot$$

Subtracting and dividing as indicated on the left side of the following equation we obtain

$$\frac{R_{2n} - R_{3n}}{R_n - R_{2n}} = (1 - x)^n,$$

whence

$$x = 1 - \sqrt[n]{\frac{R_{2n} - R_{3n}}{R_n - R_{2n}}}\cdot \tag{5}$$

As a simple illustration, suppose that a property depreciates annually (that is, $n = 1$) by a constant percentage plus a constant in accordance with the following data: $C = \$1000.00$, $R_1 = \$880.00$, $R_2 = \$772.00$, $R_3 = \$674.80$. Then, by formula (5)

$$x = 1 - \frac{772.00 - 674.80}{880.00 - 772.00} = 1/10.$$

By formula (4)

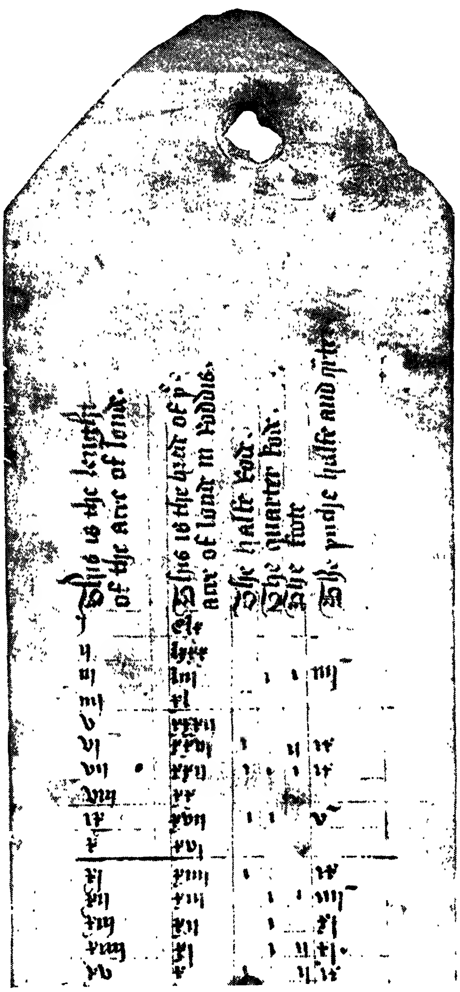
$$k = \frac{1}{10}\frac{\$880.00 - (9/10)\$1000.00}{9/10 - 1} = \$20.00.$$

As a special case, if ever $x = 0$ formula (4) takes an indeterminate form which, however, is easily evaluated in the usual way to be (1), $k = (C - S)/n$, where periodic estimates are no longer needed and R_n is replaced by S. Likewise, if $k = 0$ formula (3) becomes $S = C(1 - x)^n$, where, again, periodic estimates are no longer needed and R_n is replaced by S. Solving this equation for x we obtain formula (2).

AN INTERESTING FOURTEENTH CENTURY TABLE.

By DAVID EUGENE SMITH, Columbia University.

There has recently come into the possession of the library of Columbia University an interesting mathematical roll written apparently in the south of England about the close of the fourteenth century. It is $3\frac{9}{16}$ inches wide and $38\frac{3}{16}$ inches long and consists of two strips of parchment sewed together, only the first part of the roll being shown in the facsimile. The style of the script, the spelling, and the forms of the numerals suggest as the approximate date the year 1400. This is rather early for a mathematical manuscript in the English language, although we have others of still earlier date. English manuscripts of a mathematical nature written before the fifteenth century usually concern the interests of the less scholarly class, and, since this particular specimen relates to farm measurement, it would have been of little service had it appeared in the Latin of the church schools.



The roll consists of a table showing the widths corresponding to various lengths of a rectangular piece of land containing an acre. As the facsimile shows, the first column gives the lengths, beginning with 1 rod, the caption reading: "This is the lenght of the acre of londe." The lengths are given for every rod from 1 to 160.

The second column gives the largest number of rods in the width of the rectangular piece: "This is the brede of y[e] acre of londe in Roddis,"—in which the symbol y[e] is, as is well known to scholars, the old Anglo-Saxon form of "the," the letter resembling a small *y* (*th*) having been retained (as in this manuscript) long after the form *Th* (for the capital) was adopted.

The third and fourth columns give the fractions of a rod in halves and fourths, "The halfe Rode" and "The quarter Rode" forming an interesting relic of the ancient use of those unit fractions which seem to have been first employed in Mesopotamia in the third millennium B.C. and which are familiar in Egyptian mathematics for a period of upwards of two thousand years.

Instead of then proceeding to eighths, which would have been a little too difficult for the farm owner, the maker of the table gives, in the next column, the number of feet, or, as he writes it, "The foote." The final column gives the inches and fractions,—"The ynche halfe and q̃rter."

It will be observed that the numerals are Roman, the Hindu-Arabic forms not having as yet reached the great mass of commercial and industrial people in any country north of Italy, if, indeed, they may be said to have done so in Italy itself. The Roman numerals answered all ordinary needs so long as integers alone were involved, but they failed when elaborate fractions were demanded. This manuscript shows how the difficulty was met, both by the use of the unit fractions one half and one fourth with respect to the rod, as shown by the numerator 1 in the third and fourth columns, and with respect to the inch, as shown by the characters ($\sim$) for one half and ($\cdot$) for one fourth.

The need for such a table will become apparent if the reader will endeavor to find the width of a rectangle, say 12 rods long and containing precisely one acre, making use only of Roman numerals or the abacus in his computations. In this special case the table gives as the result 13 (xiij) rods, $\frac{1}{4}$ rod, 1 foot, $4\frac{1}{2}$ (iiij$\sim$) inches, which the reader may care to verify. The computations are accurate to a fraction of an inch, but the only fractions used in this connection are $\frac{1}{2}$ ($\sim$), $\frac{1}{4}$ ($\cdot$), and $\frac{3}{4}$ ($\underset{\cdot}{\sim}$), and in some of his results the computer has erred by more than a quarter of an inch,—an error of no practical significance in the kind of computation for which the table was intended.

RECENT PUBLICATIONS.

REVIEWS.

The Absolute Relations of Time and Space. By A. A. Robb. Cambridge, at the University Press, 1921. 8vo. 9 + 80 pages. Price 5 shillings.

Preface: "At the meeting of the British Association in 1902, Lord Rayleigh gave a paper entitled 'Does motion through the ether cause double refraction?' in which he described some experiments which seemed to indicate that the answer was in the negative. I recollect that on this occasion Professor Larmor was asked whether he would expect any such effect and he replied that he did not expect any.

"In the discussion which followed reference was made to the null results of all attempts to detect uniform motion through the aether and to the way in which things seemed to conspire together to give these null results.

"The impression made on me by this discussion was: that in order properly to understand what happened, it would be necessary to be quite clear as to what we mean by equality of lengths, etc., and I decided that I should try at some future time to carry out an analysis of this subject.

"I am not certain that I had not some idea of doing this even before the British Association meeting, but in any case, the inspiration came from Sir Joseph Larmor, either at this meeting or on some previous occasion while attending his lectures.

"Some years later I proceeded to try to carry out this idea, and while engaged in endeavouring to solve the problem, I heard for the first time of Einstein's work.

"From the first I felt that Einstein's standpoint and method of treatment were unsatisfactory, though his mathematical transformations might be sound enough, and I decided to proceed in my own way in search of a suitable basis for a theory.

"In particular I felt strongly repelled by the idea that events could be simultaneous to one person and not simultaneous to another; which was one of Einstein's chief contentions.

"This seemed to destroy all sense of the reality of the external world and to leave the physical universe no better than a dream, or rather, a nightmare.

"If two physicists A and B agree to discuss a physical experiment, their agreement implies that they admit, in some sense, a common world in which the experiment is supposed to take place.

"It might be urged perhaps that we have merely got a correspondence between the physical worlds of A and B, but if so, where, or how, does this correspondence subsist?

"It cannot be in A's mind alone, or it would not be a correspondence, and similarly it cannot be in B's mind alone.

"It seems to follow that it must be in some common sub-stratum; and this brings us at once back to an objective standpoint.

"The first work which I published on this subject was a short tract entitled *Optical Geometry of Motion, a New View of the Theory of Relativity* which appeared in 1911.

"This paper, though it did not claim to give a complete logical analysis of the subject, yet contained some of the germs of my later work and, in particular, it avoided any attempt to identify instants at different places. Later on the idea of '*Conical Order*' occurred to me, in which such instants are treated as definitely distinct.

"The working out of this idea was a somewhat lengthy task and in 1913 I published a short preliminary account of it under the title *A Theory of Time and Space*, which was also the title of a book on this subject on which I was then engaged.

"This book was in the press at the time of the outbreak of the war and was finally published toward the end of 1914.

"Unhappily at that period people were concerning themselves rather with trying to sever one another's connexions with Time and Space altogether, than with any attempt to understand such things; so that it was hardly an ideal occasion to bring out a book on the subject.

"The subject moreover was not an easy one, and I have been told more than once that my book is difficult reading.

"To this I can only reply as did Mr. Oliver Heaviside, under similar circumstances, that it was perhaps even more difficult to write.

"Be that as it may, the results arrived at fully justified my attitude towards Einstein's standpoint.

"I succeeded in developing a theory of Time and Space in terms of the relations of *before* and *after*, but in which these relations are regarded as absolute and not dependent on the particular observer.

"In fact it is not a 'theory of relativity' at all in Einstein's sense, although it certainly does involve relations.

"These relations of *before* and *after*, serving, as they do, as a physical basis for the mathematical theory, were quite ignored in Einstein's treatment; with the result that the absolute features were lost sight of.

"Even now, some six years from the date of publication of my book, comparatively few of Einstein's followers appear to realize the extreme importance of these relations, or to recognize how they alter the entire aspect of the subject.

"The theory, in so far as its postulates have an interpretation, becomes a physical theory in the ordinary sense, but these postulates are used to build up a pure mathematical structure.

"From the physical standpoint the question is: whether the postulates *as interpreted* are correct expressions of physical facts, or in some respect only approximations?

"If the postulates are not all correct expressions of the facts, then which of them require emendation and what emendation do they require?

"As regards the pure mathematical aspect of the theory: this of course remains unaffected by the physical interpretation of the postulates, and those who are interested only in pure mathematics may find that the method employed has certain advantages as a study of the foundations of geometry.

"In particular it may be noticed that by this method we get a system of geometry in which 'congruence' appears, not as something extraneous grafted on to an otherwise complete system, but as an intrinsic part of the system itself.

"I had intended making further developments of this theory, but the outbreak of the war caused an interruption of my work.

"In the meantime Einstein produced his 'generalized relativity' theory and the reader will doubtless wish to know how this work bears upon it.

"So far as I can at present judge, the situation is this: once coördinates have been introduced, the theory here developed gives rise to the same analysis as Einstein's so-called 'restricted relativity' and this latter cannot be regarded as satisfactory apart from my work, or some equivalent.

"Einstein's more recent work is extremely analytical in character.

"The *before* and *after* relations have not been employed at all in its foundation, although it is evident that, if these relations are a sufficient basis for the simple theory, they must play an equally important part in any generalization. Moreover these relations most certainly have a physical significance whatever theory be the correct one.

"A generalization of my own work is evidently possible and, to a certain extent, I can see a method of carrying this out, although I have not as yet worked out the details. (See Appendix.)

"In the meantime it seemed desirable to write some sort of introduction to my *Theory of Time and Space* which, while not going into the proofs of theorems, would yet convey to a larger circle of readers the main results arrived at in that work."

Contents—Preliminary considerations, 1–16; Conical order, 16–45; Normality of general lines having a common element, 46–55; Theory of congruences, 56–71; Introduction of coördinates, 72–75; Interpretation of results, 76–78; Appendix, 78–80.

Introduction to the Theory of Fourier's Series and Integrals. By H. S. CARSLAW. Second edition, completely revised. London, Macmillan, 1921. 8vo. 11 + 323 pp. Price 30 shillings.

Preface: "This book forms the first volume of the new edition of my book on *Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat*, published in 1906, and now for some time out of print. Since 1906 so much advance has been made in the Theory of Fourier's Series and Integrals, as well as in the mathematical discussion of Heat Conduction, that it has seemed advisable to write a completely new work, and to issue the same in two volumes. The first volume, which now appears, is concerned with the Theory of Infinite Series and Integrals, with special reference to Fourier's Series and Integrals. The second volume will be devoted to the Mathematical Theory of the Conduction of Heat.

"No one can properly understand Fourier's Series and Integrals without a knowledge of what is involved in the convergence of infinite series and integrals. With these questions is bound up the development of the idea of a limit and a function, and both are founded upon the modern theory of real numbers. The first three chapters deal with these matters. In Chapter IV the Definite Integral is treated from Riemann's point of view, and special attention is given to the question of the convergence of infinite integrals. The theory of series whose terms are functions of a single variable, and the theory of integrals which contain an arbitrary parameter are discussed in Chapters V and VI. It will be seen that the two theories are closely related, and can be developed on similar lines.

"The treatment of Fourier's Series in Chapter VII depends on Dirichlet's Integrals. There and elsewhere throughout the book, the Second Theorem of Mean Value will be found an essential part of the argument. In the same chapter the work of Poisson is adapted to modern standard, and a prominent place is given to Fejér's work, both in the proof of the fundamental theorem and in the discussion of the nature of the convergence of Fourier's Series. Chapter IX is devoted to Gibbs's Phenomenon, and the last chapter to Fourier's Integrals. In this chapter the works

of Pringsheim, who has greatly extended the class of functions to which Fourier's Integral Theorem applies, has been used.

"Two appendices are added. The first deals with 'Practical Harmonic Analysis and Periodogram Analysis.' In the second a bibliography of the subject is given.

"The functions treated in this book are 'ordinary' functions. An interval (a, b) for which $f(x)$ is defined can be broken up into a finite number of open partial intervals, in each of which the function is monotonic. If infinities occur in the range, they are isolated and finite in number. Such functions will satisfy most of the demands of the Applied Mathematician.

"The modern theory of integration, associated chiefly with the name of Lebesgue, has introduced into the Theory of Fourier's Series and Integrals functions of a far more complicated nature. Various writers, notably W. H. Young, are engaged in building up a theory of these and allied series much more advanced than anything treated in this book. These developments are in the meantime chiefly interesting to the Pure Mathematician specialising in the Theory of Functions of a Real Variable. My purpose has been to remove some of the difficulties of the Applied Mathematician."

Contents—Historical introduction, 1–15; Chapter I: Rational and irrational numbers, 16–28; II: Infinite sequences and series, 29–48; III: Functions of a single variable. Limits and continuity, 49–75; IV: The definite integral, 76–121; V: The theory of infinite series whose terms are functions of a single variable, 122–168; VI: Definite integrals containing an arbitrary parameter, 169–195; VII: Fourier's series, 196–247; VIII: The nature of the convergence of Fourier's series, 248–263; IX: The approximation curves and Gibbs's phenomenon in Fourier's series, 264–282; X: Fourier's integrals, 283–294; Appendix I: Practical harmonic analysis and periodogram analysis, 295–301; II: Bibliography, 302–317; List of authors quoted, 318–319; General index, 320–323.

Each chapter concludes with bibliographical "References." The names of the following Americans occur in the work: BÔCHER, BYERLY, FORD, GRONWALL, JACKSON, C. N. MOORE, and VAN VLECK.

Plane and Solid Analytic Geometry. By W. F. OSGOOD and W. C. GRAUSTEIN. New York, The Macmillan Company, 1921. 12mo. 17 + 614 pp. Price $3.75.

Preface: "The object of an elementary college course in Analytic Geometry is twofold: it is to acquaint the student with new and interesting and important geometrical material, and to provide him with powerful tools for the study, not only of geometry and pure mathematics, but in no less measure of physics in the broadest sense of the term, including engineering.

"To attain this object, the geometrical material should be presented in the simplest and most concrete form, with emphasis on the geometrical content, and illustrated, whenever possible, by its relation to physics. This principle has been observed throughout the book. Thus, in treating the ellipse, the methods actually used in the drafting room for drawing an ellipse from the data commonly met in descriptive geometry are given a leading place. The theorem that the tangent makes equal angles with the focal radii is proved mechanically: a rope which passes through a pulley has its ends tied at the foci and is drawn taut by a line fastened to the pulley. Moreover, the meaning of foci in optics and acoustics is clearly set forth. Again, there is a chapter on the deformations of an elastic plane under stress, with indications as to the three-dimensional case (pure strain, etc.).

"The methods of analytic geometry, even in their simplest forms, make severe demands on the student's ability to comprehend the reasoning of higher mathematics. Consequently, in presenting them for the first time, purely algebraic difficulties, such as are caused by literal coefficients and long formal computations, should be avoided. The authors have followed this principle consistently, beginning each new subject of the early chapters with the discussion of a simple special, but typical, case, and giving immediately at the close of the paragraph simple examples of the same sort. They have not, however, stopped here, but through carefully graded problems, both of geometric and of analytic character, have led the student to the more difficult applications of the methods, and collections of examples at the close of the chapters contain such as put to the test the initiative and originality of the best students.

"As a result of this plan the presentation is extraordinarily elastic. It is possible to make the treatment of any given topic brief without rendering the treatment of later topics unintelligible, and thus the instructor can work out a course of any desired extent. For example, one freshman

course at Harvard devotes about thirty periods to analytic geometry and the material consists of the essential parts of the first nine chapters. Another freshman course gives twice the time to analytic geometry (the students having already had trigonometry), taking up determinants and the descriptive properties of the quadric surfaces, and also devoting more time to the less elementary applications of the methods of analytic geometry. The advanced courses in the calculus and mechanics require the material of the later chapters. In fact, a thorough elementary treatment of the rudiments of Solid Analytic Geometry is indispensable for the understanding of standard texts on applied mathematics. It is true that these texts are chiefly Continental. But we shall never have American treatises which are up to the best scientific standards of the day until the subjects above mentioned are available in simply intelligible form for the undergraduate.

"The subject of loci is brought in early through a brief introductory chapter, and problems in loci are spread throughout the book. A later chapter is devoted to a careful explanation of the method of auxiliary variables. There is a chapter on determinants, with applications both to analytic geometry and to linear equations. Diameters and poles and polars in the plane and in space receive a thorough treatment. Cylindrical and spherical coördinates and quadric surfaces are illumined by the concept of triply orthogonal systems of surfaces. The reduction of the general equation of the second degree in space to normal forms by translations and rotations is sketched and illustrated by numerical examples.

"The question may be asked: In so extensive a treatment of analytic geometry should not, for example, homogeneous coördinates find a place? The authors believe that the student, before proceeding to the elaborate methods of modern geometry, should have a thorough knowledge both of the material and the methods which may fairly be called elementary, and they felt that a book which, avoiding the conciseness of some of the current texts and the looseness of others, is clear because it is rigorous will meet a real need.

"This book is designed to be at once an introduction to the subject and a handbook of the elements. May it serve alike the needs of the future specialist in geometry, the analyst, the mathematical physicist, and the engineer."

Contents.

Einstein's Theories of Relativity and Gravitation. A Selection of Material from the Essays submitted in the Competition for the Eugene Higgins Prize of $5,000. Compiled and edited, and introductory matter supplied, by J. M. BIRD, associate editor of the *Scientific American*. New York, Scientific American Publishing Co., 1921. 12mo. 14 + 345 pp. Price $2.00.

We have already referred [*1921*, 191–192; 269] to the prize offered by the *Scientific American*, to the conditions of its award, and to essays presented in competition for it. In the volume under review the prize essay is not given till pages 169–180 and this is followed by a dozen other essays. It was designed by the editor that "Each essay should be made easier of reading by the examination of those preceding it; at the same time each, by the choice of ground covered and by the emphasis on points not brought out sharply by its predecessors, should throw new light upon these predecessors." Chapters II–VI, pages 19–

168, are given up to the discussion of various ideas for making the reading of the essays more profitable; for this purpose, in three of these chapters, the general plan has been to make extracts from about fifty other of the competing essays, the extracts being connected by editorial comment. For example, in the chapter on "The special theory of relativity" there are extracts from competing essays of twenty-six authors, interlarded with more than twenty-five editorial comments. The following is the complete table of contents:

An Introduction to Mathematical Analysis. By F. L. GRIFFIN. Boston, Houghton, Mifflin Company, 1921. 12mo. 8 + 512 pp. Price $2.75.

Extracts from the Preface: "Under the traditional plan of studying trigonometry, college algebra, analytic geometry, and calculus separately, a student can form no conception of the character and possibilities of modern mathematics, nor of the relations of its several branches as parts of a unified whole, until he has taken several successive courses. Nor can he, early enough,

get the elementary working knowledge of mathematical analysis, *including integral calculus*, which is rapidly becoming indispensable for students of the natural and social sciences. Moreover, he must deal with complicated technique in each introductory course; and must study many topics apart from their uses in other subjects, thus missing their full significance and gaining little facility in drawing upon one subject for help in another.

"To avoid these disadvantages of the separate-subject plan the unified course presented here has been evolved. This enables even those students who can take only one semester's work to get some idea of differential and integral calculus, trigonometry, and logarithms. And specialist students, as experience has shown, acquire an excellent command of mathematical tools by first getting a bird's-eye view of the field, and then proceeding to perfect their technique.

"A regular course in calculus, following this, can proceed more rapidly than usual, include more advanced topics, and give a fine grasp: the principles and processes have become an old story. And the regular course in analytic geometry can be devoted to a genuine study of the geometrical properties of loci, since most of the type equations, basic formulas, and calculus methods are already familiar.

"The materials presented here have been thoroughly tried out with the freshman classes in Reed College during the past nine years. Problems and methods which have proved unsatisfactory have been eliminated. Care has been taken to make the concepts tangible, relate them to the familiar ideas of daily life, exhibit practical applications, and develop the attitude of investigation. . . .

"The course as given at Reed College takes four hours a week through the year, the number of lessons devoted to the several chapters, when taken complete, having run about as follows: 14, 4, 14, 8, 11, 12, 11, 16, 5, 7, 10, 6, 6, 5, 4. . . . The course is adapted to students of widely differing preparations. A knowledge of plane and solid geometry and of algebra through quadratics is the most suitable equipment; but a number of students who had had only two years of secondary mathematics have carried the course very well. On the other hand, students who have already taken trigonometry and college algebra find in the present course very little that merely duplicates their former work."

Contents—A preliminary word to students, 1–2; Chapter I: Functions and graphs (Some fundamental problems of variation: rates, mean values, extremes, zero values, formulas, etc.), 3–57; II: Some basic ideas analyzed (Instantaneous rates, tangents, areas, etc., as limits), 58–75; III: Differentiation (Derivatives of polynomials and u^n. Rates, extremes, etc.), 76–125; IV: Integration ($\int x^n dx$. Area, volume, momentum, work, fluid pressure, falling bodies, etc.), 126–155; V: Trigonometric functions (Solution of right and oblique triangles. Applications), 156–188; VI: Logarithms (Numerical calculations. Compound interest. Triangles), 189–235; VII: Logarithmic and exponential functions, 236–270; VIII: Rectangular coördinates (Mapping. Motion. *Analytic geometry:* line, circle, parabola, ellipse, hyperbola; translation, intersections), 271–325; IX: Solution of equations (Quadratics: $b^2 - 4ac$. Rational roots of higher equations. Horner's and Newton's methods), 326–342; X: Polar coördinates and trigonometric functions (Definitions. Radians. Periodic variations. Derivatives), 343–367; XI: Trigonometric analysis (Basic identities. Equations. More calculus. Involute. Cycloid. *S.H.M.* Damped oscillations. Addition formulas. Sums and products, etc.), 368–391; XII: Definite integrals (Summation of "elements": length, surface of revolution, etc. Plotting a surface. Double integration. Partial derivatives. Simpson's rule), 392–414; XIII: Progressions and series (*A.P.* and *G.P.* Investment theory. Maclaurin series. Calculation of functions. Binomia. theorem), 415–439; XIV: Permutations, combinations and probability ($P_{n,r}$; $C_{n,r}$. Chancel Normal probability curve. Least squares), 440–459; XV: Complex number system (Definition. Geometric representation. Operations. Roots of unity. Application), 460–472; Retrospect and prospect, 472–483; Appendix (Proofs for reference. Formulas. Integrals. Numerical tables: roots, natural and common logarithms, trigonometric functions for radians or degrees), 485–508; Index, 509–512.

Analytic Geometry with Introductory Chapter on the Calculus. By C. I. PALMER and W. C. KRATHWOHL. New York, McGraw-Hill Book Co., 1921. 12mo. 14 + 347 pages. Price $2.50.

Preface: "The object of this book is to present analytic geometry to the student in as natural and simple a manner as possible without losing mathematical rigor. The average student thinks visually instead of abstractly, and it is for the average student that this work has been written.

It was prepared primarily to meet the requirements in mathematics for the second half of the first year at the Armour Institute of Technology. To make it adaptable to courses in other institutions of learning certain topics not usually taught in an engineering school have been added.

"While it is useless to claim any great originality in treatment or in the selection of subject matter, the methods and illustrations have been thoroughly tested in the class room. It is believed that the topics are so presented as to bring the ideas within the grasp of students found in classes where mathematics is a required subject. No attempt has been made to be novel only; but the best ideas and treatment have been used, no matter how often they have appeared in other works on the subject.

"The following points are to be especially noted:

(1) The great central idea is the passing from the geometric to the analytic and *vice versa*. This idea is held consistently throughout the book.

(2) In the beginning a broad foundation is laid in the algebraic treatment of geometric ideas. Here the student should acquire the analytic method if he is to make a success of the course.

(3) Transformation of coördinates is given early and used frequently throughout the book, not confined to a single chapter as is so frequently the case. The same may be said of polar coordinates.

(4) Fundamental concepts are dealt with in an informal as well as in a formal manner. The informal often fixes and clarifies the ideas where the formal does not.

(5) Numerous illustrative examples are worked out in order that the student may get a clear idea of the methods to be used in the solution of problems.

(6) The conic sections are treated from the starting point of the *focus* and *directrix* definition.

(7) Because of its great importance in engineering practice the empirical equation is dealt with more completely than is usual. This treatment has been made as elementary as possible, but sufficiently comprehensive to enable one to solve the average problem in empirical equations.

(8) The fundamental concepts of the calculus are presented in a very concrete manner, and a much greater use than is usual is made of the differential. The ideas are thus more readily visualized than is possible otherwise. The applications are mainly to tangents, normals, areas, and the discussion of equations.

(9) The concluding chapter gives an adequate and careful treatment of solid geometry so necessary in the study of the calculus.

(10) The exercises are numerous, carefully graded, and include many practical applications.

(11) In the introductory chapter are found various short tables and formulas, and at the end are given four-place tables of logarithms and trigonometric functions."

Contents—Chapter I: Introduction, 1–7; II: Geometric facts expressed analytically, and conversely, 8–43; III: Loci and equations, 44–58; IV: The straight line and the general equation of the first degree, 59–85; V: The circle and certain forms of the second degree equation, 86–97; VI: The parabola and certain forms of the second degree equation, 98–116; VII: The ellipse and certain forms of the second degree equation, 117–133; VIII: The hyperbola and certain forms of the second degree equation, 134–153; IX: Other loci and equations, 154–187; X: Empirical loci and equations, 188–205; XI: Poles, polars, and diameters, 206–215; XII: Elements of calculus, 216–260; XIII: Solid analytic geometry, 261–303; Summary of formulas, 303–306; Four-place table of logarithms, 308–309; Table of trigonometric functions, 310–314; Answers, 315–340; Index, 341–347.

Plane Trigonometry. By ARNOLD DRESDEN. New York, John Wiley & Sons, 1921. 8vo. 7 + 110 pages. Price $1.60.

From the Preface: "While the importance of the function concept for elementary mathematics has become recognized by many writers of college algebra texts and of 'unified freshman mathematics' books, it has received little recognition from writers on elementary trigonometry. To emphasize this importance has been the leading motive in writing the present book. A somewhat detailed study of the graphs of the trigonometric functions (Chapter V) and of the inverse functions (Chapter VIII) has been introduced for this purpose. Much more could and should be done in this direction; perhaps the present effort may suffice as a first step.

"The opportunity afforded by the writing of a new text has been used to make some changes in the presentation of the traditional material. Circular measurement of angles is introduced in the first chapter so as to be available for use throughout the course. The fundamental theorems on projections are presented early and are used subsequently so that the student may be familiar

with them when they are applied in a general proof of the addition theorems, based on a method quite generally followed by continental writers. Recognizing the value of the 'solution of triangles,' a good deal of space has been devoted to this subject, and an attempt has been made to develop it in such a manner that the students can appreciate the reasons for the different methods that are discussed.

"On the question of 'applied problems,' I have taken a definite position. I do not think it feasible to introduce into an elementary text technical material from applied sciences, important though such material may be. Without such material, however, applications cannot well be anything but problems which use the language of the applied sciences without really belonging to them. An elementary text can render useful service, even to applied science, by stressing the fundamental concepts of trigonometry and by setting problems which connect with the student's actual experience and which suggest ways in which these concepts may be applied, leaving actual applications to the fields to which they belong.

"It has not seemed desirable to add to the number of tables of logarithms already available. The elementary treatment of logarithms in Chapter III and the problems scattered throughout the book call for the use of a set of five-place tables, of which there are many excellent ones in existence.

"No attempt at logical completeness has been made, but rather has it been my aim to adapt the treatment to the stage of logical development which may be expected of students who begin the study of trigonometry. I am aware of the fact that a fuller discussion might be made in several instances and I shall be happy if the treatment as given should arouse the critical powers of some students and develop in them a desire for more penetrating analysis.

"The material as here presented was used originally in mimeographed form by a few classes in the University of Wisconsin."

Contents—Chapter I: Positive and negative lines and angles. Coördinates. Radian measurement, 1–8; II: The trigonometric ratios. Simple identities, 9–19; III: Logarithms, 20–31; IV: Solution of right triangles. Applications, 32–40; V: The graphs of the trigonometric functions, 41–54; VI: The addition formulæ, 55–64; VII: The solution of triangles, 65–89; VIII: Inverse trigonometric functions. Trigonometric equations, 90–103; List of answers to the exercises, 105–108; Index, 109–110.

Elements of Map Projection with Applications to Map and Chart Construction. By C. H. Deetz and O. S. Adams. (Department of Commerce, U. S. Coast and Geodetic Survey, serial no. 146, special publication no. 68.) Washington, Government Printing Office, 1921. Royal 8vo. 163 pp. + 8 plates. Price $.50.

Preface: "In this publication it has been the aim of the authors to present in simple form some of the ideas that lie at the foundation of the subject of map projections. Many people, even people of education and culture, have rather hazy notions of what is meant by a map projection, to say nothing of the knowledge of the practical construction of such a projection.

"The two parts of the publication are intended to meet the needs of such people; the first part treats the theoretical side in a form that is as simple as the authors could make it; the second part attacks the subject of the practical construction of some of the most important projections, the aim of the authors being to give such detailed directions as are necessary to present the matter in a clear and simple manner.

"Some ideas and principles lying at the foundation of the subject, both theoretical and practical, are from the very nature of the case somewhat complicated, and it is a difficult matter to state them in a simple manner. The theory forms an important part of the differential geometry of surfaces, and it can only be fully appreciated by one familiar with the ideas of that branch of science. Fortunately, enough of the theory can be given in simple form to enable one to get a clear notion of what is meant by a map projection and enough directions for the construction can be given to aid one in the practical development of even the more complicated projections.

"It is hoped that this publication may meet the needs of people along both of the lines indicated above and that it may be found of some interest to those who may already have a thorough grasp of the subject as a whole."

Contents—*Part I:* General statement, 7–8; Analysis of the basic elements of map projection, 9–21; Representation of the sphere upon a plane, 22–29; Elementary discussion of various forms

of projection, 30–52. *Part II:* Introduction, 53–57; The polyconic projection, 58–66; The Bonne projection, 67–70; The Lambert zenithal (or azimuthal) equal-area projection, 71–76; The Lambert conformal conic projection with two standard parallels, 77–86; The Grid system of military mapping, 87–90; The Albers conical equal-area projection with two standard parallels, 91–100; The Mercator projection, 101–136; Fixing position by wireless directional bearings, 137–139; The gnomonic projection, 140–145. *World maps:* The Mercator projection; The stereographic projection; The Aitoff equal-area projection of the sphere; The Mollweide homalographic projection; Goode's homalographic projection (interrupted) for the continents and oceans; Lambert projection of the northern and southern hemispheres; Conformal projection of the sphere within a two-cusped epicycloid; Guyou's doubly periodic projection of the sphere, 146–160. Index, 161–163.

Examples in Differential and Integral Calculus with Answers. By the late C. S. JACKSON. (Longmans' Modern Mathematical Series.) London and New York, Longmans, Green and Co., 1921. 8vo. 8 + 142 pages. Price $3.25.

First paragraphs of preface by W. M. Roberts: "This collection of Examples in the Calculus, which was made by the late Mr. C. S. Jackson, should have been published in 1917. Mr. Jackson's sudden and regrettable death in October, 1916 [see this MONTHLY *1917*, 144] caused the publication to be delayed till after the War. The book is hampered by having to be put through the press by other hands than the author's.

"A great many of those examples which can be classed as problems were constructed by Mr. Jackson himself in connection with his work at the Royal Military Academy, Woolwich, and the many public examinations in which he took part. Many of them are very neat applications of the Calculus to practical problems, and it is hoped that these will prove particularly useful to teachers who require, in their work, a number of examples which are not mere Algebraical manipulations. Many books on the Calculus treat the subject chiefly as an extension of Algebra and Analytical Geometry. This collection should be a useful supplement to such books."

Contents—*Part I, Differential Calculus*, 1–53: Differentiation; tangents and slopes; dy/dx as a rate of increase; easy maxima and minima; velocity; differentiation of logarithms; errors and rates; approximation to roots of equations; Newton's method of approximating to the roots of an equation; maxima, minima; harder questions on tangents and normals; errors; velocity; miscellaneous examples; successive differentiation; the theorem of Leibnitz; miscellaneous expansions; indeterminate forms; Taylor's theorem and applications; curvature; examples on maps; partial differentiation. *Part II, Integral Calculus*, 54–120: Known results of differentiation; methods of integration; hyperbolic functions; integration; areas; planimeters and integraphs; areas and volumes; volumes; problems on simple integration; mean values; rectification and areas of surfaces; centers of gravity; second moments, or moments of inertia; center of pressure; pendulum; differential equations; double and triple integration. *Answers*, 121–142.

This work is one of the Series containing G. B. Mathews's *Projective Geometry* (1914), Hilda P. Hudson's *Ruler and Compasses* (1916), H. S. Carslaw's *Elements of Non-Euclidean Geometry* (1916), and H. Bateman's *Differential Equations* (1918).

Higher Mechanics. By HORACE LAMB. Cambridge, at the University Press, 1920. 8vo. 10 + 272 pages. Price 21 shillings.

Preface: "This book treats of three-dimensional Kinematics, Statics, and Dynamics in what is I think a natural, as I have found it to be a convenient, order. It may be regarded as a sequel to two former treatises[1] to which occasional reference is made; but it is not dependent on these, and will I trust be readily followed by students who are conversant with ordinary two-dimensional Mechanics.

"The subject is of course a very wide one, and some principle of selection is necessary. I have tried to confine myself to matters of genuine kinematical or dynamical importance, avoiding developments whose interest, often considerable, is purely mathematical or now mainly historical. It is owing to such considerations that whilst some account is given of the Theory of Screws, of Null-Systems, and of Least Action, on the other hand brachistochrone problems, and the general theory of the Differential Equations of Dynamics, are left untouched.

"The book does not claim to be more than an elementary one, regard being had to the nature

[1] "*Statics*, Cambridge, 1912, and *Dynamics*, Cambridge, 1914."

of the subject. The reader who wishes to carry his studies further will find ample assistance in Thomson and Tait, in Rayleigh's *Theory of Sound*, and in Whittaker's *Analytical Dynamics*. And in common with other recent writers I must mention with a special sense of obligation the works of Routh, which in their later forms are an almost inexhaustible storehouse of theorems and results, and abound in interesting historical references."

Contents—Chapter I: Kinematics of a rigid body. Finite displacements, 1–13; II: Infinitesimal displacements, 14–33; III: Statics, 34–65; IV: Moments of inertia, 66–73; V: Instantaneous motion of a body (kinematics), 74–88; VI: Dynamical equations, 89–111; VII: Free rotation of a rigid body, 112–128; VIII: Gyrostatic problems, 129–150; IX: Moving axes, 151–176; X: Generalized equations of motion, 177–207; XI: Theory of vibrations, 208–248; XII: Variational methods, 249–270; Index, 271–272.

ARTICLES IN CURRENT PERIODICALS.

American Journal of Science, fifth series, volume 2, September, 1921: "Some mechanical curiosities connected with the earth's field of force" by W. D. Lambert,[1] 129–158 [List of mechanical curiosities: " (1) The limiting level surface with the sharp edge. (2) The smooth lake with different elevations for its two ends. (3) The tendency of large bodies to 'fall' towards the equator. (4) The tendency of a rod suspended horizontally like the Eötvös balance to 'fall' by twisting about the supporting fiber. (5) The existence of great local irregularities in the curvature of the level surfaces and the interesting possibilities that the study of these irregularities seems likely to offer. "]

Annales Scientifiques de l'Ecole Normale Supérieure, third series, volume 38, August to December, 1921: "Problèmes d'hydrodynamiques relatifs aux mouvements glissants" by R. Thiry, 229–339; "Sur les ensembles abstraits" by M. Fréchet, 341–388; "Recherches sur le théorème de M. Picard" by G. Valiron, 389–429; "Rectification et complément au mémoire de la goutte liquide tournante" by J. Boussinesq, 431–437 (see *1922*, 20).

Bulletin of the American Mathematical Society, volume 27, June–July, 1921: "The April meeting of the American Mathematical Society" by R. G. D. Richardson, 389–400; "The Easter meeting of the Society at Chicago" by A. Dresden, 401–410; "The April meeting of the San Francisco Section" by B. A. Bernstein, 411–414; "The general theory of approximation by polynomials and trigonometric sums" by D. Jackson, 415–431; "The Einstein solar field" by L. P. Eisenhart, 432–434; "A covariant of three circles" by A. B. Coble, 434–437; "On skew parabolas" by Mary F. Curtis, 437–438; "The dispersion of observations" by J. L. Coolidge, 439–442; "The isomorphisms of complex algebra" by N. Wiener, 443–445; "On the generalization of certain fundamental formulas of the mathematical theory of finance" by C. H. Forsyth, 446–452; "The spread of Newtonian and Leibnizian notations of the calculus" by F. Cajori, 453–458; "Group theory reviews in the *Jahrbuch über die Fortschritte der Mathematik*" by G. A. Miller, 459–462; "Recent books on vector analysis" by J. B. Shaw, 463–465 [reviews of R. Gans's *Einführung in die Vektoranalysis, mit Anwendungen auf die mathematische Physik*, fourth edition (Leipzig and Berlin, 1921), of L. Silberstein's *Elements of Vector Algebra* (New York, 1919), of C. Runge's *Vektoranalysis*. Vol. I. *Die Vektoranalysis des dreidimensionalen Raumes* (Leipzig, 1919), of R. Leveugle's *Précis de Calcul géométrique* (Paris, 1920)]; Reviews by R. D. Carmichael of *Œuvres de G. H. Halphen* (2 vols., Paris, 1916–1918), 466–468, of G. H. Hardy's *Some famous problems of the theory of numbers and in particular Waring's problem* (Oxford, 1920), 471–475, and of O. Stolz and J. A. Gmeiner's *Theoretische Arithmetik* (2d ed., 2 vols., Leipzig, 1911–1915), 485; Reviews by D. E. Smith of F. Cajori's *A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* (Chicago, 1919), 468–470, of P. Kirchberger's *Mathematische Streifzüge durch die Geschichte der Astronomie* (Leipzig, 1921), 479–480, and of Emma Gifford's *Natural Tangents* (Manchester, England, 1920), 480; Review by J. B. Shaw of H. Lacaze's *Cours de Cinématique Théorique* (Paris, 1920), 470; Reviews by E. B. Wilson of H. Lamb's *Statics, including Hydrostatics and the Elements of the Theory of Elasticity* (Cambridge, 1916), 475–477, and of J. R. Eccles's *Advanced Lecture Notes on Light* (Cambridge, 1919), 485–486; Reviews by C. L. E. Moore of M. d'Ocagne's *Principes usuels de Nomographie* . . . (Paris, 1920),

[1] On page 141 he shows that a rod suspended horizontally (Eötvös torsion balance of the first kind) will tend to point east and west because the meridian sections of level surfaces in the earth's field of force are more curved than prime vertical surfaces. This tendency would seem to more than balance the tendency to point north and south that is due directly to the centrifugal force according to the statement in *Nature* of October 20, page 240 (see *1922*, 22).

477–478, and of K. P. Williams's *Dynamics of the Airplane* (New York, 1921), 483; Review by H. L. Rietz of *Œuvres Complètes de Christiaan Huygens* (vol. 14, The Hague, 1920), 481–482; Review by J. W. Young of D. E. Smith's *The Sumario Compendioso of Brother Juan Diez*: *The Earliest Mathematical Work of the New World* (Boston, 1921), 484; Review by H. H. Mitchell of *Materialien für eine wissenschaftliche Biographie von Gauss*, Heft 8, *Zahlbegriff und Algebra bei Gauss* by A. Fraenkel (Leipzig, 1920), 486–487; Notes, 488–495; New publications, 496–499; Thirtieth annual list of papers, 500–507; Index of volume 27, 508–516.

L'Education Mathématique, volume 24, November 1, 1921: "La géométrie du compas" (fin.), 17–18.

Engineering News Record, volume 87, November 17, 1921: "Some weaknesses of aerial photo maps" by P. J. Barry, 828 ["The principles of astronomy, trigonometry, geometry, geodesy and perspective cannot be assigned to oblivion by merely pressing a button. It is not as easy as that."]—November 24: "Improving the terminology in mechanics" by J. H. Griffith, 865—December 1: "Triangulating under difficulties" by H. W. Bradstreet, 899–900.

Jahresbericht der Deutschen Mathematiker-Vereinigung, volume 30, nos. 1–4 (issued October, 1921): "Mathematik und räumliche Anschauung" by R. Baldus, 1–15 [Translated extract: The oldest definition of mathematics designated it as the theory of magnitudes. This definition is to be avoided (even if the notion of magnitude be made clear) because there are wide domains of mathematics which it fails to embrace. One need only mention analysis situs, the theory of aggregates, and projective geometry. Were one, as has been done more recently, to describe mathematics as the science of drawing logically necessary conclusions, then mathematics and logical thought would coincide; this definition extends beyond mathematics. . . . These (Euclid's 13 Books of Elements) even as works of art influenced the non-mathematician very strongly. "Euclid's *Elements* seems to me almost as beautiful as Homer's *Iliad*" said H. S. Chamberlain in his *Foundations of the Nineteenth Century*]; "Die Mathematik in der Schulreform" by W. Lietzmann, 59–68 [Section titles: 1. Grundschule; 2. Die bisherigen höheren Knabenschulen; 3. Die Ziele des mathematischen Unterrichts in den höheren Schulen; 4. Die deutsche Oberschule; 5. Die Aufbauschulen; 6. Das höhere Mädchenschulwesen; 7. Die Mittelschulen; 8. Die Gabelungstendenzen; 9. Die Lehrerbildungsfrage; 10. Fortbildungseinrichtungen.]—Supplement: "Mathematische Gesellschaft in Göttingen," 32 [November 23, 1920: "Runge, Amerikanische Arbeiten über Sternhaufen und die Milchstrasse"; February 21–22, 1921: "Hilbert, Eine neue Grundlegung des Zahlbegriffes."]; "Mathematisches Kränzchen in Prag," 32 [February 4, 1921: "Winternitz, Die Knoppsche Erzeugungsweise der Kurven von Peano, Osgood und v. Koch"]; "Einstein-Preis des Scientific American," 34 [See, *1921*, 191]—Nos. 5–8 (issued November, 1921): "W. R. Hamiltons Bedeutung für die geometrische Optik" by G. Prange, 69–82 [An appreciation and exposition of the fundamental and extensive results in geometrical optics obtained by W. R. Hamilton nearly one hundred years ago, only particular parts of which are generally familiar, such as the Hamilton-Jacobi theory in analytical mechanics, Hamilton's formula in the theory of congruences of lines in differential geometry, and the discovery of conical refraction in physics]; "Zur projektiven Differentialgeometrie der Ebene" by L. Berwald, 110–121 [A study of the "accompanying triangle," with reference to E. J. Wilczynski, S. W. Reaves, and others]—Supplement: "Programm der Jahresversammlung in Jena" (September 18–24, 1921), 45–47 [three papers on postulational treatment: "Fraenkel, Über die Zermelosche Begründung der Mengenlehre"; "Bernays, Über die Hilbertsche Grundlegung der Arithmetik"; "Hertz, Über die Minimalzahl von Axiomen für ein System von Sätzen und den Begriff des idealen Elementes."]; "Preisaufgaben und gekrönte Preisschriften," 52 [reannouncement of Wolfskehl prize of 100,000 marks "für denjenigen . . . dem es zuerst gelingt, den Beweis des grossen Fermatschen Satzes zu führen."]

Journal of the Indian Mathematical Society, volume 13, August, 1921: "On the cartesian oval" (second paper) by W. R. Aiyar, 121–132; "On the expansion of certain functions (with properties of associated co-efficients)" by C. Krishnamachari, 133–146; "Short notes: Linear systems of the third order on a conic" by R. Vaidyanathaswami, 147–150; Questions and Solutions, 151–160.

Mathematical Gazette, volume 10, December, 1921: "A school course in surveying" by H. R. Vernon, 353–358 ["The field-work and plan-drawing are done during Geography lessons, while the calculation of area, height finding and resection are undertaken by the mathematics classes." For school children of ages 10–13, graded]; "Missing-figure problems" by W. E. H. Berwick, 359–362 [A number of new and interesting problems in "skeleton" addition, multiplication and division. See *1921*, 37, 278]; "Sign in elementary analytical geometry" by F. G. Brown,

363–368 ["It is not claimed that the conventions indicated in this brief paper are entirely satisfactory; Bôcher in his *Plane Analytic Geometry* indicates that neither his own convention 'nor any other one which could be made' can be satisfactory. . . . Immediately you try to frame satisfactory rules for the perpendicular *from a point* to a line you run up against the fundamental convention of Cartesians, which requires (x, y) to be arrived at by travelling from the axes and not towards them."]; "Gleanings far and near," 358, 362, 368 [No. 88. "The American Cocker—'I have heard that devils can be raised with Daboll's arithmetic. . . . That's my small experience as far as the Massachusetts calendar, and Bowditch's navigator, and Daboll's arithmetic go.'—p. 515, Herman Melville's *Moby Dick* (World's Classics)"]; Review of W. F. Osgood's *Elementary Calculus* by C. Jones, 381–382.

MATHEMATICS TEACHER, volume 14, May, 1921: "Why it is impossible to trisect an angle or to construct a regular polygon of 7 or 9 sides by ruler and compasses" by L. E. Dickson, 217–223; "College entrance requirements in mathematics: preliminary report of National Committee on Mathematical Requirements," 224–245; "Comments on the teaching of geometry" by F. C. Touton, 246–251; "Some ideals in teaching mathematics" by E. W. Schreiber, 252–254; "Computation in junior high school mathematics" by D. W. Werremeyer, 255–260; "The slide rule in business" by S. L. Shelley, 261–263; "A program of investigation and coöperative experimentation in the mathematics of the seventh, eighth and ninth school years" by R. Schorling and J. R. Clark, 264–275; "Testing as a means of improving the teaching of high school mathematics" by E. R. Breslich, 276–291; New Publications, 292–295—October: "The aims of mathematical education" by J. H. Minnick, 297–304; "Empirical results in the theory of numbers" by R. D. Carmichael, 305–310; "Teaching pupils how to study mathematics" (to be continued) by A. Davis, 311–320; "*La Disme* of Simon Stevin—The first book on decimals" by Vera Sanford, 321–33; "No homework for mathematics pupils" by H. C. Wright, 334–336; "The future of secondary instruction in geometry" by H. E. Webb, 337–341; "The slide rule as a subject of regular class instruction in mathematics" by W. E. Breckenridge, 342–343; News and Notes, 344–348; Book Reviews, 349–354.

NATURE, volume 108, October 27, 1921: "A system of space-time co-ordinates" by J. L. Synge, 275—November 3: "Psychological tests for vocational guidance," 321–323 [A report of a conference. "Mr. D. Kennedy Fraser (lecturer in education at the University of Edinburgh) spoke upon similar lines. He described from personal experience the use of intelligence tests in America. . . . He strongly urged the execution of similar research in this country"]; "University and educational intelligence," 323–324 ["The Rhodes Trust has issued a statement for the academic year 1920–21 dealing with the scholarships it administers. . . . 129 (Rhodes scholars were in residence) from the United States . . . mathematics had six Rhodes Scholars" (altogether)]—November 17: "Reflection 'Halo' of (semi-) cylindrical surfaces" by J. H. Shaxby, 369; "Problems of physics" by O. W. Richardson, 372–377 ["Abridged from the presidential address delivered to Section A (Mathematics and Physics) of the British Association at Edinburgh on September 9"]; "Notes," 380 ["The Copley Medal (was presented in 1921 by the Royal Society) to Sir Joseph Larmor, for his researches in mathematical physics"]—November 24: "The tendency of elongated bodies to set in the north and south direction" by E. H. Grove-Hills, 403 [A reply to Sir Arthur Shuster. "The whole matter is fully discussed in an article by Mr. W. D. Lambert of the United States Coast and Geodetic Survey in the *American Journal of Science*" (see above)]; "Societies and Academies: Royal Society, November 17," 421 ["The design of repeating patterns" by P. A. MacMahon and W. P. D. MacMahon (Quotation: "The study and classification of repeating patterns in space of two dimensions is founded upon the simplest geometrical forms which happen to be repeats. These are employed as bases and are subjected to specified transformations which depend upon certain contact systems. . . . Repeats are of three varieties, the block, the 'stencil,' and the 'archipelago.' "), "The mathematical foundations of theoretical statistics" by R. A. Fisher]; "Philosophical Society, October 31," 421 ["Convex solids in higher space" by W. Burnside; "The fifth book of Euclid's 'Elements' " by M. J. M. Hill.]

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES, volume 7, March, 1921 (received October 1): "Normalized geometric systems" by A. A. Bennett, 84–89; "Problems of potential theory" by G. C. Evans, 89–98—May: "An overlooked infinite system of groups of order pq^2" by G. A. Miller, 146–148—June: "A formula for the viscosity of liquids" by H. B. Phillips, 172–177—September: "The average of an analytic functional" by N. Wiener, 253–260; "Semi-covariants of a general system of linear homogeneous differential equations" by E. B. Stouffer, 273–276; "An algorism for differential invariant theory" by O. E. Glenn, 276–279—

October: "The average of an analytic functional and the Brownian movement" by N. Wiener, 294–298; "An integral equation and its applications" by E. Hille, 303–305.

Revue Générale des Sciences, volume 32, October 15, 1921: "Application des méthodes interférentielles aux mesures astronomiques," 531–532.

School Science and Mathematics, volume 21, December, 1921: "The value and method of the historical element in the teaching of secondary mathematics" by J. T. Vallandingham, 817–822; "Worth while work with algebra failures" by Helen I. Westley, 822–825; "Teaching proportions in geometry and algebra" by J. A. Nyberg, 868–874; "Note on prime numbers" by G. A. Miller, 874 [Giving a formula proved by Tschebychef, and then a modified form. "The main object of this note is to warn the readers thereof not to adopt the former formula, which appears in various reliable works, including Landau's *Primzahlen*, volume 1, 1909, page 22. The present writer employed this formula several years ago in Miller, Blichfeldt, Dickson *Finite Groups*, 1916, page 167, where the latter formula would have been more useful. Hence this warning is the more earnest especially since the latter is such a direct consequence of the former and the notion of prime number is so very elementary. The note may also serve to illustrate the fact that obvious improvements are sometimes overlooked by the best authors."]

Science, new series, volume 54, October 7, 1921: "A new graphic analytic method" by R. von Huhn, 334–336—October 28: "The American Mathematical Society" by R. G. D. Richardson, 416—November 11: "A notable mathematical gift" [the subsidy promised to the Mathematical Association of America by Mrs. Mary Hegeler Carus] by G. A. Miller, 456—November 25: "The eclipse expeditions to Christmas Island," 513 ["It is hoped to confirm the results obtained by the British expeditions at Principe and Sobral during the eclipse of May, 1919"]; Review by C. J. Keyser of B. Russell's *The Analysis of Mind* (New York, 1921), 518–520.

Science Progress, volume 16, October, 1921: "Recent advances in pure mathematics" by Dorothy M. Wrinch, 173–178; Review by G. B. J. of H. S. Carslaw's *Introduction to the Theory of Fourier's Series and Integrals* (London, 1921), 324.

Scientific Monthly, volume 14, January, 1922: "Certain unities in science" by R. D. Carmichael, 41–59; "Thomas Hariot—1560–1621" by F. V. Morley, 60–66.

Sigma Xi Quarterly, volume 9, September, 1921: "What should the Society of Sigma Xi preach?" by G. A. Miller, 64–66.

Sphinx-Œdipe, volume 16, September, 1921: "Notice sur Charles Ange Laisant" (suite) by H. Brocard, 129–133; "Conférences Math. à Bruxelles (1920–1921)," 133–134; "Tables de M. Kraïtchik," 135; "Questions et Réponses," 135–143; "Histoire de Sciences" [account of an early French periodical, *Le Géomètre*, see below], 143–144; "Divers: Télescope géant," 144—October: "Arithmetic of potentials" by T. J. Barniville, 145–147; "Sur les racines primitives relatives au mod. P^n dans un corps algébrique quelconque" by G. Métrod, 147–150; "Construction d'une polygraphie de cavalier avec Sphinx en partant d'un circuit complet et fermé" (suite et fin.) by "Un Abonné," 150–154; "Sur l'équation (1) $x^4 + ax^2y^2 + y^4 = z^2$, a positif" by M. Rignaux, 154–155; "Questions et Réponses," 156–160; "Divers: Astronomie," 160 [L'astronome américain Pickering vient de faire des observations, qui, d'après lui, prouveraient que la vie existe à la surface lunaire."]

["'Le Géomètre' était un Recueil de mathématiques, à l'usage des candidats aux Ecoles Royales Polytechnique, de St. Cyr, de la Marine des Eaux et Forêts, etc. publié par M. Guillard, ancien élève de l'Ec. Normale, Agrégé de l'Université, Prof. de Math. Elém. au Collège Royal de Louis le Grand, Paris, chez l'éditeur Rue St. Jacques N° 121 à partir du 15 Mars 1836. Ce volume de 224 pp. (13 x 21) et 9 pl., était composé chez Ch. Eberhart, imprimeur du Collège R. de France, 12 Rue du Foin St. Jacques, et l'on y trouve la liste de 245 souscripteurs, dont certains se sont fait un nom, depuis cette époque: Gerono, ex-principal du Collège de Lorient; Sturm, prof. au Collège Rollin; le baron Reynaud, examinateur; Miquel, élève de l'Institution Barbet; Catalan, régent de Math. au Collège de Châlons-sur-Marne; Le Febure de Fourcy, examinateur; Terquem, bibliothécaire du dépôt central d'artillerie; Chasles anc. élève de l'Éc. Polyt., à Chartres; . . . .

"Ce journal, dont nous avons ainsi les 14 premieres feuilles parues donnait des mémoires, des questions et des réponses. Je signalerai les solutions de Chauwin (conc. général math. elém. 1822), Vanéechout (conc. général de 1809, prem. et 2e classes de math. des lycées de Paris), Bruyère (conc. de 1817, Elém.), Giorgini (1812, Spéc.), Gerono (1818, Elém.), Latour et de Privezac (concours de 1814, Spéciales). . . .

"A signaler aussi: un théorème remarquable de Miquel (p. 166) et la Quest. 3 (p. 14, rép. p. 164–166): *Diviser un triangle scalène en quatre parties égales, par 2 lignes qui se coupent à angles droits.* Cette question a été reprise en 1894 dans l'*Interm. des Math.*"]

["TÉLESCOPE GÉANT: M. MACAFEE, bien connu aux U. S. A. pour ses travaux astron., en collab. avec le Prof. DAVID TODD, de Harvard s'occupe actuellement en France d'établir les plans du plus gigantesque télescope connu. On verra Mars aussi aisément que si l'on n'était qu'à 2 km. 500 de distance.—Pour obvier aux difficultés que présente la construction d'un colossal miroir réflecteur, M. MacAfee remplace celui-ci par une cuvette de métal de 50 pieds de diamètre, remplie de mercure; un mouvement rotatif imprimé a cette cuvette lui donne la concavité voulue, et lui fait jouer l'office de réflecteur. On announce la livraison pour 1924."]

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, 3012 West Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, Providence, R. I.

[*1918*, 33; *1919*, 167; *1920*, 28, 223.]

The officers for the session 1921–1922 are the following: Chairman, Professor R. G. D. Richardson; Committee on program, Professor Burgess, Professor Gilman, Frances Merriam, Gr., Elizabeth Stafford '23, Charles Hopkins '22, George Sauté '24; Committee on arrangements, Mr. C. D. Wentworth, instructor, Nellie Stokes '23, Frances Wright '23, Clarence Eddy '22, Phillip Welch '23.

In a printed program, the following papers are announced:

October 27, 1921: "The story of a mathematical book" by Professor D. E. Smith, of Columbia University.

December 9: "Lewis Carroll" by Dorothy Rickenbacker '23; "Number systems on other bases than ten" by Clarence Bennett '23; "Various proofs of the Theorem of Pythagoras" by George Sauté '24.

January 13, 1922: "Mathematics of the calendar" by Henry Bodwell '24; "The three point problem" by Frances Merriam, Gr.

February 17: "Skeleton division" by Clarence Eddy '22; "Cardan and the solution of the cubic" by Elizabeth Stafford '23; "The inscribed regular heptagon" by Frances Wright '23.

March 10: "Some special graphical methods" by Professor O. D. Kellogg, of Harvard University.

April 28: "Cryptography" by Donald MacPherson, Gr.; "Galois" by Katherine Colton '22; "Measuring the diameter of Betelgeuse" by Charles Hopkins '22.

May: Picnic.

MATHEMATICS CLUB OF COLUMBIA UNIVERSITY, New York.

[*1918*, 227; *1919*, 262; *1920*, 425.]

The following officers were elected for the year 1920–21: President, Albert Meder, Jr. '22; vice-president, Oscar Bodansky '22; secretary, William Thompson '23. The following papers were read:

October 1, 1920: "Some graphical methods" by Albert Meder, Jr. '22.
October 15: "Mathematical bits from Leacock" by Oscar Bodansky '22.
October 29: "Formulæ of investments" by Professor L. P. Siceloff.
November 12: "The probability equation" by A. Preisman '22.
November 26: "Constructibility" by O. Frink, Jr. '22.
December 10: "Analytic methods" by R. Kronig '22.
January 12, 1921: "Relations" by Professor C. J. Keyser.
February 18: "Volumes and areas with empirical boundaries" by W. Skeats '23.
March 4: "Convergent series" by Professor W. B. Fite.
March 18: "Measuring Betelgeuse" by M. Schwartzschild '22.
April 8: "Discontinuous functions" by Professor T. S. Fiske.

April 22: "The mathematics of wound healing" by William Thompson '23.
May 6: "Types of geometry" by Dr. K. Lamson.

The average attendance was about fifteen.

(Reported by Mr. Thompson.)

The Grinnell College Mathematics Club, Grinnell, Iowa. [*1918*, 449.]

November 2, 1920: "Archimedes, his life and work" by Louise Pinkerton '22. Election of officers.
November 16: "Fallacies" by Elizabeth Pace '22; "Simultaneous equations" by Burton Jones '23. Problems.
November 30: "Proofs of the Pythagorean theorem" by Raymond Weigen '22.
December 6: "The mathematical treatment of certain classes of scientific data" by Professor H. L. Rietz, of The State University of Iowa. This was a joint meeting with the Science Club.
January 11, 1921: "Chinese swan pan" by Ruth Raffety '23; "Mathematics of the calendar" by Margaret Divelbess '23. The constitution presented by Louise Pinkerton '22.
March 8: "First mathematical work printed in the New World" by Ruth Boyce '22; "Plato's general attitude toward mathematics" by Frances Nelson '22.
March 22: Social meeting.
April 5: "Projective geometry" by Professor O. W. Albert.
April 19: "Newton, his life and work" by Robert Hannelly '23; "The adding machine" by Leonard Paulu '22.
May 3: "Constructions on the circle" by Frances Harshberger '23; "College as a training school for mathematics teachers" by Dorothy Rollette '23.
May 10: "Euclid's parallel postulate" by Lillie Mann '23.
May 23: Picnic. Election of officers.

(Reported by Miss Harshberger.)

Undergraduate Mathematics Club, University of Illinois, Urbana, Ill. [*1918*, 404; *1919*, 309.]

The Club was organized under a new constitution March 24, 1920, which provided that members be undergraduates belonging to one of the following classes: (1) Students with an average of 2/3 A and 1/3 B in mathematics grades or the equivalent; (2) Juniors or seniors majoring in mathematics; (3) Students unanimously elected by a committee of the Club. There is, however, one faculty member, acting as adviser. There are no dues. The Club is for work or mathematical recreation only.

The regular meetings are from 7 to 8 p.m., weekly; and are open to the public,—the business meetings being separate. The first quarter of the hour is devoted to problems. An address, about one-half hour long, follows. Discussion concludes the program. Each member is expected to give one address during the year. The general theme for the year was "The applications of mathematics."

The average attendance is about twenty.

The following officers were elected for 1920–21: President, John Arnold '21; vice-president, Lorraine Conrad '21; secretaries, Lloyd Baker '22, Vincent Duvigneaux '22, Nelson Sowers '24.

The papers presented were:

October 6, 1920: "Applications of mathematics to chemistry" by Vincent Duvigneaux '22.
October 13: "Mathematical aspects of statistics" by Professor A. R. Crathorne.
October 20: "Applications of mathematics to insurance" by Lorraine Conrad '21.
October 27: "Applications of mathematics to physics" by Professor A. W. Williams.
November 3: "Slide rules" by Dr. H. M. Westergaard, associate.
November 10: "The influence of mathematical theories in the history of psychology" by Professor C. A. Ruckmick.
November 17: "Logarithm tables" by Professor Joel Stebbins.
November 24: "Mathematics in electricity" by Lloyd Baker '22.
December 1: "Mathematics in sound" by Dorothy Briggs '21.
December 8: "Einstein's restricted theory of relativity" by John Arnold '21.

December 15: "The physical basis of Einstein's theory" by Dr. W. H. Hyslop, instructor in Physics.
December 22: "Mathematics in electrical engineering" by A. R. Knight, of the Department of Electrical Engineering.
January 19, 1921: "Linkages" by Arthur Dixon '23.
February 22: "The importance of mathematics in genetic problems" by Professor J. A. Detlefsen, of the College of Agriculture.
March 1: "Mathematics in geology" by Professor T. T. Quirke.
March 8: "The quantitative determination of examinations" by Dr. C. R. Griffith, instructor.
March 15: "Codes and ciphers" by Major G. F. N. Dailey.
March 22: "Professor Michelson's interferometer" by John Worley '23.
April 5: "The determination of π" by Dorothy Briggs '21.
April 12: "The history of the calendar" by Nelson Sowers '24.
April 19: "Mathematical short cuts" by Arthur Dixon '23.
April 26: "Whole numbers" by Professor A. J. Kempner.
May 10: "Calculating machines" by Frank Hutchins '24.
May 17: "Mathematics in music" by John Arnold '21.

By constitutional revision May 10, 1921, the Club decided to separate into two divisions, a men's and a women's division, each self-governed, but closely federated under the old name. While both are concerned with pure mathematics, the men, in addition, will be interested more generally in engineering fields or other scientific work, while the women will be interested in teaching.

The officers of the Men's Mathematics Club for 1921–22 are: President, Vincent Duvigneaux '22; vice-president, Edgar Leach '22; secretaries, Nelson Sowers '24 and Frank Hutchins '24. The officers of the Women's Mathematics Club are: President, Doris Engle '22; vice-president, Lois Trogdon '22; secretary, Nellie Hunt '23; treasurer, Viola Judy '22.

(Reported by Mr. Sowers.)

Mathematics Club of the University of Montana, Missoula, Mont. [*1918*, 408.]

The officers for the year 1920–21 were: President, William Walterskirchen '21; vice-president, Mayme Carney '22; secretary, Cecil Phipps '21; treasurer, Catherine Hauck '23.

The following papers were read:

October 27, 1920: "A proof of the transcendence of e" by Professor E. A. F. Carey.
November 11: "The impossibility of the trisection of the angle" by Hugh Norville '21.
December 9: "$e^{i\pi}$" by Hilda Benson '22.
January 12, 1921: "Developments from the exponential series" by Professor A. S. Merrill.
January 26: "Methods of sailing" by Harry Rooney '22.
February 23: "The fundamental concepts of relativity" by Cecil Phipps '21.
March 9: "Probability in a game of dice" by Ruth McQuay '22.
March 23: "Experimental verification of relativity" by Cecil Phipps '21.
April 27: "DeMoivre's theorem" by Kay McKoin '22.
May 25: "The simple pendulum" by William Walterskirchen '21.

(Reported by Mr. Phipps.)

Pi Mu Epsilon, University of Pennsylvania, Philadelphia, Pa. [*1918*, 455.]

The chapter of Pi Mu Epsilon recently organized at the University of Pennsylvania is the successor to the "Vinculum." Members of the Mathematics Department, graduates, and undergraduates majoring in mathematics are eligible to membership. The organization has about thirty members, eight of whom are of the faculty. During the past year the majority of the meetings were of a purely business nature.

The officers for 1921–22 are: Director, Professor E. S. Crawley; vice-director, Professor J. H. Minnick, dean of the School of Education; secretary, Ella Rosentoor '22; treasurer, Esther Bernstein '23; librarian, Mabel Kessler '21.

The following were open meetings:

February 18, 1921: "Einstein's theory" by Professor C. Richards, of the Department of Mathematical Physics.
April 22: "Astronomy," an illustrated lecture by Professor S. G. Barton, of the Department of Astronomy.

(Reported by Miss Rosentoor.)

Pi Mu Epsilon, Syracuse University, Syracuse, N. Y.
[*1918*, 271.]

The officers for the year 1920–21 were: Director, Professor J. J. Nassau; vice-director, Professor Mary Harwood; secretary, Marion French '21; treasurer, Sterling Pugh '21; librarian, Leda Gast '21. Two of the faculty, one graduate and eight undergraduates were elected to membership. There was a Christmas party and a picnic in addition to the following meetings:
October 11, 1920: "Computation of the orbits of comets" by Professor L. Lindsay.
November 8: "History and ideals of Pi Mu Epsilon" by Professor E. D. Roe, Jr.
December 13: "Research work" by Professor W. H. Metzler; "A slide rule for the solution of quadratic equations" by Professor Nassau.
February 7, 1921: "Quaternions" by Professor Roe; "The geometry of point space of n dimensions" by Professor Nassau.
March 21: "Some theorems on determinants" by Jung Sun, Gr.
April 25: "Relativity" by Dean Graham, professor of Electrical Engineering. Open meeting.

(Reported by Professor Nassau.)

The Mathematical and Physical Society of the University of Toronto, Toronto, Ont.
[*1918*, 229; *1919*, 169.]

Officers for the session 1920–21 were elected as follows: Honorary president, Professor Lachlan Gilchrist; president, Aylmer Paisley '21; vice-president, James Phillips '22; secretary, Myra McLean '22; treasurer, William Webster '23; corresponding secretary, Eva Henry '21; representatives—fourth year, Lawrence Rentner '21; third year, Eric Horwood '22; second year, Max Bell '23; first year, Grizilda Bovard '24.

Meetings were held on alternate Thursdays at 4:15 p. m.
October 21, 1920: "How to build up atoms" by Professor J. C. McLennan.
November 4: "Physics, ancient and modern" by Percy Lowe, Gr.; "A college education for business" by Dallas Bates, Gr.
November 18: Social evening.
December 2: "Early life assurance in Britain" by Hudson Stowe '22; "Newton" by George Tuck '23.
December 16: "The notion of correspondence in mathematics" by Professor Samuel Beatty.
January 6, 1921: "My first impressions of the M. P. course" by Dorothy Gavin '22; "Laplace" by Harley Dewey '22.
January 20: Skating party.
February 3: "The life and significance of Galois" by Professor A. T. DeLury. Open meeting.
February 17: Debate between First and Second Years—"Resolved that individual initiative is more apt to bring success than is state control in the development of industrial enterprises."
March 3: "A magnetic survey" by Joseph Pearce, Gr.; "The history of mathematics" by Ewen Armstrong '21.
March 17: "Examination howlers" by Professor J. Satterly. Election of officers. The closing meeting.

(From a printed program, supplemented by Professor Beatty.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2951. Proposed by B. F. FINKEL, Drury College.

What is the average linear velocity of a point on the periphery of a locomotive driving wheel 2 feet in diameter and making 100 revolutions per minute?

2952. Proposed by R. M. MATHEWS, Wesleyan University.

Problem 2900 (*1921*, 277) suggests the following: Derive formulas for the products:

$$\cos\theta\cos\frac{1}{2}\theta\cos\frac{1}{2^2}\theta\cdots\cos\frac{1}{2^n}\theta,$$

$$\sin\theta\sin\frac{1}{2}\theta\sin\frac{1}{2^2}\theta\cdots\sin\frac{1}{2^n}\theta,$$

and

$$\tan\theta\tan\frac{1}{2}\theta\tan\frac{1}{2^2}\theta\cdots\tan\frac{1}{2^n}\theta.$$

2953. Proposed by C. F. GUMMER, Queen's University.

An algebraic equation being known to have exactly r real roots, all simple, is it possible to find an equation of degree $r-1$ whose roots separate those of the given equation, and whose coefficients are rational functions of the coefficients of the given equation?

2954. Proposed by C. N. MILLS, Heidelberg University.

A machine-gun is placed on an armored train which is moving with a velocity of v feet per second along a straight horizontal track. The muzzle velocity of the bullets is v feet per second. Find the greatest range, (1) in front of the train and (2) behind the train.

2955. Proposed by the late L. G. WELD.

Find the proportions of an anchor ring such that its section by a plane parallel to its axis and tangent to its inner circle (circle of the gorge) shall be a lemniscate.

2956. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

A particle is constrained to move on the outer surface of a sphere whose radius equals the mean radius of the earth and which is rotating with the same angular velocity, ω, as the earth. The particle is acted upon by a force equal to $m\omega^2\sin\phi\cos\phi$, which is directed to the nearer pole, m being the mass of the particle and ϕ the latitude. This force balances the equatorward tendency of a particle initially at rest relative to the surface; on the earth its place is supplied by the slight departure from sphericity. Show that for small velocities relative to the rotating surface the particle when acted upon by no forces except the one mentioned and by the constraint, will describe a curve differing slightly from a small circle on the sphere.

Discuss the form of the curve when the velocity relative to the surface is increased. Would it be possible to lay out a railroad track from New York to Chicago such that its departure from the great circle connecting those points would counterbalance the pressure on the rails due to the earth's rotation?

2957. Proposed by J. L. WALSH, Harvard University.

The envelope of the circles of curvature of a curve is, in part at least, the curve itself. What further curves, if any, are parts of this envelope?

2958. Proposed by R. P. BAKER, University of Iowa.

Over a frictionless pulley a weightless cord sustains at one end a mass M, while the other end is wound on the axle of a wheel of mass M and moment of inertia N. At the zero of time the wheel revolves with angular velocity ω and tends to wind up the cord. Describe the motion neglecting friction.

SOLUTIONS

2813 [1920, 81]. Proposed by PAUL CAPRON, U. S. Naval Academy.

An ellipse having the major-axis $2a$ and the eccentricity ϵ, is revolved first about its major axis, forming a prolate spheroid, then about its minor axis forming an oblate spheroid. Show that the surfaces of these spheroids are, respectively,

$$2\pi a^2(1/\epsilon)(\sqrt{1-\epsilon^2}\sin^{-1}\epsilon + 1)$$

and

$$2\pi a^2\left[2 + (1/\epsilon)(1-\epsilon^2)\log\left(\frac{1+\epsilon}{1-\epsilon}\right)\right].$$

SOLUTION BY H. S. UHLER, Yale University.

Case I. Prolate spheroid.

Let the equation of the ellipse be

$$b^2x^2 + a^2y^2 - a^2b^2 = 0. \tag{1}$$

We may take as element of surface the lateral area of the frustum of a right circular cone having the following specifications: planes of bases normal to the x-axis and at distances x and $x + dx$ from the origin, and generatrix of lateral surface tangent to the revolving ellipse at the point: (x, y). Slant height $\equiv dt$.

Then

$$dt = dx\sqrt{1 + (dy/dx)^2}.$$

From (1)

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y};$$

hence

$$dt = \frac{bdx}{a^2y}\sqrt{a^4 - (a^2-b^2)x^2}.$$

Accordingly, the element of surface, $d\sigma$, is given by

$$d\sigma = 2\pi y dt = \frac{2\pi b dx}{a^2}\sqrt{a^4 - (a^2-b^2)x^2},$$

and the required area of the prolate spheroid,

$$A_1 = \frac{4\pi b}{a^2}\int_0^a dx\sqrt{a^4 - (a^2-b^2)x^2}. \tag{2}$$

By making use of the standard formula,

$$\int d\varphi\sqrt{n^2-\varphi^2} = \frac{1}{2}\varphi\sqrt{n^2-\varphi^2} + \frac{1}{2}n^2\sin^{-1}\frac{\varphi}{n},$$

it is merely a matter of simple reductions to change equation (2) to the following form,

$$A_1 = 2\pi b\left[b + \frac{a^2}{\sqrt{a^2-b^2}}\sin^{-1}\left(\frac{\sqrt{a^2-b^2}}{a}\right)\right]. \tag{3}$$

Now, by the definition of e, $b^2 = a^2(1-e^2)$; so that, by elimination of b, equation (3) may be expressed as the following function of a and e:

$$A_1 = 2\pi a^2(e^{-1}\sqrt{1-e^2}\sin^{-1}e + 1 - e^2). \tag{4}$$

Case II. Oblate spheroid.

Proceeding as above, it will be found that

$$A_2 = \frac{4\pi a}{b^2}\int_0^b dy\sqrt{(a^2-b^2)y^2 + b^4}.$$

The following well-known relation

$$\int d\varphi\sqrt{\varphi^2+n^2} = \tfrac{1}{2}\varphi\sqrt{\varphi^2+n^2} + \tfrac{1}{2}n^2\log\left(\varphi+\sqrt{\varphi^2+n^2}\right)$$

facilitates the reduction of the preceding integral to

$$A_2 = 2\pi a\left[a + \frac{b^2}{\sqrt{a^2-b^2}}\log\left(\frac{a+\sqrt{a^2-b^2}}{b}\right)\right],$$

or

$$A_2 = 2\pi a^2\left[1 + \frac{1}{2}e^{-1}(1-e^2)\log\left(\frac{1+e}{1-e}\right)\right]. \tag{5}$$

Remarks. Both of the proposed expressions are incorrect. When $e = 1$ $(b = 0)$ the given formula for the prolate spheroid leads to $2\pi a^2$ instead of zero. Again, when $e = 0$ $(b = a)$ the series

$$\frac{1}{2}e^{-1}\log\left(\frac{1+e}{1-e}\right) = 1 + \frac{1}{3}e^2 + \frac{1}{5}e^4 + \cdots$$

shows that the given expression for the oblate spheroid reduces to $8\pi a^2$ instead of $4\pi a^2$ (sphere). Formulas (4) and (5) are equivalent to those given in Williamson's *Integral Calculus*, page 258.

Note. Contributors named below observed that the results as corrected are given in Lamb's *Infinitesimal Calculus*, 1902, page 273 and in Czuber's *Integralrechnung*, page 284. It is also given in *The Encyclopaedia Britannica*, ninth edition, volume 13, page 55.—EDITORS.

Also solved by NORMAN ANNING, J. A. BULLARD, NATHAN DEUTSCHMAN, A. FAYDER, L. G. FEMAN, MAURICE KRAUT, BENJAMIN LEVINE, I. H. MARANTZ, MOSES NISSENBAUM, H. L. OLSON, ARTHUR PELLETIER, H. W. REDDICK, H. A. RHODES, J. L. RILEY, D. H. RICHERT, J. B. REYNOLDS, and T. R. THOMSON.

2815 [1920, 134]. Proposed by the late L. G. WELD.

A right circular cone is laid upon an inclined plane so that its element of contact makes a given angle with the slant line of the plane. Assuming that there is no slipping and that the rolling friction is negligible, find the time of oscillation of the cone.

I. SOLUTION BY J. B. REYNOLDS, Lehigh University.

The center of gravity of the cone moves in a circle of radius $\frac{3}{4}h\cos\alpha$ at a perpendicular distance $\frac{3}{4}h\sin\alpha$ from the plane, h being the height of the cone and 2α its vertical angle. When the radius of this circle makes an angle θ with the line of slope of the plane, the center of gravity of the cone has been raised a height $\frac{3}{4}h\cos\alpha(1-\cos\theta)\sin\beta$ above its lowest position, β being the angle of slope of the plane. Then the potential energy, *P.E.*, is $\frac{3}{4}mgh\cos\alpha\sin\beta(1-\cos\theta)$, m being the mass of the cone.

Since the line of contact is the instantaneous axis of rotation, the kinetic energy, *K.E.*, of the cone is $\frac{1}{2}I\dot\varphi^2$, I being the moment of inertia with respect to the instantaneous axis or an element of the cone and $\dot\varphi$ the angular velocity about this axis.

Now the moment of inertia of a thin disc of radius a and thickness t with respect to a line through its center making an angle α with a perpendicular to its plane is

$$t\int_0^{2\pi}\int_0^a r^2(\cos^2\theta\cos^2\alpha+\sin^2\theta)r\,dr\,d\theta = \frac{\pi a^4}{4}(1+\cos^2\alpha)t$$

and for the moment of inertia of the disc about a parallel line at a distance $a\cos\alpha$, we have

$$\frac{\pi a^4}{4}(1+\cos^2\alpha)t + \pi a^2t(a\cos\alpha)^2 = \frac{\pi a^4}{4}(1+5\cos^2\alpha)t.$$

So that, since the radius of any circular differential disc at distance x from the vertex of the cone

is $x \tan \alpha$, we have

$$I = \int_0^h \frac{\pi}{4}(1 + 5\cos^2\alpha)\tan^4\alpha\, x^4 dx = \frac{\pi}{20} h^5(1 + 5\cos^2\alpha)\tan^4\alpha$$

$$= \tfrac{3}{20} m h^2 \tan^2\alpha(1 + 5\cos^2\alpha),$$

since

$$m = \tfrac{1}{3}\pi h^3 \tan^2\alpha.$$

By the principle of the conservation of energy

$$K.E. + P.E. = \text{constant}$$

or

$$\tfrac{1}{2}\tfrac{3}{20} m h^2 \tan^2\alpha(1 + 5\cos^2\alpha)\dot{\varphi}^2 + \tfrac{3}{4} mgh \cos\alpha \sin\beta(1 - \cos\theta) = \text{constant};$$

or since $\dot{\varphi} = \dot{\theta} \cot \alpha$, we get

$$\tfrac{1}{10} h^2(1 + 5\cos^2\alpha)\dot{\theta}^2 + gh \cos\alpha \sin\beta(1 - \cos\theta) = \text{constant}.$$

Differentiating this equation with respect to t and cancelling out $\dot{\theta}$ we get,

$$\ddot{\theta} = -\frac{5g \cos\alpha \sin\beta}{h(1 + 5\cos^2\alpha)} \sin\theta,$$

from which we have, if θ is small, for the time T for an oscillation

$$T = 2\pi \sqrt{\frac{h(1 + 5\cos^2\alpha)}{5g \cos\alpha \sin\beta}}.$$

If the oscillations are not small and γ is the given angle that the element of contact initially makes with the line of greatest slope

$$T = 2\pi \sqrt{\frac{g(1 + 5\cos^2\alpha)}{5g \cos\alpha \sin\beta}} \left\{ 1 + \frac{1}{4}\sin^2\frac{\gamma}{2} + \frac{9}{64}\sin^4\frac{\gamma}{2} + \cdots + \left[\frac{n\lfloor 2n - 1}{2^{2n-1}(\lfloor n)^2}\right]^2 \sin^{2n}\frac{\gamma}{2} + \cdots \right\}.$$

II. Solution by William Hoover, Columbus, Ohio.

Assume the vertex of the cone as fixed in the inclined plane; then we have the motion of a solid of revolution with one point fixed; and the motion is represented by Euler's equations, employing the usual notation, in the form

$$\begin{aligned} A\dot{\omega}_1 + (C - A)\omega_2\omega_3 &= L, \\ A\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= M, \\ C\dot{\omega}_3 &= N, \end{aligned} \tag{1}$$

the two equal moments of inertia about axes through the vertex of the cone and perpendicular to the axis of the cone and to each other being given by A, and the moment of inertia about the axis of the cone by C.

Also take the line of contact of the cone and plane as an instantaneous axis, and let its direction angles be λ, μ, ν, and the vertical angle of the cone $= 2\alpha$, $\varphi =$ the angle through which the cone rotates after any time t from the beginning of motion; so that, the x-axis being initially in the inclined plane, φ also gives the rotary motion of the x-axis; and $\omega =$ the instantaneous angular velocity about the instantaneous axis.

Let $\rho =$ the distance of any point in the instantaneous axis from the origin; then as in solid analytic geometry,

$$\cos\lambda = \frac{x}{\rho} = \sin\alpha \sin\varphi, \qquad \cos\mu = \frac{y}{\rho} = \sin\alpha\cos\varphi, \qquad \cos\nu = \cos\alpha; \tag{2}$$

and, by the theory of rotary motion,

$$\omega_1 = \omega \sin\alpha \sin\varphi, \qquad \omega_2 = \omega\sin\alpha\cos\varphi, \qquad \omega_3 = \omega\cos\alpha. \tag{3}$$

We have

$$\omega = \dot{\varphi}. \tag{4}$$

If $m =$ the mass of the cone, and $h =$ its altitude,

$$A = \tfrac{3}{20} m h^2(4 + \tan^2\alpha), \qquad C = \tfrac{3}{10} m h^2 \tan^2\alpha. \tag{5}$$

Let $\dot{\theta}$ = the angular motion of the instantaneous axis on the plane; this is the angular motion about the line (n) which is normal to the inclined plane and passes through the vertex of the cone. A plane perpendicular to the inclined plane includes this normal, the instantaneous axis (a_i), the axis of the cone, and a perpendicular (p) to the axis of the cone. The angle between n and p is α and between a_i and p is $\pi/2 + \alpha$. The components of $\dot{\theta}$ and ω about p are $\dot{\theta}\cos\alpha$ and $\omega\cos(\pi/2 + \alpha) = -\omega\sin\alpha$, but these denote the same motion; then

$$\dot{\theta}\cos\alpha = -\omega\sin\alpha \quad \text{or} \quad \omega = -\dot{\theta}\cot\alpha. \tag{6}$$

Now the moment of the external forces about the instantaneous axis

$$= L\cos\lambda + M\cos\mu + N\cos\nu; \tag{7}$$

also

$$= \tfrac{3}{4}mgh\sin\alpha\sin\beta\sin\theta \tag{8}$$

β being the inclination of the plane.

Substituting (4) in (3) and differentiating, we have

$$\dot{\omega}_1 = \sin\alpha(\cos\varphi\cdot\dot{\varphi}^2 + \sin\varphi\cdot\ddot{\varphi}), \quad \dot{\omega}_2 = \sin\alpha(-\sin\varphi\cdot\dot{\varphi}^2 + \cos\varphi\cdot\ddot{\varphi}), \quad \text{and} \quad \dot{\omega}_3 = \cos\alpha\cdot\ddot{\varphi}, \tag{9}$$

Now put the values given by (3), (5), (9) in (1), and we have the values of L, M, N; and using these, with (2), (7) and (8), we have the single equation of motion

$$\frac{3m}{20}(6 + \tan^2\alpha)h^2\sin^2\alpha\cdot\ddot{\varphi} = \frac{3m}{4}gh\sin\alpha\sin\beta\sin\theta. \tag{10}$$

Now from (4) and (6),

$$\ddot{\theta} = -\frac{g}{h}\frac{5\sin\beta\cos\alpha}{1 + 5\cos^2\alpha}\sin\theta. \tag{11}$$

If θ be so small that θ may displace $\sin\theta$, we have the required time of oscillation

$$T = 2\pi\sqrt{\frac{1 + 5\cos^2\alpha}{5\sin\beta\cos\alpha}\cdot\frac{h}{g}}.$$

Also solved by F. L. WILMER, and discussed by C. H. ECKART.

2820 [1920, 134]. Proposed by C. B. HALDEMAN, Ross, Ohio.

Given one angle and the radii of the inscribed and circumscribed circles, to construct the triangle geometrically.

SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec.

With the usual notation, A, R, r are given. Draw a circle of radius R, and let KB be any diameter. (The reader is requested to draw the figure). Make the angle, $BKC = A$. Draw XY parallel to BC and at a distance r from it on the same side as K. Bisect the arc BC at V, and let the circle, center V, radius VC, cut XY at I and I'.

These points will be the incenters for the two (symmetrical) solutions. Let VI meet the circle again in A. Then ABC is the required triangle. To show this, it is only necessary to prove that the straight line CI bisects the angle C. Since $VI = VC$, $\angle VIC = \angle VCI$. But $\angle VIC = (A/2) + \angle ICA$ and $\angle VCI = (A/2) + \angle BCI$. Hence $\angle ICA = \angle BCI$ and I is the intersection of two bisectors of the angles of the triangle ABC.

Also solved by L. C. MATHEWSON, H. L. OLSON, ARTHUR PELLETIER, J. B. REYNOLDS, JOSEPH ROSENBAUM, C. N. SCHMALL, and the Proposer.

2829 [1920, 226]. Proposed by E. S. PALMER, New Haven, Conn.

Given a set of arbitrary pairs of positive integers (a_p, b_p), $(p = 1, 2, \cdots, n)$: (a) Is it always possible to find a set of positive integers k_p, $(p = 1, 2, \cdots, n)$ such that

$$k_pa_p + k_pb_p > \sum_{r=1}^{r=n} k_ra_r, \qquad (p = 1, 2, 3, \cdots, n).$$

(b) If or when possible, show how to find k_p.

Solution by Albert A. Bennett, University of Texas.

The system is homogeneous, so that if one solution exists, an infinite number exist. For convenience write c_p for $a_p + b_p$ and write in place of the inequalities the following equalities:

$$k_p c_p = \sum_{r=1}^{n} k_r a_r + e_p, \qquad e_p > 0, \qquad (p = 1, 2, 3, \cdots, n).$$

Solving for k_p, one has, provided that the denominators do not vanish,

$$k_p = \frac{e_p}{c_p} + \frac{1}{c_p} \frac{\sum_{r=1}^{n} \frac{e_r a_r}{c_r}}{1 - \sum_{r=1}^{n} \frac{a_r}{c_r}},$$

as may be verified at once by substitution.

Three cases occur.

1. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} > 0$. In this case, solutions are obtained readily by inserting arbitrary e's restricted only so that the k's shall be integers.

2. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} = 0$. There are no solutions in this case.

3. $1 - \sum_{r=1}^{n} \frac{a_r}{c_r} < 0$. In this case, also, there are no solutions. For if there were solutions, these would be in the form given above, and the expression, $\sum_{p=1}^{n} k_p a_p$, obtained by multiplying each solution, k_p by a_p and adding together would be

$$\sum_{p=1}^{n} k_p a_p = \sum_{p=1}^{n} \frac{e_p a_p}{c_p} + \sum_{p=1}^{n} \frac{a_p}{c_p} \frac{\sum_{r=1}^{n} \frac{e_r a_r}{c_r}}{1 - \sum_{r=1}^{n} \frac{a_r}{c_r}},$$

$$= \sum_{r=1}^{n} \frac{e_r a_r}{c_r} \frac{1}{1 - \sum_{r=1}^{n} \frac{a_r}{c_r}}.$$

Since the right-hand member would be negative under the hypothesis of this case, there could be no set of positive members, k_p, $p = 1, 2, \cdots, n$, of the form required for a solution. (The writer is indebted to Prof. H. P. Manning for the treatment here given of this third case.)

2833 [1920, 227]. Proposed by W. H. ECHOLS, University of Virginia.

In *Engineering*, London, September 28, 1917, appeared the following equations concerning the stability of ships; they are employed by the naval constructors in the Norfolk (Virginia) Navy Yard, and they are of importance:

$$\tan^3 \theta_0 + \frac{2m}{\rho} \tan \theta_0 - \frac{2x_0}{\rho} = 0, \qquad \tan^3 (\theta_0 + \theta_1) + \frac{2m}{\rho} \tan (\theta_0 + \theta_1) - \frac{2x_0}{\rho} - \frac{2x}{\rho} = 0,$$

$$\tan^3 (\theta_0 - \theta_2) + \frac{2m}{\rho} \tan (\theta_0 - \theta_2) - \frac{2x_0}{\rho} + \frac{2x}{\rho} = 0.$$

The required unknowns are θ_0, x_0 and m. The constants have values as follows: θ_1 and θ_2 are positive angles ranging from 15′ to 10°, x is positive and less than 5, and ρ is positive with a considerable range of values. A rapid solution involving small labor is desired, determining x_0 within the same limits as given for x.

Solution by the Proposer.

It may be of interest to note that in the practical application of the problem, ρ is the metacentric radius when the vessel is upright, x is the shift of the center of gravity of the ship when a weight w is placed on the side at a distance d from the center line and is equal to wd/W, where W is the weight of displaced water. In a typical ship of 4000 tons, w equal to 2.5 tons, $\rho = 16$ ft.,

$x = 0.025$ ft., the constant $2x/\rho$ is 0.003125. The required m is the metacentric height, or the distance from the center of gravity of the ship to the metacenter when the vessel is upright. The "angle of heel," θ, is the angle through which the ship turns when the weight w is shifted transversely, and is small in the case of testing an ordinarily well designed ship. Under these circumstances the following solution appears to be valid.

Let

$$X = 2m/\rho, \qquad Y = 2x_0/\rho, \qquad a = 2x/\rho.$$

Then eliminating X and Y there results

$$\begin{vmatrix} \tan^3\theta_0, & \tan\theta_0, & 1 \\ \tan^3(\theta_0+\theta_1) - a, & \tan(\theta_0+\theta_1), & 1 \\ \tan^3(\theta_0-\theta_2) + a, & \tan(\theta_0-\theta_2), & 1 \end{vmatrix} = 0. \tag{1}$$

On expanding the determinant, (1) becomes

$$[\tan(\theta_0+\theta_1) - \tan\theta_0][\tan(\theta_0-\theta_2) - \tan\theta_0][\tan(\theta_0+\theta_1) - \tan(\theta_0-\theta_2)][\tan(\theta_0+\theta_1) + \tan(\theta_0-\theta_2) + \tan\theta_0] - a[\tan(\theta_0+\theta_1) + \tan(\theta_0-\theta_2) - 2\tan\theta_0] = 0. \tag{2}$$

In (2) put $z = \tan\theta_0$, $p = \tan\theta_1$, $q = \tan\theta_2$. On simplifying, the equation becomes

$$pq(p+q)\frac{(1+z^2)^2}{(1-pz)^2(1+qz)^2}\{pqz^3 + 2(p-q)z^2 - (3+2pq)z - (p-q)\} - a\{2pqz + (p-q)\} = 0, \tag{3}$$

a factor

$$\frac{1+z^2}{(1-pz)(1+qz)} \equiv K \tag{4}$$

being omitted since it does not involve the solution. Interchanging p and q in (3) merely changes the sign of z, therefore we may assume $p \geqq q$.

The angle of heel, θ_0, of the unweighted ship, due to faulty construction by which the material is not so distributed as to keep the center of gravity in a vertical plane of symmetry, is in general small. The expression represented by K in (4) is nearly unity under the conditions.

The roots of equation (3) are the abscissas of the points of intersection of the straight line

$$y = 2apqz + a(p-q), \tag{5}$$

and the curve

$$y = pq(p+q)K^2\{pqz^3 + 2(p-q)z^2 - (3+2pq)z - (p-q)\}. \tag{6}$$

The curve (6) cuts the z-axis in the same points as does the cubic

$$y = pq(p+q)\{pqz^3 + 2(p-q)z^2 - (3+2pq)z - (p-q)\}, \tag{7}$$

and it has vertical asymptotes $z = 1/p$ and $z = -1/q$. The ordinates of (7) at $z = -1/q$ and $z = 1/p$ are respectively

$$+pq(p+q)\left(\frac{p}{q^2} + \frac{1}{q} + p + q\right) \quad \text{and} \quad -pq(p+q)\left(\frac{q}{p^2} + \frac{1}{p} + p + q\right).$$

The cubic has three real roots γ, α, β in the respective intervals

$$-\infty, \qquad -1/q, \qquad +1/p, \qquad +\infty,$$

and cuts the y-axis at $y = -pq(p^2 - q^2)$. It has a concavo-convex inflection at

$$z = -\frac{2}{3}\frac{p-q}{pq},$$

the ordinate there being positive. The middle root α lies between the origin and the z-intercept of the tangent at $z = 0$, which is

$$z = -\frac{p-q}{3+2pq}.$$

The straight line (5) cuts the z-axis midway between the asymptotes at a point

$$A = -(p-q)/2pq.$$

The slope of (5) is positive and this line (as we shall show) cuts (6) in only one point between the asymptotes. The abscissa of this point is the root we seek.

Represent by Q the cubic cofactor of K^2 in (6). Take the logarithm of both sides of that equation and differentiate. Then

$$\frac{1}{2y}\frac{dy}{dz} = \frac{2z}{1+z^2} + \frac{(p-q)+2pqz}{(1-pz)(1+qz)} + \frac{1}{2Q}\frac{dQ}{dz}. \tag{8}$$

The ordinate y is positive between A and α, so also is Q. The first term on the right is negative when z is negative. The derivative DQ is negative from $z = 0$ to its negative root which is easily seen to be less than $-1/q$. The inflection of $y = Q$ occurs at $(4/3)A$.

The numerator obtained by adding the first two terms on the right is

$$(p-q) + 2(1+pq)z - (p-q)z^2.$$

This is clearly negative for $z = -(p-q)/[2(1+pq)]$ and all smaller values, and therefore from A to $-\frac{1}{2}(p-q)$, inclusive. Hence the derivative Dy is negative throughout this interval.

Consider the interval from $-\frac{1}{2}(p-q)$ to α. The derivative DQ decreases in absolute value as x varies from $-\frac{1}{2}(p-q)$ to 0. The greatest value of Q in the interval from $-\frac{1}{2}(p-q)$ to α is at $-\frac{1}{2}(p-q)$. The numerical value of the last term in (8) is greater than that of DQ taken at $z = 0$ divided by twice that of Q at $-\frac{1}{2}(p-q)$, or

$$\frac{3+2pq}{(p-q)\{1+2pq+(1-\frac{1}{4}pq)(p-q)^2\}},$$

which is greater than

$$\frac{3}{p(1+2p^2)}.$$

The second term on the right in (8) is 0 at A and increases (the numerator increasing and the denominator decreasing) through positive values from $z = A$ to $z = 0$ when it is greatest and equal to $p - q$. Therefore this term is less than p throughout the interval from $-\frac{1}{2}(p-q)$ to α. The sum of the last two terms on the right in (8) is certainly negative for values of p which make $p^2(1+2p^2) < 3$, which is true if $p < 1$, or $\theta_1 < 45°$.

These results show that y in (6) is a one-valued continuously decreasing function of z throughout the interval A to α and therefore there is only one root of (3) in this interval.

The conditions of the problem are such that the required root of (3) is small; a first and practically sufficient approximation

$$z_1 = -\frac{p-q}{pq}\,\frac{a+pq(p+q)}{2a+(p+q)(3+2pq)}, \tag{9}$$

is obtained by making K provisionally equal to 1 and neglecting terms in Q containing powers of z above the first. A new approximation can be obtained by putting z_1 for z in K and in z^3 and z^2 in Q, and solving again for z.

It is of interest to note that the original equations can be written

$$Y = \alpha X + \alpha^3, \qquad Y + a = \beta X + \beta^3, \qquad Y - a = \gamma X + \gamma^3,$$

regarding α, β, γ as variable parameters. These represent three straight lines which envelope, respectively, the semi-cubic parabolas

$$27Y^2 + 4X^3 = 0, \qquad 27(Y+a)^2 + 4X^3 = 0, \qquad 27(Y-a)^2 + 4X^3 = 0.$$

These straight lines make given angles with each other. The second makes $+\theta_1$ with the first and the third makes $-\theta_2$ with the first.

A graphical solution of the equations consists in fitting a figure composed of three straight lines, meeting in a point and making given angles with each other, so that each is tangent to the corresponding semi-cubic parabola. The intersection of these three tangents has the required coördinates X, Y. The slope of the tangent to the first parabola is the required $\tan\theta_0$.

An actual navy yard experiment gave the data

$$\theta_1 = 4°\,49'\,14'', \qquad \rho = 2.63, \qquad \theta_2 = 4°\,39'\,36'', \qquad x = 0.0568.$$

Application of the computations indicated above gives $p = 0.0843$, $q = 0.0815$, $a = 0.0432$, $\tan\theta_0 = -0.03124$, $\theta_0 = -(1°\,47'\,21'')$. $X = 0.511$, $Y = -0.016$, $m = 1.344$, $x_0 = -0.021$.

2835 [1920, 273]. Proposed by J. L. RILEY, Stephenville, Texas.

If x, y, z, u are finite, and not all zero, and satisfy the equations

$$x = by + cz + du, \qquad y = ax + cz + du, \qquad z = ax + by + du, \qquad u = ax + by + cz,$$

and if none of the quantities a, b, c, d have the value -1, then will

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1.$$

Solution by H. L. Olson, University of Michigan.

The necessary and sufficient condition that these equations shall have a set of solutions not all zero is that the determinant of the coefficients,

$$\begin{vmatrix} -1, & b, & c, & d \\ a, & -1, & c, & d \\ a, & b, & -1, & d \\ a, & b, & c, & -1 \end{vmatrix} = 0.$$

If we expand this determinant and divide by $(a+1)(b+1)(c+1)(d+1)$, which is not zero by hypothesis, we find that the condition can be written

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1.$$

Also solved by P. J. da Cunha, Arthur Pelletier, A. V. Richardson, C. H. Richardson, H. S. Uhler, and C. C. Wylie.

2857 [1920, 377]. Proposed by the late L. G. WELD.

A savings bank offers to pay 3% interest on deposits, the said interest to be continuously compounded, *i.e.*, compounded at infinitesimal intervals of time. What would be the amount of $1.00 for one year?

Solution by E. J. Oglesby, Washington Square College, New York University.

The amount of one dollar at the percentage, denoted by the fraction r, compounded n times per year is $(1 + r/n)^n = [(1 + r/n)^{n/r}]^r$. Hence

$$A = \lim_{n \doteq \infty} \left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^r = e^r = e^{.03} = 1.03045.$$

Also solved by H. C. Bradley, H. N. Carleton, Elmer Latshaw, Arthur Pelletier, W. T. Stone, and A. L. Wechsler.

2858 [1920, 428]. Proposed by C. P. SOUSLEY, Pennsylvania State College.

A boy can split wood as fast as his father can saw, and the father can split twice as fast as the son can saw. How should the money received for their labor be divided?

Solution[1] by C. A. Barnhart, University of New Mexico.

Let x = the number of cords of wood that the boy saws in 1 hour, and kx = the number of cords of wood that the boy splits in 1 hour. Then, $2x$ = the number of cords of wood that the father splits in 1 hour, and kx = the number of cords of wood that the father saws in 1 hour. Then, $\dfrac{1}{x} + \dfrac{1}{kx} = \dfrac{k+1}{kx}$ = the number of hours in which the boy will saw and split 1 cord of wood, and $\dfrac{1}{2x} + \dfrac{1}{kx} = \dfrac{k+2}{2kx}$ = the number of hours in which the father will saw and split 1 cord of wood.

[1] The question is rather vague, but probably this solution makes the most natural interpretation of it: to use the time it takes each to split and saw a cord of wood as a basis for comparison of their work.—Editors.

Assuming that for each dollar received by the son, the father will receive z dollars, and noting that the money received will vary inversely as the number of hours required to saw and split 1 cord of wood, we have

$$z = \frac{k+1}{kx} \bigg/ \frac{k+2}{2kx} = 1 + \frac{k}{k+2}.$$

From which it is evident that the money received by the father depends directly upon k. Since $\lim_{k \doteq 0} z = 1$ and $\lim_{k \doteq \infty} z = 2$, then for $k > 0$, we have $1 < z < 2$.

Also solved by C. E. Bardsley, H. C. Bradley, H. N. Carleton, Michael Goldberg, and Daniel Kreth.

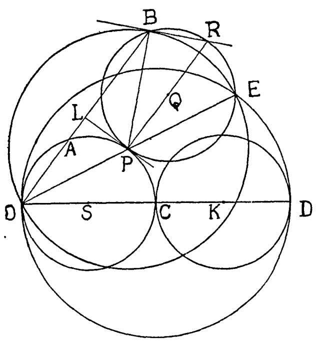
2861 [1920, 428]. Proposed by B. F. Finkel, Drury College.

Obtain by plane geometry, *i.e.*, without the use of calculus, a construction for finding points on the envelope of a system of circles whose diameters are chords of a fixed circle passing through a given point on it. Also determine geometrically the nature of the locus.

I. Solution by Augustus Bogard, College of St. Theresa, Winona, Minn.

The problem of finding the envelope of a system of curves is that of finding a curve that is at every point tangent to some curve of the system. That is, at any point on the envelope the envelope and some curve of the system have a common tangent, and they lie on the same side of it.

Let O be the fixed point, C the center of the given circle, and OCD the diameter from O, and let OE be any other chord from O, making with OD the angle θ. OE is then a diameter and P, its mid-point, is the center of one of the circles of the given system. At P, on the further side of OE, construct an angle $EPB = 2\theta$, the side of the angle cutting the circle OE at B. Then B is a point of the envelope required.



Proof. Draw the circles having OC and CD as diameters, and the circle PEB. Let Q be the center and PQR the diameter from P of PEB. PQR bisects the angle EPB and this circle is tangent at P to the circle OC. Further, chord PB = chord PO and so diameter PR = diameter OC = diameter CD. Hence the circle whose center is at Q may be thought of as rolling without slipping on the circle OC, and P is the center of the curvature at B of the curve described by this point. It follows that the line BR is tangent to the curve at B. It is also tangent at B to the circle OE, the circle of the given system, passing through B. Hence B is a point on the envelope. This analysis shows that the curve is described in the same manner as the cardioid, and is therefore a cardioid. Its polar equation is easily derived as follows: Considering O as the pole, OD as the initial line, the angle $DOB = \phi$ as the vectorial angle of B, and OB its radius vector, we have $OB = OE \cos\theta = OD \cos^2\theta = a(\cos 2\theta + 1) = a(1 + \cos\phi)$, where $a = OC$. That is, $\rho = a(1 + \cos\phi)$, which is the equation of a cardioid.

II. Solution by Otto Dunkel, Washington University.

Using the notation in A. Bogard's solution, the middle point P of the chord OE of the fixed circle, of radius OC, lies on the circle OC. It may be shown easily that the tangent at P to the circle OC is parallel to the tangent to the given circle at E. Hence the construction given for the point B in the Monthly (*1921*, 182) is equivalent to the following: Draw the tangent at E to the given fixed circle, then the point B is the foot of the perpendicular from O to the tangent. Let A be the point in which OB cuts the circle OC, then $AB = CE = a$. Hence the point B is found by the usual method for constructing a cardioid, *i.e.*, by drawing chords OA to the circle OC and prolonging each a length $AB = OC = a$.

III. Solution, Historical Notes, and Remarks by R. C. Archibald, Brown University.

In an article by T. de St-Laurent, *Annales de Mathématiques Pures et Appliquées*, July, 1826, vol. 17, page 15, it is found that the cardioid is the catacaustic of a reflecting circle with respect to a luminous point on its circumference. In a footnote to this result Gergonne remarked that the catacaustic is the envelope of the space traversed by a moving circle of variable radius, constantly having its centre on the circumference of a circle concentric with the reflecting circle [and of one third the radius], and passing through a fixed point of this circumference.

But the result is only a very special case of a well-known theorem by Quetelet:[1] The envelope of the circles passing through a fixed point, and whose centers lie on a given curve, C, is the pedal with respect to the fixed point of a curve similar to C but of double its linear dimensions.

Colin Maclaurin, in a memoir[2] written in 1718, at twenty years of age, was not only the first to show that the cardioid is the first positive pedal of a circle, with respect to a point on its circumference, but also the first to find the equations of the positive and negative pedals of the cardioid with respect to its cusp.

That the envelope of the circles in B. F. Finkel's problem is a cardioid was shown by B. Price who gave an analytic proof[3] in 1852. A geometrical proof of this result is readily derived by inversion[4] of the theorem: If the vertex of a right angle move along a straight line and one side pass through a fixed point O, the envelope of the other side will be a parabola with focus at O.

Conversely, if a series of circles through the cusp of a cardioid are tangent to it, their cuspidal diameters will be chords of the cardioid's axial circle (Stubbs,[5] 1843).

It is interesting to note the relation of Quetelet's theorem to Otto Dunkel's solution above, and to his paper "The relation of caustics to certain envelopes" (1921, 182–183).

Also solved by Joseph Rosenbaum, A. V. Richardson, and F. L. Wilmer.

2864 [1920, 482]. Proposed by C. B. HALDEMAN, Ross, O.

If S is a side of a regular undecagon inscribed in a circle of radius unity, show that

$$S^5 - (S^4 - 3S^2 - 1)\sqrt{11} - 11S = 0.$$

Solution by A. M. Harding, University of Arkansas.

It is evident that $S = 2 \sin \pi/11$. Hence our problem is to find an equation one of whose roots is $2 \sin \pi/11$. The complex eleventh roots of unity, which occur in conjugate pairs, viz.

$$\cos\frac{2k\pi}{11} \pm i \sin\frac{2k\pi}{11}, \qquad k = 1, 2, 3, 4, 5.$$

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[1] *Nouveaux Mémoires de l'Académie de Bruxelles*, vol. 3, 1826, p. 91 (memoir presented February 3, 1823); the theorem is stated as follows: La caustique par refléxion pour une courbe quelconque éclairée par un point brillant, est la developpée d'une autre courbe, laquelle a la propriété d'être l'envelope de tous les cercles qui ont leurs centres sur la courbe réfléchissante, et qui passent par le point brillant." Quetelet showed also that the "autre courbe" was a "secondary caustic," the locus determined by dropping a perpendicular, OL, from the luminous point, O, on the tangent to the reflecting curve, and producing this perpendicular an equal distance to B.

[2] C. Maclaurin, "Tractatus de curvarum constructione & mensura ubi plurimae series curvarum infinitae vel rectis mensurantur vel ad simpliciores curvas reducunter," *Philosophical Transactions of the Royal Society* of London, vol. 30, pp. 803–812; abridgment, vol. 6, 1809, pp. 356–362. The geometrical construction for points on the cardioid and its pedals is touched upon by Maclaurin in his *Treatise of Fluxions*, 1742, vol. 2.

[3] B. Price, *Infinitesimal Calculus*, vol. 1, 1852, p. 417. It was also proposed and solved in *Educational Times*, June and October, 1853, and June, 1856. There are many other discussions of the problem.

[4] C. Taylor, *An Introduction to the Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. 357.

[5] J. W. Stubbs, "On the application of a new geometric method to the geometry of curves and surfaces," *Philosophical Magazine*, vol. 23, pp. 338–347.

satisfy the equation

$$\sum_{i=0}^{i=10} x^i = 0 \qquad \text{or} \qquad \sum_{i=1}^{i=5}\left(x^i + \frac{1}{x^i}\right) + 1 = 0. \tag{1}$$

By the substitution of y for $x + 1/x$ this equation in its second form may be reduced to[1]

$$y^5 + y^4 - 4y^3 - 3y^2 + 3y + 1 = 0. \tag{2}$$

By taking $k = 1$ it will be seen that one root of (2) is

$$y = 2\cos\frac{2\pi}{11} = 2 - 4\sin^2\frac{\pi}{11} = 2 - S^2, \qquad \text{or} \qquad -S^2 = z = y - 2.$$

Reducing by 2 the roots of (2) by synthetic division we obtain the equation for z

$$z^5 + 11z^4 + 44z^3 + 77z^2 + 55z + 11 = 0.$$

If we now substitute $-S^2$ for z and group the terms of the resulting equation we find

$$S^{10} - 22S^6 + 121S^2 = 11(S^8 - 6S^6 + 7S^4 + 6S^2 + 1)$$

which may be written

$$S^5 - 11S = \pm(S^4 - 3S^2 - 1)\sqrt{11}.$$

Since the quantity in the parenthesis is negative for $S = 2\sin\pi/11$, the positive sign must be chosen. This gives the result

$$S^5 - (S^4 - 3S^2 - 1)\sqrt{11} - 11S = 0.$$

Also solved by T. M. BLAKSLEE, J. S. BROWN, J. B. FAUGHT, and A. V. RICHARDSON.

2866 [1920, 482]. Proposed by NORMAN ANNING, University of Michigan.

Equilateral triangles whose sides are 1, 3, 5, 7, $\cdots$ are placed so that their bases lie corner to corner in a straight line. Show that the vertices lie upon a parabola and are all at integral distances from its focus.

SOLUTION BY G. W. SMITH, University of Kansas.

If the base line of the triangles is taken as the x-axis and the y-axis is made to pass through the vertex of the first triangle of the series, the coördinates of the vertices of the triangles are $(0, \pm\frac{1}{2}\sqrt{3})$, $(2, \pm\frac{3}{2}\sqrt{3})$, $(6, \pm\frac{5}{2}\sqrt{3})$, $\cdots$. The vertex of the nth triangle is given by the equations $x = n(n-1)$, $y = \pm[(2n-1)/2]\sqrt{3}$. Eliminating n from these, gives $y^2 = 3(x + \frac{1}{4})$, a parabola whose focus is $(\frac{1}{2}, 0)$. The distance from the vertex $\{n^2 - n, [(2n-1)/2]\sqrt{3}\}$ to $(\frac{1}{2}, 0)$ is $n^2 - n + 1$ which is integral. It is worth noticing above that the second difference in the abscissas of the vertices is constant while it is the first difference in the ordinates that is constant, a property of the rational integral function of the second degree, such as x is of y.

Also solved by W. E. ANDERSON, T. L. BENNETT, AUGUSTUS BOGARD, H. C. BRADLEY, E. F. CANADAY, P. J. DA CUNHA, J. F. DOVE, L. H. DUBE, W. C. EELLS, J. B. FAUGHT, MICHAEL GOLDBERG, A. M. HARDING, LESTA HOEL, G. R. LIVINGSTON, GERTRUDE I. MCCAIN, I. MAIZLISH, E. W. MARTIN, L. C. MATHEWSON, H. L. OLSON, ARTHUR PELLETIER, R. V. PRITCHARD, R. V. RICHARDSON, and T. O. WALTON.

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[1] See Burnside and Panton, *The Theory of Equations*, eighth edition, vol. 1, 1918, p. 90 where the necessary reductions are given; on page 100, ex. 3 the resulting equation (2) is also given.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

At Stevens Institute of Technology Mr. W. E. F. APPUHN has been promoted to the rank of assistant professor.

At Davis and Elkins College, Mr. H. W. WHETSELL has been promoted from an instructorship to a professorship of mathematics.

At Sweet Briar College, Miss EUGENIE M. MORENUS has been appointed professor of mathematics and Miss MARY SEARLE instructor.

At Milwaukee-Downer College, Miss FRANCES A. ATWATER has been appointed instructor in mathematics and head of the department.

At the University of Georgia, Mr. A. H. STEVENS has been appointed instructor in mathematics.

At Colgate University, Mr. H. A. D. BELL has been appointed instructor in mathematics.

At Hope College, Holland, Mich., Mr. CLARENCE KLEIS has been appointed assistant in mathematics.

At Denison University, Mr. R. A. SHEETS has been promoted from instructor to assistant professor of mathematics and Mr. H. B. LEMON has been appointed instructor.

At the University of North Dakota, Professor R. R. HITCHCOCK, head of the department of mathematics, is on leave of absence pursuing studies in Columbia University. Associate Professor R. F. CASTNER has been appointed acting head of the department and J. D. LEITH instructor.

At the University of Wisconsin, Mr. M. L. MACQUEEN and Miss CAMILLA HAYDEN have been appointed instructors in mathematics. Mr. L. H. BUNYAN, Mr. R. H. MARQUIS, Mr. W. E. ARMENTROUT and Mr. D. W. MCLENEGAN have been appointed assistants.

At Hunter College, Miss EVELYN WALKER has been appointed assistant professor of mathematics.

At the Georgia School of Technology, Mr. W. V. SKILES has been promoted from an associate professorship of mathematics to a professorship. Mr. D. M. SMITH and Mr. A. B. MORTON have been promoted from assistant professorships to associate professorships and Mr. G. T. TRAWICH has been appointed instructor.

At Randolph-Macon Woman's College, Lynchburg, Va., Miss GILLIE A. LAREW (*1921*, 236) has been promoted from associate professor of mathematics to professor. Miss M. K. BOWEN, instructor, is on leave of absence pursuing graduate studies at the University of Chicago. Her place being temporarily filled by Miss VIRGINIA WATTS (*1921*, 236).

At Beloit College, Mr. H. H. CONWELL (*1921*, 43) has been promoted from an associate professorship of mathematics to a professorship and Mr. R. M. ROBINSON has been appointed instructor.

At the Municipal University of Akron, O., Miss W. H. LIPSCOMBE has been appointed instructor of mathematics.

At the University of Montana, Professor N. J. LENNES is on leave of absence for the current academic year. Miss GERTRUDE CLARK has been appointed assistant in mathematics.

At Ottawa University, Mr. H. W. BAILEY has been appointed instructor of mathematics.

At Tufts College Mr. P. D. WILKINS has been appointed instructor of mathematics.

At the University of Rochester, Professor A. S. GALE is serving in the newly created office of freshmen dean in addition to his duties as professor of mathematics and Mr. T. R. LONG has been appointed assistant in mathematics.

At Lake Forest College, Miss MARIE M. JOHNSON has been appointed instructor of mathematics.

At Creighton University, Mr. DAVID HICHES has been appointed professor of mathematics.

At Kenyon College, Mr. F. L. WHITE has been appointed instructor of mathematics.

At Miami University, Professor A. E. YOUNG has resigned to take a position with the Standard Oil Company in Pittsburgh, Pa. Professor W. E. ANDERSON has been promoted from an associate professorship of mathematics to a professorship and made head of the department. Assistant Professor J. C. BEEKLEY has resigned to pursue studies in physical chemistry at Princeton University. Mr. G. W. SPENCELY and Mr. C. D. EHRMAN (*1921*, 42) have been appointed to assistant professorships in mathematics.

At Union College, Mr. LAW BOWMAN, instructor in mathematics, has resigned.

At Lehigh University, Mr. H. S. BUNN has been appointed instructor in mathematics.

At Allegheny College, Mr. R. H. SKELTON has been appointed assistant professor of applied mathematics to succeed Professor K. A. MILLER.

At Cornell College, Mount Vernon, Iowa, Mr. EDMUND INGALLS has been appointed instructor in mathematics.

At the University of Maine, Messrs. F. S. BEALE, W. E. LORING, and EDWARD BROW have been appointed instructors. Mr. M. F. JORDAN, instructor in mathematics, has resigned to accept a graduate fellowship in astronomy at Harvard University. Mr. A. S. ADAMS, instructor, has resigned to become principal of eastern Maine Conference Seminary, and Mr. T. H. JOHNSON, instructor, has accepted a position in mathematics at the Moses Brown School.

At Syracuse University, Professor W. H. METZLER, head of the department of mathematics, has been made dean of the College of Liberal Arts.

At the University of Pittsburgh, Mr. Z. N. HOLLER has been appointed instructor in mathematics. Mr. PAUL FRANK and Mr. W. J. BRAZLER, instructors in mathematics, have resigned.

At Carleton College, Dr. C. H. GINGRICH of the department of mathematics and astronomy is on leave of absence for the current year pursuing work at the Mt. Wilson Observatory.

From June 20 to August 9, 1921, Professor E. R. HEDRICK, of the University of Missouri, delivered addresses, on behalf of the National Committee on Mathematical Requirements, at the following Institutions: University of Texas, University of Oklahoma, University of Nebraska, State Normal Schools at Peru and Kearney (Nebraska), University of Chicago, University of Iowa, Iowa State Teachers' College, University of Michigan, and Northwestern University.

Associate Professor EMMA KONANTZ, of Ohio Wesleyan University, who has spent the last two years in China, has in addition to teaching in Peking University, been "studying the history of Chinese mathematics, and assisting Dr. C. T. Chen, head of the department of mathematics in Peking University, in the preparation of a translation with explanatory notes, of a Chinese algebra of the thirteenth century. This algebra shows the solution of equations of higher degree, in four unknown quantities, and contains processes approximately the same as Horner's method of root extraction, the Chinese thus antedating Western mathematics several centuries."—*Ohio Wesleyan Alumni Quarterly.*

At the meeting in Boston, November 9, 1921, of the American Academy of Arts and Sciences, A. G. WEBSTER, of Clark University, delivered an address on "Hermann von Helmholtz and his significance for a century of science."

In the meetings of the American Mathematical Society at Columbia University, October 29, 1921, 11 papers were presented by the following members: G. A. CAMPBELL, T. H. GRONWALL, EDWARD KASNER, R. L. MOORE (2), L. H. RICE, J. F. RITT (2), L. B. ROBINSON, J. L. WALSH, and J. K. WHITTEMORE. Details concerning the papers may be found in *Bulletin of the American Mathematical Society,* March, 1922.

At the nineteenth annual meeting of the Association of Teachers of Mathematics in New England held at Massachusetts Institute of Technology on December 3, 1921, the following papers were read: "Geometry understood, not memorized" by R. R. SMITH of the Newton Classical High School; "Modern methods in junior high schools" by E. R. SMITH of Park School, Baltimore, Md.; "Graphic solutions of certain systems of equations" by R. E. BRUCE of Boston University.

At the fall meeting of the Connecticut Valley Section of the Association of Teachers of Mathematics in New England, at Hartford, November 5, 1921, the following papers were presented: "An introduction to trigonometry" by H. E. WEBB; "The work of the examiners of the College Entrance Examination Board" by A. E. BOOTH; and "The new statement of college entrance requirements in mathematics" by W. R. LONGLEY. Professor P. F. SMITH is president of the Section, Mr. M. M. S. MORIARTY, vice-president, Professor ELEANOR C. DOAK, treasurer, and Mr. H. B. MARSH a director.

The mathematics department of the Virginia Educational Conference met in annual session, November 23, 1921, in John Marshall High School, Richmond, Va.

The following papers were read: "Fractions is fractions" by E. G. SCHAKELFORD; "An experiment in classroom method" by EVELYN WHITMORE; "The coördination of high school and first year college mathematics" by J. J. LUCK and J. D. RIDDICK; "The practical use of mathematics" by J. E. ROWE. The following officers were reëlected: T. MCN. SIMPSON, Jr., president, and C. W. GIVENS, secretary.

At the Toronto meetings of the American Mathematical Society, December 28–29, 1921, 32 papers by the following mathematicians were presented: S. BEATTY, R. W. BURGESS, W. L. CRUM, LOUISE D. CUMMINGS, D. R. CURTISS, E. L. DODD, L. P. EISENHART and OSWALD VEBLEN, ARNOLD EMCH, H. S. EVERETT, C. H. FORSYTH, J. S. C. GLASHAN (2), O. E. GLENN, C. F. GUMMER, OLIVE C. HAZLETT, EINAR HILLE, E. V. HUNTINGTON, W. A. HURWITZ, M. H. INGRAHAM, JOSEPH LIPKA, H. F. MACNEISH, G. A. MILLER (2), E. H. MOORE, I. R. POUNDER, I. J. SCHWATT (2), J. L. WALSH and NORBERT WIENER, E. J. WILCZYNSKI, C. E. WILDER (2), and S. D. ZELDIN. At the joint session with the American Physical Society and with Sections B and C of the American Association for the Advancement of Science, the American Physical Society was represented by SAUL DUSHMAN, Sections B and C by J. C. MCLENNAN and R. C. TOLMAN respectively, and the American Mathematical Society by H. B. PHILLIPS. At the joint session with the Mathematical Association of America and with Section A of the Association for the Advancement of Science addresses were made by R. D. CARMICHAEL, D. R. CURTISS, H. E. SLAUGHT, and R. M. YERKES. Details concerning these meetings may be found in this MONTHLY and in the *Bulletin of the American Mathematical Society.*

Published August 1, 1922.

DESIRED COPIES OF THE MONTHLY

Through an oversight the Association has not reserved a sufficient number of copies of the Monthly for October, 1920 and for August–September and October, 1921. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, A. A. BENNETT, University of Texas, Austin, Texas.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December, 1922

The following are dates of Section meetings of the Association in 1921 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29, 1922

IOWA, Simpson College, Indianola, April 30; Des Moines, November 4

KANSAS, Topeka, January 22; Topeka, January 21, 1922

KENTUCKY, University of Kentucky, May 7; Georgetown College, April 8, 1922

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, May 13, 1922; Washington, December, 1922

MINNESOTA, St. Paul, June 4; Macalester College, St. Paul, May 27, 1922.

MISSOURI, Washington University and Soldan High School, St. Louis, November 25–26; Kansas City, November, 1922

OHIO, Columbus, March 25–26; Columbus, April 14–15, 1922

ROCKY MOUNTAIN, Denver, March 25–26; Greeley, Colo., April 14–15, 1922

TEXAS, Dallas, November 25; Houston, December 1–2 1922,

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SIXTH ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

By invitation of the University of Toronto and the Royal Canadian Institute, the sixth annual meeting of the Mathematical Association of America was held at the University of Toronto on Thursday and Friday, December 29 and 30, 1921, in affiliation with the American Association for the Advancement of Science and in connection with the annual meeting of the American Mathematical Society. There were 110 in attendance at the sessions, including the following 88 members of the Association:

R. C. Archibald, Brown University.
G. N. Armstrong, Ohio Wesleyan University.
C. S. Atchison, Washington and Jefferson College.
Clara L. Bacon, Goucher College.
L. A. Bauer, Dept. of Terrestrial Magnetism, Washington, D. C.
Samuel Beatty, University of Toronto.
R. D. Beetle, Dartmouth College.
O. F. H. Bert, Washington and Jefferson College.
G. A. Bliss, University of Chicago.
J. W. Bradshaw, University of Michigan.
R. W. Burgess, Brown University.
W. D. Cairns, Oberlin College.
H. C. Carver, University of Michigan.
Louise D. Cummings, Vassar College.
C. H. Currier, Brown University.
D. R. Curtiss, Northwestern University.
A. T. De Lury, University of Toronto.
W. W. Denton, University of Michigan.
Sister Mariola Dobbin, St. Clara College.
B. F. Dostal, University of Denver.
L. H. Dube, Ottawa University.
Arnold Dresden, University of Wisconsin.
Peter Field, University of Michigan.
J. C. Fields, University of Toronto.
William Findlay, McMaster University.
B. F. Finkel, Drury College.
T. M. Focke, Case School of Applied Science.
W. B. Ford, University of Michigan.
J. L. Gibson, University of Utah.
D. C. Gillespie, Cornell University.
O. E. Glenn, University of Pennsylvania.
J. W. Glover, University of Michigan.
R. A. Gray, Oakwood Collegiate Institute, Toronto.
C. F. Gummer, Queen's University.
Olive C. Hazlett, Mt. Holyoke College.
E. R. Hedrick, University of Missouri.
H. A. Howe, University of Denver.
E. V. Huntington, Harvard University.
W. A. Hurwitz, Cornell University.
M. H. Ingraham, University of Wisconsin.
G. H. Jamison, Kirksville (Mo.) State Teachers College.
L. C. Karpinski, University of Michigan.
G. W. Keith, Parkdale Collegiate Institute, Toronto.
J. P. Kelly, Boston College.
Jennie A. Kinnear, Port Colborne, Ont.
H. R. Kingston, Western University.
N. J. Lennes, University of Montana.
Florence P. Lewis, Goucher College.
E. J. Maurus, Notre Dame University.
G. A. Miller, University of Illinois.
Norman Miller, Queen's University.
G. R. Mirick, East High School, Rochester, N. Y.
E. H. Moore, University of Chicago.
F. R. Moulton, University of Chicago.
B. L. Newkirk, General Electric Co., Schenectady, N. Y.
H. L. Olson, University of Michigan.
F. W. Owens, Cornell University.
Mrs. F. W. Owens, Cornell University.
W. J. Patterson, Western University.
Mrs. Anna J. Pell, Bryn Mawr College.
H. B. Phillips, Mass. Institute of Technology.
A. D. Pitcher, Western Reserve University.
I. R. Pounder, University of Toronto.
P. L. Rea, Marietta College.
A. V. Richardson, Bishop's College.
R. G. D. Richardson, Brown University.
H. L. Rietz, University of Iowa.
G. M. Robison, Cornell University.
E. D. Roe, Jr., Syracuse University.
W. H. Roever, Washington University.
N. E. Sheppard, University of Toronto.
Mary E. Sinclair, Oberlin College.
H. E. Slaught, University of Chicago.
W. A. Skirrow, Parkdale Collegiate Institute, Toronto.
Clara E. Smith, Wellesley College.
E. R. Smith, State College of Iowa.
I. W. Smith, North Dakota Agric. College.

P. F. SMITH, Yale University.
H. S. UHLER, Yale University.
OSWALD VEBLEN, Princeton University.
G. L. WAGAR, Mt. Hermon School.
C. W. WATKEYS, Rochester University.
A. G. WEBSTER, Clark University.
E. T. WHITE, London Normal School.
C. E. WILDER, Northwestern University.
A. H. WILSON, Haverford College.
E. B. WILSON, Mass. Institute of Technology.
JESSICA M. YOUNG, Washington University.

The meetings of the American Association were held during the week, beginning with the address of the Retiring President on Tuesday evening, and continuing through Saturday of that week. There was a total attendance of over 1800 at the various sessions of the American Association and affiliated societies. The University authorities and the Royal Canadian Institute made exceptionally fine preparations for the entertainment of visitors, and an attractive combination of scientific programs and social diversions was furnished for the various groups of scientists. On Wednesday evening in Convocation Hall, Professor William Bateson, director of the John Innes Horticultural Institute of Surrey, England, delivered an address on "Evolutionary Faith and Modern Doubts." This address appears in full in *Science* for January 20, 1922. At the close of this address a Convocation of the University of Toronto was held, presided over by Sir Robert Falconer, at which the degree of Doctor of Science *honoris causa* was conferred on Professor Bateson, Retiring President A. A. Howard, and President E. H. Moore. On Wednesday afternoon at the close of the mathematical program, a reception was given in Hart House at which the mathematicians were cordially received by Sir Robert and Lady Falconer, and Professor A. T. De Lury. At a general session on Thursday afternoon, Sir Adam Beck, Chairman of the Hydro-Electric Commission of Ontario, gave an address on "Hydro-Electric Developments in Ontario," illustrating by a series of moving pictures the various hydro-electric projects in Ontario. The largest of the social functions was the Thursday evening conversazione in Hart House where the many guests of the University and the Royal Canadian Institute were entertained in a variety of fascinating ways in the large halls and theater of Hart House. At the close of the mathematical program on Friday evening, an exhibition of fancy skating and a hockey match between the Varsity Intermediates and the St. Helens Intermediates was held at the Arena. All arrangements for the convenience of guests were ably made by the Committee on Arrangements, under the chairmanship of Professor De Lury. A special committee of ladies of the University paid constant attention to the entertainment of the ladies among the visiting guests, and a general dinner for the ladies was given Friday evening at the same time as the dinners of the various groups of scientists. The mathematicians were housed in Knox College and in Annesley Hall on the Campus, and enjoyed thus the same social conveniences that have characterized a number of the summer meetings of the Association and Society. The joint dinner of mathematicians, astronomers, and physicists on Friday evening was attended by about 150 guests. During the dinner the visitors were entertained by the Adanac quartet, Miss Vera McLean, and Mrs. Collinshaw, of Toronto. At the closing session of the Association a resolution of thanks to the mathematical

staff and the authorities of the University of Toronto, to the Royal Canadian Institute, and to the Program Committee, was unanimously adopted. A very extensive account of the features of the Toronto meeting of the American Association is given in *Science* for January 13 and 20, 1922.

Many members attended the joint symposium on Quantum Theory on Thursday morning, at which Section C of the American Association was represented by Professor R. C. Tolman, the American Physical Society by Dr. Saul Dushman, and the American Mathematical Society by Professor H. B. Phillips.

The program was formulated by the Program Committee under the chairmanship of Professor C. F. Gummer of Queen's University. An innovation that was generally commended was the institution of a series of papers giving outlines of certain fields of research. It is frequently urged that College and University teachers should be engaged in some form of productive work, but many college instructors do not know promising lines of investigation, and do not know how and where to find the literature which will inform them of what has already been done in various lines. A suggestion has been made that the Association can do a valuable service if on its programs and through the MONTHLY, university teachers map out for college teachers possible lines of research growing readily out of college courses. The papers by Professors Veblen and Bliss afford a beginning of such suggestions.

The following papers were given. Abstracts of these papers follow, the numbers corresponding to the numbers in the lists of titles.

JOINT SESSION OF THE ASSOCIATION WITH THE AMERICAN MATHEMATICAL SOCIETY AND SECTION A OF THE AMERICAN ASSOCIATION.

(1) "Algebraic guides to transcendental problems," retiring address by Professor R. D. CARMICHAEL, University of Illinois, chairman of the Chicago Section of the Society, and a vice-president of the Association.

(2) "A mechanical analogy in the theory of equations" by Professor D. R. CURTISS, Northwestern University, retiring vice-president of Section A.

(3) "Subsidy funds for mathematical projects" by Professor H. E. SLAUGHT, University of Chicago.

(4) "Informational service of the National Research Council" by Dr. R. M. YERKES, Chairman Research Information Service.

1. Professor Carmichael stated that algebraic guides in transcendental equations had been hitherto used in a very limited number of fields, notably in that of integral equations where the method was used by Volterra and Fredholm as a heuristic guide, but by Hilbert as a method of deducing the results in integral equations by limiting processes from algebraic processes as the number of variables becomes infinite. He instanced earlier uses of analogous methods by Cauchy which were clarified by Lipschitz, by Sturm who passed from the solutions of difference equations to infinitely small differences, from the treatment of linear difference equations to that of differential equations, by Porter who, in the *Annals of Mathematics* in 1902, carried out some of the essential work done more than

two years later by Hilbert. Professor Carmichael discussed two types of problems, in the one class oscillation and comparison theorems, in the other theorems of expansion in orthogonal functions and their generalizations, and in successive sections he formulated algebraic and transcendental oscillation, comparison, and expansion theorems. His paper has been offered to the *Bulletin of the American Mathematical Society*.

2. Professor Curtiss's paper appears in *Science* for February 24, 1922, pp. 189–194.

3. Professor Slaught displayed in graphic manner some ways in which mathematics needs funds for carrying out its projects, by exhibiting the following table and commenting upon it item by item. The paper was printed in full in *Science* for February 10, 1922, pp. 146–148.

SUBSIDY FUNDS NEEDED FOR MATHEMATICAL PROJECTS.

	Annual	Lump Sum	Endowment
1. Revolving book fund for the publication of treatises		$ 25,000	
2. Mathematical Dictionary, already carefully planned by the Association		100,000	
3. Publication of Historical Journal—Continuation of *Bibliotheca Mathematica*	$ 2,000		$ 40,000
4. Enlargement of all the research journals to full capacity	2,000		40,000
Extra volume of the *Transactions* of the American Mathematical Society		4,000	
5. Two extra numbers of the AMERICAN MATHEMATICAL MONTHLY for expository papers and expansion of the regular volume by 80 pages	2,000		40,000
6. Publication of Mathematical Monographs—Funds already provided by Mrs. Mary Hegeler Carus	1,200		24,000
7. *Mathematical Abstract Journal*, approved by the National Research Council	15,000		300,000
8. Preparation of a Bibliography of Bibliographies in Mathematics		5,000	
9. For prizes and Research Fellowships	25,000		500,000
10. Honorary Stipends for Executive Officers of the Society and Association	4,000		80,000
Totals	$51,200	$134,000	$1,024,000

4. Dr. Yerkes stated that the Informational Service of the National Research Council early decided to have not a storehouse but a clearing house of information. They are attempting to develop a personnel file of research activities and achievements of all who have been trained to research, or have attained to research ability, news of centers of interest, progress of current work, information as to the laboratories and institutions which are equipped for various lines of work; and they purpose to keep this information classified under various useful headings. They are seeking to coördinate the existing facilities in our libraries and to make these available through various source-books, and through lists of published and unpublished bibliographies, and thus to avoid the lamentable loss resulting from the constant duplication of effort in finding what has been done in any

particular line. The Service is trying to create a special division to take care of needs as to scientific apparatus. But the principal part of the machinery is the corps of specialists with particular knowledge of the subject and with the insight necessary for interpreting or clarifying the needs of those applying for aid. Mathematicians like other scientists are urgently invited to use the Service for whatever purposes they may wish of a scientific sort.

SESSION OF THE ASSOCIATION.

(5) Outlines of Fields of Research: (*a*) "Foundations of geometry" by Professor OSWALD VEBLEN, Princeton University.

(6) Outlines of Fields of Research: (*b*) "Calculus of variations" by Professor G. A. BLISS, University of Chicago.

(7) "Courses in mechanics for students majoring in mathematics" by Professor E. V. HUNTINGTON, Harvard University.

(8) "Topology of three-dimensional manifolds in three dimensions" by Professor NORMAN MILLER, Queen's University.

5. Professor Veblen's paper was devoted to a discussion of the field for research in the foundation of elementary geometry. He first referred to the study of the axioms of order by Pasch, Peano, E. H. Moore, and others, and called attention to a set of axioms of betweenness given by C. Müntz (*Jahresbericht der Deutschen Math. Ver.*, 1914) which seem very satisfactory. Aside from further improvements of these axioms (in the direction of "complete independence" for example) there is much to be done in the way of demonstrating the fundamental topological theorems. Among these are to be counted not only problems involving refined continuity considerations such as have been studied by R. L. Moore, N. J. Lennes, J. R. Kline and others, but such combinatorial questions as the classification of all polygons in space. A solution of this problem would be equivalent to a solution of the well-known knot problem of analysis situs. There is thus a whole series of problems, some easy and some very difficult, for anyone who will start with the axioms of order and try to build up the theorems which should immediately follow.

In the researches referred to above there is little if any apparent relation to the problem of teaching elementary geometry. The time has now arrived, however, when a study of the foundations of geometry should be made with the object of obtaining a logical formulation of the subject which can be used in elementary teaching. This is a problem of research because no suitable formulation of the subject yet exists, and it is one in which the routine experience of college and secondary school teachers can be directly useful. The sort of axioms to look for would be less like the order axioms of Pasch than like those given at the beginning of Veblen and Young's *Projective Geometry*. Among the general principles which should be followed in such a treatment of geometry are:

(*a*) The introduction of coördinates and the methods of analytical geometry should come very early. Further assumptions and definitions could then be stated in analytical terms, and the development of algebra and geometry go hand in hand.

(*b*) It should be kept continually in mind that geometry is a branch of natural science. The abstract logical structure should be compared carefully at each stage with the facts of nature. This comparison is in general possible only with a certain percentage of accuracy. It is not enough to verify the axioms and then feel sure that all theorems are true, for it might easily happen that all the axioms are verified within one one hundredth of a per cent. and yet that some of the theorems fail by several per cent. Every important theorem should be tested by experiment or experience just as in physics and in chemistry.

(*c*) A thoroughly skeptical attitude should be preserved toward the facts of geometry. The parts of space about which we speak with confidence are here and now. As soon as we speak of distant parts of space or of space as it was in the distant past we encounter difficulties with regard to the evidence. Those who teach geometry, if not those who are learning it, should be conscious that it gives only a partial description of nature and that this description may have to undergo serious modifications before it can be made a part of a more adequate description.

6. According to Professor Bliss research is the discovery of a problem on which to work rather than the working out of an assigned topic. One needs enthusiasm for such a task, a moderate acquaintance with the literature, and a good stock of imagination. In speaking of the calculus of variations, which was his particular chosen topic, he stated that the earlier stage was that of problem solving, or finding problems like those already solved; the present stage is one in which the general methods of the theory have been or are being attacked, together with the problem of fitting the general theory to special cases, etc. For example: the solution of Newton's solid of revolution of minimum resistance does not provide for the case where the curve is to begin at the x-axis; Bolza found by a not very difficult analysis the solution for this special case.

Professor Bliss gave samples of problems in the calculus of variations which are partly solved, but which still afford opportunity for original work. Four problems of this kind are the shortest line, the brachistochrone (curve of quickest descent), minimum surface of revolution, minimum area between a curve, its evolute and the bounding normals. As an outgrowth from this last problem, Rider and Dunkel have treated the minimum area between a curve and its caustic. Under the other problems, Mason has treated the problem of passing a wire through two points so that the moment of inertia about the origin is a minimum; and under the old problem of minimum surfaces of revolution, MacNeish and Miss Sinclair made extensions of various sorts. Professor Bliss closed with the statement that one can still find examples under the old problems which admit of new extensions, if only one reads up the subject of the calculus of variations and investigates what has been done on these old problems.

Professor Hurwitz corroborated what the two speakers said in reference to the beginnings of research, instancing various problems and discussions which have appeared in recent numbers of the MONTHLY.

Professor Glover stated that the *Transactions of the Actuarial Society of*

America has published a list of about fifty different subjects for research, a list which was obtained from the members of that society, and suggested that a similar procedure might readily produce valuable results for the Association.

7. After pointing out the futility of trying to teach kinetics without the use of the calculus, Professor Huntington expressed the belief that any course in theoretical mechanics ought to aim: (*a*) to provide the student with an opportunity to extend and solidify his knowledge of mathematical methods, especially the calculus; and (*b*) to teach him the practical principles of dynamics as such, unencumbered by technical mathematical details.

Attempts sometimes made to separate these two aims are in danger of leading (in the hands of "pure mathematicians") to a loss of all mechanical perspective in a maze of mathematical manipulation; or else (in the hands of "practical engineers") to a loss of all scientific generality in a multiplicity of numerical cases treated more or less by rule of thumb.

Since the theory of the subject is extremely brief, a course in mechanics is made or marred by its choice of problems. The whole question turns on the selection of good problems—problems which reveal the power and scope of mathematical methods without degenerating into mere mathematical puzzles—problems which yield interesting mechanical results, without becoming merely a series of detached numerical cases. The main part of the paper consisted in a detailed discussion of several examples of typical problems of this sort.

Professor Webster said that mathematicians on going out from college should know the leading problems of historical interest in mechanics, not for the purpose of helping one's calculus, but in order to know what the most important problems of mechanics are, and to know the leading methods of mechanics, whether elementary or advanced. He therefore did not agree with Professor Huntington in the advisability of using such fanciful problems as that of the squirrel and the cage.

8. In the search for conditions for the equivalence of two manifolds in the sense of analysis situs various invariants have been defined. Dr. Miller considered the case of regions of three-dimensional space and showed how certain invariants present themselves naturally in their physical construction. Any region may be built up, in the manner of a clay model, from pieces each equivalent to a sphere. At any stage in the construction an added piece may touch the region already formed along one or more faces, each of which may be bounded by one or more curves. The operations are, however, reducible to three kinds, viz.: (*a*) adding a piece which touches along a simply connected face; (*b*) adding a piece which touches along two simply connected faces, "putting a handle on a surface;" (*c*) adding a piece which touches along a doubly connected face. A region constructed by operations (*a*) alone is simply connected. A region constructed by operations (*a*) and (*b*) only is characterized topologically by one invariant, its connectivity, which may be defined as one more than the number of (*b*) operations used. A region with a single boundary which requires operations (*c*) is not equivalent to a region of the former type. An example of such a region

is the interior of a sphere from which a knotted tube has been removed. The least number of (*c*) operations which must be used in the construction of a region may be called the degree of knottedness of the region. In order that two regions be equivalent topologically this number must be the same for both. The definition of connectivity is now extended to apply to regions of this second type and the two invariants are defined for regions with several boundaries.

Session of the Association.

(9) "Functionality in mathematical instruction in schools and colleges" by Professor E. R. Hedrick, University of Missouri.

(10) "An example in the inversion of upper limits and bounds" by Professor Samuel Beatty, University of Toronto.

(11) "New mathematical periodicals" by Professor G. A. Miller, University of Illinois.

(12) "Proof of the fundamental theorem regarding the length of a curve" by Professor J. L. Synge, University of Toronto, by invitation.

9. In this paper, Professor Hedrick explained and expounded in some detail the principle announced by the National Committee on Mathematical Requirements concerning the central position in all mathematical teaching that is occupied by the function concept. He described the contents of the report of the same committee on The Function Concept in Secondary School Mathematics, which was prepared for the committee by the speaker and was published by the U. S. Bureau of Education as *Secondary School Circular No. 8*, June, 1921. He then proceeded to extend these ideas to higher secondary courses and to collegiate courses in mathematics. He pointed out by the citation of many instances the need for the formation of accurate habits of quantitative thinking and of functional thinking on the part of the great masses of our population, not only for the control of situations that arise in their private lives, but also for efficient judgment of such great public questions as the control of public service corporations, life insurance, railroad rates, pensions, and the like, all of which necessarily involve quantitative relationships that will be misunderstood by voters and legislators unless they have formed correct habits of functional thinking. Finally, the necessary reform of mathematical teaching to insure such training was discussed. Colleges and Universities, as well as secondary schools, must fundamentally alter their traditional mathematical courses if the advantages of mathematical study pointed out in this paper are to be realized.

10. Professor Beatty showed that if $f_1(t), f_2(t), \cdots$ is a sequence of real functions defined for a continuous, closed interval (t), and if $\operatorname*{limit}_{n} \operatorname*{bound}_{(t)} f_n(t)$ is finite, then $\overline{\operatorname*{bound}}_{(t)} \operatorname*{limit}_{n} f_n(t)$ is equal to or less than it; moreover, a sufficient condition for equality is that, corresponding to each pair t, $\epsilon > 0$, there exists a pair n, $\delta > 0$ such that

$$f_{n+p}(t + \theta\delta) - \overline{\lim_{n}}\, f_n(t) < \epsilon,$$

for every positive integer p and every fraction θ between 1 and -1. This finds application in the general theory of the complex power-series. For suppose that $a_0, a_1, a_2, \cdots$ is a sequence for which

$$\overline{\lim_n} \, |a_n{}^{1/n}| = 1,$$

and that A_n of the sequence $A_0, A_1, A_2, \cdots$ denotes

$$a_n + \binom{n+1}{1} a_{n+1}\sigma e^{it} + \binom{n+2}{2} a_{n+2}\sigma^2 e^{2it} + \cdots,$$

σ being an assigned proper fraction and t ranging over the interval $(0, 2\pi)$; then it appears that

$$\overline{\lim_n} \, \overline{\text{bound}}_{(t)} \, |A_n{}^{1/n}| = \overline{\text{bound}}_{(t)} \, \overline{\lim_n} \, |A_n{}^{1/n}|.$$

This leads to a demonstration of the existence of a singularity on the circle of convergence of the complex power-series

$$a_0 + a_1 z + a_2 z^2 + \cdots.$$

11. Professor Miller said that various mathematical journals have been disappearing, others are in great difficulties and are in danger of dying off. While a number of mathematical journals have lived to a ripe and useful old age, the average journal has been short-lived. Among the recently inaugurated journals which are of promising character and are contributed to by many noted and able writers are:

Zeitschrift für Mathematik; Zeitschrift für Angewandte Mathematik und Mechanik; Stanford University Publications (Series in Mathematics and Astronomy) begun in January, 1921; *The Mathematics Teacher,* conducted under the auspices of the National Council of Teachers of Mathematics; the Italian journal *Periodico,* which began a new series this present year of 1921; a Dutch periodical which is to appear six times a year; and others.

An inquiry from Professor Curtiss brought out the fact that the *Archiv* and the *Bibliotheca Mathematica* have apparently been discontinued, but that the *Jahrbuch* is expected to continue. It is, of course, obvious that the status of some European journals is uncertain, and that there is slight information of a definite sort concerning some of the periodicals which have been for many years familiar to American mathematicians.

12. Professor Synge gave an alternative proof of the existence theorem that if a curve is given in the parametric form

$$x = x(u), \qquad y = y(u), \qquad z = z(u),$$

where $\dot{x}$, $\dot{y}$, and $\dot{z}$ exist and are continuous, not all vanishing simultaneously, then the length of the curve from $A(u=u_A)$ to $B\ (u=u_B)$ exists and is given by

$$\int_{u_A}^{u_B} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du.$$

Meeting of the Board of Trustees of the Association.

Ten members of the Board were present at the meetings.

The following 58 persons and 4 institutions, on application duly certified, were elected to membership:

To individual membership.

Frank Ayres, Jr., B.S. (Washington Coll.). Head of dept. of math., Ogden College, Bowling Green, Ky.

W. E. Bagley, M.S. (Washington). Head of dept. of science, Central Academy and College, McPherson, Kans.

R. W. Barnard, M.S. (Michigan). Instr., Univ. of Michigan, Ann Arbor, Mich.

V. E. Beaupré, B.A. (Manitoba). Prof., Ecole Polytechnique, Montreal, Canada.

Rachel Blodgett, Ph.D. (Radcliffe). Instr., Wellesley College, Wellesley, Mass.

J. E. Burnam, A.M. (Texas). Asso. prof., Simmons College, Abilene, Tex.

May B. Carter, A.M. (Brown). Asso. prof., The Western College for Women, Oxford, Ohio.

H. C. Carver, B.S. Prof., math. and insurance, Univ. of Michigan, Ann Arbor, Mich.

Jung Lu Chin, graduate of Peking Teachers College. Graduate stud., Univ. of Chicago.

W. A. Coit, Ph.D. (Boston). Prof., Acadia Univ., Wolfville, N. S.

J. P. Dalton, M.A., D.Sc. (St. Andrews); D.Sc. (Cape). Prof., Univ. of the Witwatersrand, Johannesburg, S. A.

C. H. Davis, American Railway Express Co., Providence, R. I.

A. T. De Lury, M.A. (Toronto). Prof., University of Toronto, Toronto, Canada.

Leila Eason, A.B. (Texas). Palestine, Tex.

J. H. Eichelberger, Student, Univ. of Chicago. Freeport, Ill.

Iva Ernsberger, A.M. (Nebraska). Instr., Non Collegiate, Iowa State College, Ames, Ia.

David Forsyth, B.A. (Toronto). Principal Emeritus, Kitchener and Waterloo Collegiate Inst., Kitchener, Ont.

R. M. Fréchet, Professeur d'Analyse Supérieure, Faculté des Sciences de Strasbourg, Strasbourg, France.

Margaret L. Furrey, B.S. (Northwestern). Asst. instr., Northwestern Univ., Evanston, Ill.

R. A. Gray, B.A. (Toronto). Principal, Oakwood Collegiate Inst., Toronto, Ont.

E. S. Hammond, Ph.D. (Princeton). Asst. prof., Bowdoin College, Brunswick, Me.

Paula F. Henry, A.M. (Texas). Teacher, Fort Worth High School, Fort Worth, Tex.

H. C. HICKS, Ph.B. (Chicago). Grad. student, Univ. of Chicago, Chicago, Ill.
F. C. HOUSEHOLDER, A.B. (Kansas). Asso. prof., N. Dak. Agricultural Coll., Fargo, N. Dak.
MARIE M. JOHNSON, M.S. (Iowa). Instr., Lake Forest Coll., Lake Forest, Ill.
G. W. KEITH, B.A. (Toronto). Math. master, Parkdale Colleg. Inst., Toronto, Ont.
J. T. KENDRIGAN, B.S. (Maine). Prof., Northland Coll., Ashland, Wis.
JENNIE A. KINNEAR, B.A. (Queen's Univ.). Port Colborne, Ont.
LOUISE LANGE, Ph.D. (Göttingen). Prof., Oxford College, Oxford, Ohio.
ARTHUR LÉVEILLÉ, B.A. (London). Prof., Univ. of Montreal, Montreal, Canada.
M. A. MACKENZIE, M.A. (Cantab.). Prof., Univ. of Toronto, Toronto, Ont.
MAX MORRIS, A.M. (Harvard). Instr., Case School of Applied Science, Cleveland, Ohio.
D. S. MORSE, A.M. (Harvard). Instr., Cornell Univ., Ithaca, N. Y.
THIRZA A. MOSSMAN, A.B. (Nebraska). Instr., St. Xavier Coll. for Women, Chicago, Ill.
W. J. PATTERSON, M.A. (Queen's). Prof., Western Univ., London, Canada.
EDNA P. PEPPER, A.B. (Western Coll.). Instr., Western Coll., Oxford, Ohio.
H. A. ROBINSON, Senior, Univ. of Ga., Athens, Ga.
KATHLEEN M. ROSE, B.A. (Queen's Univ.). Teacher, Sudbury High School, Sudbury, Ont.
S. M. SEWELL, M.S. (Chicago). Prof., Southwest Texas State Normal Coll., San Marcos, Tex.
LEONARD SHAFFER, A.B., B.S. (Penna.). Instr., New Hampshire Coll., Durham, N. H.
H. C. SHAUB, A.B. (Ohio State). Instr., Dartmouth Coll., Hanover, N. H.
R. A. SHEETS, A.M. (Denison). Asst. prof., Denison Univ., Granville, Ohio.
N. E. SHEPPARD, M.A. (Toronto). Lecturer, Univ. of Toronto, Toronto, Ont.
W. A. SKIRROW, M.A. (Queen's). Asst. master, Parkdale Colleg. Inst., Toronto, Ont.
G. R. SMITH, B.A. (Toronto). Ottawa, Ont.
G. W. SPENCELEY, A.M. (Harvard). Asst. prof., Miami Univ., Oxford, Ohio.
C. W. STROM, A.B. (Luther Coll.). Instr., Luther Coll., Decorah, Ia.
MARGARET F. TEMPLE, A.B. (Hamline). Teacher, High School, Jackson, Minn.
R. J. W. TEMPLIN, A.M. (Bucknell). Instr., Lafayette Coll., Easton, Pa.
W. N. THOMPSON, B.S. (Drury); A.B. (Northwestern). Asst. prof., Drury Coll., Springfield, Mo.
ALTHÉOD TREMBLAY. Prof., Laval Univ., Quebec, Canada.
W. J. WAGNER, A.M. (Illinois). Asst., Univ. of Illinois, Urbana, Ill.
ELIZABETH V. WATTS, A.M. (Chicago). Instr., Randolph-Macon Woman's Coll., Lynchburg, Va.
E. T. WHITE, D. Paed. (Toronto). Math. master, Normal School, London, Canada.
D. E. WHITFORD, A.M. (Brown). Instr., Univ. of Rochester, Rochester, N. Y.

R. R. Wood, B.S. (Haverford). Instr., Whittier College, Whittier, Calif.
E. W. Woolard, Meteorologist, U. S. Weather Bureau, Washington, D. C.
A. L. Young, B.S. in M.E. (Iowa State). Instr., Iowa State Coll., Ames, Ia.

To Institutional Membership.

Crane Junior College, Chicago, Ill.
St. Xavier College for Women, Chicago, Ill.
Michigan State Normal College, Ypsilanti, Mich.
St. Benedict's College, St. Joseph, Minn.

The following have been appointed Associate Editors of the Monthly for the year 1922:

Julian L. Coolidge, Edward L. Dodd, Otto Dunkel, Benjamin F. Finkel, Walter B. Ford, David C. Gillespie, Cuthbert F. Gummer, Leander M. Hoskins, Raymond E. McClenon, Francis D. Murnaghan, David E. Smith.

Professor H. L. Rietz and the Secretary were appointed the representatives of the Association on the Council of the American Association for the Advancement of Science for 1922.

In order to facilitate the counting of ballots for the annual election, the Secretary was appointed a permanent member of the Committee of Tellers, this committee to consist of three members.

Professor Slaught reported that Chancellor A. B. Chace, of Brown University, had promised a contribution of $200.00 for 1922 toward the expenses of the Association. This was accepted with thanks and Professor Slaught was authorized to formulate a suitable reply to be sent to him and to be filed in duplicate with these minutes.

A revised report was made by the Association's Committee on Life-Membership, recommending that the life-membership fee be fixed at the exact value—at the attained age of the member—of the dues payable during his lifetime according to the McClintock 4% Male Table, using a deduction of 5% as the probable saving in collection expenses. The question was raised whether so complicated conditions should be made a part of the By-Laws, and a modification of the amendment proposed earlier, whereby the membership fee should be $105 - x$ dollars, was referred to the Association's committee for approval or suggestions.

A committee was appointed to consider the time and place for the summer meeting in 1922; also a committee to consider the general policy with reference to the Association library. Other matters of current business were transacted by the outgoing Board and the new Board for 1922 held an organization meeting after the election of new members by the Association on Friday.

Annual Business Meeting of the Association.

The Secretary-Treasurer announced the names of those elected to membership by the Board. He reported also the death of the following members:

Paul Arnold, Professor of mathematics, University of Southern California (February 24, 1921);
J. E. Clark, Professor of mathematics, Emeritus, Yale University (January 3, 1921);
N. F. Davis, Professor of mathematics, Emeritus, Brown University (May 17, 1921);
A. M. Kenyon, Professor of mathematics, Purdue University (July 27, 1921);
H. R. Park, Teacher, Junior College, Riverside, California (November 12, 1919);
Alexander Pell, Bryn Mawr, Pa. (January 26, 1921);
Anna I. Young, Professor of mathematics, Agnes Scott College (September 30, 1920).

The election of officers for the year 1922 was conducted by mail and in person at this meeting, as provided in the By-Laws. The tellers (C. S. Atchison and A. D. Pitcher) reported the result of the balloting, 437 ballots having been cast, 2 of these for names not nominated on the official ballot:
For President: R. C. Archibald, 292 votes; J. L. Coolidge, 143 votes.
For Vice-Presidents: W. H. Bussey, 199 votes; R. D. Carmichael, 317 votes; B. F. Finkel, 240 votes; Paul Saurel, 89 votes.
For additional members of the Board of Trustees (to serve until January, 1925): Clara L. Bacon, 181 votes; L. P. Eisenhart, 253 votes; E. V. Huntington, 277 votes; W. D. Lambert, 190 votes; D. N. Lehmer, 232 votes; Helen A. Merrill, 214 votes; G. A. Miller, 269 votes; W. R. Ransom, 121 votes.
For member of the Board to serve until January, 1924, in the place of Professor Bennett (appointed Editor-in-Chief): Dunham Jackson, 187 votes; E. J. Wilczynski, 243 votes.

The following were accordingly declared elected:
President: R. C. Archibald, Brown University.
Vice-Presidents: R. D. Carmichael, University of Illinois; B. F. Finkel, Drury College.
Additional members of the Board of Trustees: L. P. Eisenhart, Princeton University; E. V. Huntington, Harvard University; D. N. Lehmer, University of California; G. A. Miller, University of Illinois.
For member of the Board to serve until January, 1924: E. J. Wilczynski, University of Chicago.

The Secretary-Treasurer made his financial report for the year, giving an account of all the business transacted for the Association up to December 6th, 1921. The report of the Auditing Committee (C. H. Yeaton, H. E. Slaught, and Mrs. Anna J. Pell, chairman) was then made. The financial report is printed in full below.

Report of the Secretary-Treasurer as Treasurer, Dec. 6, 1921.

Receipts.

Balance Dec. 2, 1920		$3,781.76
1920 indiv. dues	$ 131.80	
1920 instit. dues	32.50	
1921 indiv. dues	4,394.76	
1921 instit. dues	564.60	
1921 subscriptions	696.03	
Contributions to 1921 expenses	202.60	
Initiation fees	322.00	
Sale copies of MONTHLY	96.98	
Advertising	493.00	
Register	2.50	
Interest State Savgs. Bk.	74.91	
Interest Peoples Bk.	49.15	
Interest Liberty Bonds	45.00	
Total 1921 receipts		$7,105.83
Total assets to the end of 1921 business		$10,887.59
Total expenditures		$7,445.71
Balance to the end of 1921 business		$3,441.88
Received on 1922 and later business		572.10
Book balance Dec. 6, 1921		$4,013.98

Expenditures.

Publisher's bills (Nov. '20–Sept. '21)		$4,688.94
President's office		60.75
Manager's office		25.24
Editor-in-Chief's office		1,198.46
Other editors' postage		7.76
Secretary-Treasurer's office:		
Postage	$193.52	
Bond	5.00	
Office supplies	34.92	
Express, tel., etc.	12.23	
Clerical work	308.00	
Part expenses Register	27.00	
Printing	546.83	
Chicago annual meeting	55.73	
Wellesley meeting	73.84	
Paid copies of MONTHLY	6.25	
Paid to sections from initiation fees	51.24	
		$1,314.56
Annals subvention		150.00
Total expenditures		$7,445.71
Cash on hand		13.05
Checking account		83.66
State Savgs. Bk. Co. account		1,810.24
Peoples Bkg. Co. account		1,107.03
Liberty Bond		500.00
Victory Bond		500.00
Bank balance Dec. 6, 1921		$4,013.98

When the accounts were closed on December 6, 1921, in order to furnish the auditing committee a complete record, there remained on the total business for the year 1921, the following items:

Bills Receivable.

1921 dues unpaid	$150.00
Advertising	100.00
Interest Liberty Bonds	10.00
	$260.00

Bills Payable.

(Either paid in December or estimated.)

Publisher's bills (Oct.–Dec.)	$1,500.00
Manager's office	30.00
Editor-in-chief's office	80.00
Other editors' postage	20.00
Secretary-treasurer's office	190.00
Annals subvention	100.00
Init. fees due to sections	100.00
Printing annual ballots, programs, etc.	150.00
	$2,170.00

If to the balance on 1921 business shown in this report, $3,441.88, there be added the amount of bills receivable, $260, and there be subtracted the esti-

mated amount of bills payable, \$2,170, there results an estimated final balance on 1921 business of approximately \$1,530. The corresponding estimated final balance one year ago on 1920 business was \$1,360. The apparently better showing of \$170 in 1921 over 1920 is more than accounted for by the "special contributions" of \$202.60 in 1921; thus the Association *has just been able to match expenditures with income.* It is only by the most careful attention to the affairs of the Association, by soliciting additional advertising, by repeated appeals to part of our members to pay their dues, and by similar time-consuming efforts, that the officers are able to make this showing. A greater margin of safety must be attained and this can be done through the coöperation of all members, especially in seeking steadily for new members.

W. D. CAIRNS, *Secretary-Treasurer.*

SERRET'S ANALOGUE OF DESARGUES'S THEOREM.

By H. S. WHITE, Vassar College.

In the course of a discussion of the eight points that lie on three quadric surfaces but not on any gauche cubic curve, P. Serret announced (*Géomêtrie de Direction*, 1869, page 325) an extension of Desargues's theorem concerning a quadrangle inscribed in a conic. This extension states an involutional property of a complete hexagon whose six vertices lie on a gauche cubic. Such a hexagon has 20 plane faces, or 10 pairs of opposite plane faces, each pair containing all six vertices. Any chord of the gauche cubic meets the cubic in points P and Q, and is cut by the 20 planes in 10 pairs of points. The theorem is: *The 10 pairs of points lie in a quadric involution, and P, Q form an 11th pair in the same involution.*

The proof, being inclusive of other things, is unnecessarily long for this particular purpose, and it is worth while to notice that the ordinary synthetic proof of the theorem on conics can be applied with very little change to this extension. Indeed, the only change required is to use pencils of planes through two chords of a cubic curve in place of the flat pencils of lines through two points of a conic.

Let P and Q denote two points on a gauche cubic curve. Six other points shall be called A, A_1, B, B_1, C, C_1. Consider the curve as generated by three projective pencils of planes, two of which have for axes the chords AA_1 and BB_1 respectively. Four pairs of corresponding planes will be

$$\begin{Bmatrix} AA_1P \\ BB_1P \end{Bmatrix}, \begin{Bmatrix} AA_1Q \\ BB_1Q \end{Bmatrix}, \begin{Bmatrix} AA_1C \\ BB_1C \end{Bmatrix}, \begin{Bmatrix} AA_1C_1 \\ BB_1C_1 \end{Bmatrix}.$$

Permute the four planes through BB_1 from the order $PQCC_1$ to the projective order QPC_1C. Then take sections of the two sets of planes by the chord PQ.

On this chord are found two projective quadruples of points:

$$\begin{array}{llll} P,\ Q, & \text{one in } AA_1C, & \text{one in } AA_1C_1; \\ Q,\ P, & \text{"} \quad \text{"}\ BB_1C_1, & \text{"} \quad \text{"}\ BB_1C. \end{array}$$

Owing to the double correspondence of P and Q, these are three pairs in involution. Aside from P and Q, two points of a pair are found in two planes determined by mutually exclusive triads taken from the given 6 points.

As any two chords can be chosen for axes, we may use AA_1 and B_1C_1. The resulting tetrads of points on PQ are then:

$$\begin{array}{llll} P,\ Q, & \text{one in } AA_1C, & \text{one in } AA_1B; \\ Q,\ P, & \text{"} \quad \text{"}\ B_1C_1B, & \text{"} \quad \text{"}\ B_1C_1C. \end{array}$$

From the first three pairs it is found that this involution is identical with the former, hence the fourth pair lies in the same involution. That fourth pair, in planes AA_1B and CC_1B_1, is determined like the former pairs by two mutually exclusive (or supplementary) triads of points in the given sextette. By repetition of this permutation process we can show that any desired pair of supplementary triads give, on the chord PQ, a pair in this same involution. In all, there are ten such pairs of planes, giving ten pairs of points in involution in addition to the pair P, Q.

As Serret points out, it is sufficient to show that the eight faces of a simple octahedron on the six points cut a line in four pairs of points in involution, and from this can be inferred the remainder. Hence there may be written down two equations in line-coördinates which are satisfied by chords (double secants) of a gauche cubic through the six vertices of the octahedron.

A CERTAIN TWO-DIMENSIONAL LOCUS.

By J. L. WALSH, Harvard University.

The writer has recently had occasion to consider the following problem, in connection with the approximate determination of the roots of certain types of polynomials:[1] If two points z_1 and z_2 have as their respective loci the interiors (boundaries included) of two circles, determine the locus of a point z given by the relation $z = (m_2z_1 + m_1z_2)/(m_1 + m_2)$, when m_1 and m_2 are real or complex constants. A closely allied problem is found by considering the locus of the point z determined as before, but where in addition m_1 and m_2 also vary, and take all

[1] See several papers already published and others about to be published in the *Transactions of the American Mathematical Society*.

The following interpretation can be given to the problem of the present note: Let the points $\alpha_1, \alpha_2, \cdots \alpha_n$ vary independently and have as common locus the interior and boundary of a circle C_1, and let the points $\beta_1, \beta_2, \cdots \beta_n$ vary independently and have as common locus the interior and boundary of a circle C_2; determine the locus of the roots of the equation

$$(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n) = A(z - \beta_1)(z - \beta_2) \cdots (z - \beta_n)$$

where A takes all the values such that $|A| = 1$.

values such that the relation $|m_1| = |m_2|$ is satisfied.[1] Otherwise expressed, the problem is to determine the locus of a point z which is equidistant from points z_1 and z_2 varying as described. It is the purpose of this note to present a solution of that problem; the answer is contained in the

THEOREM. *Let the loci of two points z_1 and z_2 be, respectively, the interiors (boundaries included) of two circles C_1 and C_2. Then the locus of a point z, equidistant from z_1 and z_2, is either the entire plane or the exterior of a certain hyperbola whose foci are the centers A_1 and A_2 of C_1 and C_2, according as the loci of z_1 and z_2 have or have not a point in common.*

We understand by the *exterior* of a hyperbola the region of the plane bounded by and lying between the two branches of the hyperbola, including the points of the hyperbola itself. The *locus* of z is taken to consist of all points equidistant from two points z_1 and z_2 satisfying the given conditions.

If C_1 and C_2 are not entirely external to each other—that is, if the loci of z_1 and z_2 have a point in common—the two points z_1 and z_2 may be chosen to coincide. Then any point of the plane is equidistant from z_1 and z_2 chosen at this common position, so every point of the plane is a point of the locus.

If the loci of z_1 and z_2 have no common point, C_1 and C_2 are mutually external. There are some points of the plane, such as on the perpendicular bisector of the segment A_1A_2, which belong to the locus of z. There are some points, such as on the line A_1A_2 but not on the segment A_1A_2, which do not belong to that locus; for *all* the points z_1 (or z_2) are nearer to one of these latter points than *any* point z_2 (or z_1). We must determine the boundary separating the points z of the locus from the points not of the locus.

Let z' be a point on the boundary of the locus of z and be equidistant from points z_1' and z_2' in their proper loci. Then z_1' must be actually on C_1, for otherwise we could move z_1 all over a small area interior to C_1 surrounding z_1', and we could consider z to be determined from z_1 and z_2' so that the triangle z_1zz_2' remains constantly similar to the triangle $z_1'z'z_2'$. Then z would move over a small area completely surrounding z', every point of this small area would be a point of the locus of z, and hence z' could not be on the boundary of that locus. We know, then, that z_1' and z_2' must lie on C_1 and C_2 respectively.

Moreover, z_1' and z_2' must lie on C_1 and C_2 in such a way that z_1' is the point of C_1 nearest to z' and z_2' the point of C_2 farthest from z', or so that z_1' is the point of C_1 farthest from z' and z_2' the point of C_2 nearest to z'. Thus, if z_1' satisfies neither of these conditions, there can be chosen a point z_1'' *interior to* C_1 but such that the distances $z'z_1''$ and $z'z_1'$ (and hence $z'z_1''$ and $z'z_2'$) are equal, so by the reasoning already given z' is not on the boundary of the locus of z. If z_1' and z_2' are the points of C_1 and C_2 farthest from z', there are two points z_1'' and z_2'' interior to C_1 and C_2, respectively, and on the lines $z'z_1'$ and $z'z_2'$ which are

[1] If we consider the allied problems using the relations $|m_1/m_2| = \rho$, a constant, or $(m_1/m_2)/|m_1/m_2| = e^{i\phi}$, a constant, we are led in the first case to a locus which is a doubly connected region bounded by a quartic, and in the second case to a simply connected region bounded by a curve of the eighth degree. The precise equations of these boundaries may easily be found by the methods of this note.

equidistant from z', so z' is not on the boundary of the locus of z. Similarly we may show that z_1' and z_2' cannot be the points of C_1 and C_2 nearest to z', so z_1' and z_2' must satisfy the conditions stated.

The points z', z_1', A_1 are collinear and similarly the points z', z_2', A_2. The point A_1 is on the segment $z'z_1'$ if and only if A_2 is not on the segment $z'z_2'$. The distances $z'z_1'$ and $z'z_2'$ are equal by hypothesis, so the distances $z'A_1$ and $z'A_2$ differ by the sum of the distances A_1z_1' and A_2z_2', that is, by the sum of the radii of C_1 and C_2. Then z' lies on the hyperbola whose foci are A_1 and A_2 and whose "constant difference" is the sum of the radii of C_1 and C_2. The locus of z is not the entire plane and therefore has a boundary; the locus contains the perpendicular bisector of A_1A_2 but no point of the line A_1A_2 not on the finite segment A_1A_2. Hence this locus must be bounded by the entire hyperbola and is the exterior of the hyperbola. This completes the proof of the theorem.

Denote by B_1' and B_1'' and by B_2' and B_2'' the intersections of the line A_1A_2 with C_1 and C_2, respectively, determined so that B_1' separates B_1'' and B_2'' but B_2' does not separate B_1'' and B_2''. The hyperbola cuts the line A_1A_2 at the midpoints of the segments $B_1'B_2'$ and $B_1''B_2''$. The reader will easily prove that the asymptotes of the hyperbola are the perpendicular bisectors of the transverse tangents to C_1 and C_2.

For the problem just considered, the point z' can never lie on the segment $z_1'A_1$; otherwise z_2' would be within C_1, and C_1 and C_2 are supposed to be mutually external. But if we modify our problem by assigning to z_1 as locus the region of the plane *exterior to* C_1, and if the locus of z is not the entire plane, the point z' always lies between A_1 and z_1'. The locus of z can be shown to be the region of the plane exterior to a certain ellipse whose foci are A_1 and A_2.

If we modify our problem by assigning to z_1 as locus a *half plane*, while the locus of z_2 remains the interior of C_2, the locus of z is either the entire plane or the exterior of a parabola whose focus is A_2 and whose directrix is parallel to the boundary of the locus of z_1.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

20. Babbage Visits Mme. Laplace.

Sir John Herschel, speaking of the status of mathematics and astronomy in Great Britain at the opening of the nineteenth century, remarked that "Mathematics were at the last gasp, and Astronomy nearly so." It was for this reason that he, in conjunction with George Peacock and Charles Babbage, formed the so-called "Analytical Society", the purpose of which was to introduce into Cambridge the Continental type of the calculus and, in general, to revivify the mathematics of England. The same three scholars were influential in establishing the Astronomical Society of London, and each was among the leaders in other efforts of a similar nature, one of Babbage's most important papers being entitled "Reflections on the Decline of Science in England" (1830).

Although we commonly think of Babbage with relation to his calculating engine (part of which, curiously enough, found its way to the Dudley Observatory at Albany), he held the chair of Lucasian professor of mathematics at Cambridge, and contributed worthily to the science of astronomy, to higher algebra, and to physics.

In 1840 he was in Paris, and M. Alexis Bouvard (1767–1843) took him to see the widow of Laplace. Bouvard, who became connected as astronomer with the national observatory at Paris in 1793, had assisted Laplace in his computations for the *Mécanique Céleste*, and was therefore in friendly relations to the latter's household. Bouvard was then seventy-three years old, while Babbage was only about fifty.

As Babbage was leaving Paris he wrote to M. Bouvard a letter of appreciation and thanks, acknowledging the gift of a portrait of Laplace which had long been in Bouvard's study. The letter, now in my collection, shows something of the personal side of Babbage and his natural courtesy on such an occasion, and is as follows:

(M. Alexis Bouvard.)
À Paris

My dear Sir:

I cannot leave Paris without thanking you for the very delightful day I spent with you in the society of Madame de Laplace. I shall preserve with the respect it deserves the portrait of her illustrious husband and if any circumstance could have rendered the gift more valuable it is the fact that it has so long adorned the study of that chosen friend by whose unremitted labors the Mecanique Coeleste received such valuable assistance.

I am My Dear Sir
with the greatest respect and Regard
Ever very sincerely Yours,
C. Babbage

Paris 2 Sep. 1840

P.S. I enclose two copies of the drawing of the Engine of Differences on [*sic*] to replace your own and one which I beg your nephew to do me the favour to accept.

21. De Morgan and the Libri Controversy.

If ever there was an eccentric genius in mathematics, and one who scattered his energies so recklessly as to render notable success in any one line impossible, that man was Augustus De Morgan (1806–1871). He did a great deal of good by means of his articles in the *Penny Cyclopedia*, in the *British Almanac and Companion*, and in Smith's *Dictionary of Greek and Roman Biography* (with some curious errors), and also by means of his works on the calculus, logic, double algebra, insurance, and trigonometry, and he gave the mathematical world much food for pleasant thought in his *Budget of Paradoxes;* but at his best he was an eccentric man. He stood rigidly for principle, which the cynic might say was in itself an evidence of eccentricity, and it must be confessed that some of his acts seem to show that he was such a slave to his principles as to justify such a cynic's belief.

An evidence of this eccentricity, perhaps excusable under the particular circumstances, may be seen in the letter which is given below, and which is now

among my autographs. There is nothing to show to whom it was written, but it refers to one phase of the famous Libri controversy which aroused great interest and very bitter feeling at the beginning of the second half of the nineteenth century. Libri, the well-known historian of mathematics, had been accused of stealing many of the books that made up his extensive library. The French courts had, in his absence, convicted him of the crime. He had fled to London and had become an intimate friend of the De Morgans, and this letter shows the faith that De Morgan himself had in the innocence of the accused. It also shows the esteem in which De Morgan held Chasles, the French geometer, and the lengths to which a question of principle could carry him in what was, as regards the scientists of the time, an international dispute.

The letter, which has not before been published to my knowledge, is as follows:

7 Camden Street
March 18/54

My dear Sir:

I am fully of opinion that M. Chasles has high claims on the Royal Society—But, though I am aware it is useless, I for one, will never give an opinion on his claims except in connexion with those of another, whose claims I hold still higher—I mean *Libri*. What I should say of Chasles as an historian, I should say of Libri in a higher degree, and both are truly original mathematicians. I am well aware that though the malignant accusations made against Libri have been refuted in a way in which charges very seldom are refuted—and though the evidence of determination to prevent his having a fair trial is as clear as any evidence can be—yet the power of ignoring notorious facts which bodies of men possess in their collective capacity,—and the disposition of our country to acknowledge all results of foreign law, however little it may agree with our own in fairness of procedure—would make it useless to propose Libri to the R.S. I myself should not entertain a sufficiently high opinion of any English Society to venture on such a step.

This being understood, and speaklg [*sic*, apparently for "speaking"] of Chasles absolutely—I should describe him as one of the very few mathem^{ns}—and the only one of French birth—who attend to the history of the science. In this subject he is a man of real learning—deep in original sources. His *Aperçu* &c is but one of his contributions—and all of them throw new light on the older history of geometry & arithmetic.

In mathematics—as you know—he is among the foremost of those who have developed the powers of geometry to that extent which would have left algebra behind as an instrument—if algebra itself had not received new developments in the mode of applying it to geometry.

Chasles, I have no doubt, has been underrated by his countrymen. In truth, who was there in France (except Libri) to appreciate him? But if ever history and geometry revive in that country—Chasles will be looked upon as the founder of a school.

Chasles should have been elected years ago—He is now in the Institute—You might have given his passport—you can now only *viser* it—This is the great fault of our medals, associateships &c—and springs partly from moral fear, partly from it being nobody's business to form an independ^t opinion on reputations.

Yours very truly,
A. De Morgan

QUESTIONS AND DISCUSSIONS.

Edited by C. F. Gummer, Queen's University, Kingston, Ont., Canada.

REPLIES.

Of the two replies to Question 44, that of Professor Gilman contains a solution of the problem proposed, in the shape of a simple formula for the co-factors in the given "binomial" determinant.

Miss Hazlett does not consider the simplification of the co-factors in the particular case, but establishes a certain general formula for the co-factors in a determinant whose elements are all zero to the right of the principal diagonal. This is essentially the same as evaluating a determinant in which all elements to the right of the principal diagonal are zero with the exception that those in the partial diagonal next to the principal diagonal are all unity. Particular determinants of this class are continuants[1] and the expressions for the sums of like powers in terms of elementary symmetric functions.[2]

44. (*1921*, 260). Is there any known formula for the co-factor of the element a_{ij} in the "binomial" determinant here shown?

$$\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ C_1^2 & 1 & 0 & \cdots & 0 & 0 \\ C_2^4 & C_1^4 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ C_n^{2n} & C_{n-1}^{2n} & C_{n-2}^{2n} & \cdots & C_1^{2n} & 1 \end{vmatrix}.$$

Calling the co-factors A_{ij}, it is plain that $A_{ii} = 1$, and $A_{ij} = 0$, $(i > j)$. The co-factors of the zero elements, $(i < j)$, are undoubtedly expressible in a formula; the writer would like to know if any reader of the MONTHLY has met with such a formula anywhere.

I. Reply by R. E. GILMAN, Brown University.

Solving for $\cos (2j - 2)\theta$ the system of equations[3]

$$\begin{aligned} \tfrac{1}{2} \cos^0 \theta &= \tfrac{1}{2} \\ 2 \cos^2 \theta &= 1 + \cos 2\theta \\ 2^3 \cos^4 \theta &= 3 + 4 \cos 2\theta + \cos 4\theta \\ &\cdots\cdots\cdots \\ 2^{2n-1} \cos^{2n} \theta &= \tfrac{1}{2}C_n^{2n} + C_{n-1}^{2n} \cos 2\theta + \cdots + \cos 2n\theta, \end{aligned}$$

we have

$$\cos (2j - 2)\theta = A_{1j}(\tfrac{1}{2} \cos^0 \theta) + A_{2j}(2 \cos^2 \theta) + \cdots.$$

Hence A_{ij} is the coefficient of $(2 \cos \theta)^{2i-2}$ in the expansion of $2 \cos (2j - 2)\theta$ in powers of $\cos \theta$, which may be written[4] for $j > i$

$$A_{ij} = (-1)^{j-i} \frac{(2j-2)(j+i-3)!}{(j-i)!(2i-2)!}.$$

II. Reply by OLIVE C. HAZLETT, Mount Holyoke College.

It is not difficult to derive a set of formulæ for the co-factors, A_{ij}, of any element, a_{ij}, of the determinant

$$a = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & a_{k2} & a_{k3} & \cdots & 1 \end{vmatrix}.$$

As the correspondent remarks, $A_{ii} = 1$ and $A_{ij} = 0$ when $i > j$. For the cases when $i < j$, a formula can be proved by induction.

The proof of this last is based on two facts: (1) the co-factor of a_{ij} (when $j \leqq k$, the order of the determinant) is independent of k; (2) the product of the matrix a by A' (the transpose of the matrix, A, of the co-factors A_{ij}) is the unit matrix, I. Statement (2) is well known, and (1) follows from the fact that all elements to the right of the main diagonal are zero. Accordingly, we first find a formula for A_{12} for the determinant of general order k by finding A_{12} when the

[1] L. G. Weld, *The Theory of Determinants*, New York, 1893, p. 178.

[2] L. E. Dickson, *Elementary Theory of Equations*, New York, 1914, p. 169.

[3] Loney, *Plane Trigonometry*, Cambridge, 1893, Part II, p. 342.

[4] *Ibid.*, p. 349.

order of the determinant is 2. Then we consider the determinant, a, of order 3 and use the formula for A_{12} just found. The necessary and sufficient conditions that $aA' = I$ give us the formulæ for A_{13} and A_{23}. Next we consider the determinant a of order 4 and thus derive formulæ for A_{14}, A_{24} and A_{34}. Finally, we proceed as in a proof by induction.

From the determinant a of order 2, we see that $A_{12} = -a_{21}$. Substituting this in the determinant a of order 3 (as we have a right to do, by statement (1)), we have

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -a_{21} & 1 & 0 \\ A_{13} & A_{23} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

But this is true if and only if

$$\begin{cases} 0 = a_{31}A_{11} + a_{32}A_{12} + a_{33}A_{13}, \\ 0 = a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23}. \end{cases}$$

Hence

$$\begin{cases} A_{13} = -a_{31} + a_{32}a_{21}, \\ A_{23} = -a_{32}. \end{cases}$$

Substituting these formulæ for A_{12}, A_{13} and A_{23} in A', formed for the determinant a of order 4, we find that the necessary and sufficient conditions that $aA' = I$ are

$$0 = a_{41}A_{i1} + a_{42}A_{i2} + a_{43}A_{i3} + a_{44}A_{i4} \quad (i = 1, 2, 3);$$

and hence

$$\begin{cases} A_{14} = -a_{41} + a_{42}a_{21} + a_{43}a_{31} - a_{43}a_{32}a_{21}, \\ A_{24} = -a_{42} + a_{43}a_{32}, \\ A_{34} = -a_{43}. \end{cases}$$

Now proceed by induction. Assume that, for $f \leqq l$, it is true that whenever $g < f$,

$$A_{gf} = -a_{fg} + \Sigma a_{fi}a_{ig} - \Sigma a_{fi_1}a_{i_1i_2}a_{i_2g} + \cdots + (-1)^{f-g}a_{fi_1}a_{i_1i_2}a_{i_2i_3}\cdots a_{g+1,g},$$

where the summation signs are to be interpreted in a special way. The first summation, $\Sigma a_{fi}a_{ig}$, denotes the sum of all terms of the type indicated for which $f > i > g$. In the next summation, $f > i_1 > i_2 > g$. The third summation, which is not written above, is $\Sigma a_{fi_1}a_{i_1i_2}a_{i_2i_3}a_{i_3g}$ and means the sum of all terms of the type indicated for which $f > i_1 > i_2 > i_3 > g$. Similarly with all the summation signs. In each succeeding summation there is an additional factor in each term. If there is an odd number of factors, a_{st}, in a term, the sign to be used is "minus"; otherwise, "plus." In the last summation, there is necessarily only one term and in this case $i_1 = f - 1$, $i_2 = f - 2$, and more generally $i_r = f - r$.

Now let the order of the determinant a be $l + 1$. Since the elements of a and A' outside of the last row and column are the same as in the case of the determinant of order l and in both the elements of the last column are 0, 0, $\cdots$, 1, it follows from our assumption that the product aA' is just like the unit matrix of order $l + 1$ in its first l rows, and to make the last row the same it is only necessary that the l equations

$$0 = \sum_{f=1}^{l+1} a_{l+1,f}A_{gf} \quad (g \neq l + 1)$$

be satisfied. Substituting in these equations the expressions for the A_{gf} in which $f \leqq l$, we derive formulæ for the $A_{g,l+1}$.

It is not hard to see that the formula for $A_{g,l+1}$ is of the same form as those for A_{gf} $(f \leqq l)$. First, every term is a product of a number of a_{ij}'s of the general form

$$a_{l+1,i_1}a_{i_1i_2}\cdots a_{i_{s-1}i_s}a_{i_sg}, \tag{I}$$

in which $l + 1 > i_1 > i_2 > \cdots > i_s > g$. This follows from the fact that every term comes from a product $a_{l+1,f}A_{gf}$ and from the fact that every term of A_{gf} $(f \leqq l)$ is a product of a's of the form

$$a_{fj_1}a_{j_1j_2}\cdots a_{j_{t-1}j_t}a_{j_tg}, \tag{II$_f$}$$

in which $f > j_1 > j_2 > \cdots > j_t > g$. Moreover, we get every term of the type I in which $s = 1, 2, \cdots, l - g$, since each A_{gf} contains every term of the type II$_f$ for which $t = 1, \cdots, l - g - 1$. Also, we get no term twice. Finally, the signs are as they should be, for in the expression for $A_{g,l+1}$ each A_{gf} $(f \leqq l)$ is multiplied by an $a_{l+1,f}$ and the sign of the product is changed. Thus the induction is complete.

III. Note by the Editor.

Miss Hazlett's formula for the co-factor of a_{ij} $(i < j)$ in the determinant a is also readily obtained as follows:

The system of equations

$$\begin{array}{ll} x_1 & = y_1 \\ a_{21}x_1 + x_2 & = y_2 \\ a_{31}x_1 + a_{32}x_2 + x_3 & = y_3 \\ \cdots\cdots\cdots\cdots & \cdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + x_k & = y_k \end{array}$$

has the unique solution $x_j = \Sigma_i A_{ij} y_i$, $j = 1, \cdots, k$.

But we also find by solving the equations in succession $x_1 = y_1$, $x_2 = y_2 - a_{21}y_1$, $x_3 = y_3 - a_{31}y_1 + a_{32}a_{21}y_1 - a_{32}y_2$, etc. Hence $A_{12} = -a_{21}$, $A_{13} = -a_{31} + a_{32}a_{21}$, $A_{23} = -a_{32}$, etc., which agree with Miss Hazlett's formula; and, if the formula is assumed to hold for $i < j < n$, it is easy, by solving the nth equation, to extend it to $j = n$.

DISCUSSIONS.

In the first of the following discussions Mr. Webb presents a method of deriving the addition formulæ of plane trigonometry in which the various functions involved appear as line-segments rather than as the ratios of sides of right triangles. The author maintains that certain advantages in the way of clearness and concreteness result from this treatment, based on the idea of "arc in unit circle," and not of "angle." It is hardly necessary to point out that the ancient preference for the arc as independent variable is by no means out of date, and is favored particularly by French writers. A well-known example is Serret's treatise,[1] in which the properties of the functions are developed in connection with the arc, and the definition of angle (as the measure of the arc in a unit circle) only introduced on page 92 in preparation for the work on the triangle. On the whole, modern teachers are disposed to lay more stress on the relation to similar figures, to encourage the sense of angle as difference in direction of lines (independent of the scale of measurement), and to regard the graduated circle, of any convenient radius, simply as a useful device for the measurement of angles at the center. This is perfectly natural, since it is seldom the arc of the circle in which we are really interested, and the idea of angle, whatever the geometric basis, has a certain imaginative quality not to be found in the arc of the one-inch circle. Thus the consideration that the short hand of the clock turns twice as quickly as the earth takes on a peculiarly unsuggestive aspect if we are required to measure the two angular speeds on the same circle, whether of three inches or of four thousand miles.

It must be remembered however that the difference between the view of an angle as arc in unit circle and the view of it as ratio of arc to radius disappears entirely when we consider that the unit adopted is perfectly arbitrary, and that in fact one of us is comparing the arc with the radius and the other is comparing it with a certain portion of the foot-rule used in constructing the figure.

It may be well to add that Mr. Webb's paper is not primarily an argument in favor of the "arc method," but is more particularly a method of presenting the addition formulæ.

[1] J.-A. Serret, *Traité de Trigonométrie*, Paris, 1908.

In the second discussion Professor Reynolds outlines a method for teaching the mathematics of investment without the use of the geometric series. In view of the recommendation of the National Committee on Mathematical Requirements that this subject should be studied (by election) in the Secondary School, these suggestions will be of interest to others besides college teachers.

I. A Method of Deriving Formulæ for the Expansion of $\sin(x+y)$ and $\cos(x+y)$.

By H. E. Webb, Central High School, Newark, N. J.

For some time past the writer, in beginning with his classes the subject of plane trigonometry, has defined the sine and the cosine of a *number* as lines in a unit circle (the number being measured in linear units on the circumference), while the tangent and cotangent are defined as the ratios of these to each other, and the cosecant and secant as their respective reciprocals. At a later stage these functions are related to ratios between the sides of a right triangle, when the number is less than $\pi/2$, and are regarded as functions of the central angle which intercepts the arc.[1] This procedure has numerous advantages, among which may be mentioned the presentation of a clear idea of the limiting cases, and of the nature of the discontinuity of the four functions last named; an early acquaintance with "radian measure"; an escape from the necessity of two sets of definitions, one for the acute angle and the other for other angles; and the opportunity for presenting clearly the notions of negative angles and of positive angles in excess of 180°.

In line with this general plan, the following development of the expansion of $\sin(x+y)$ and $\cos(x+y)$ is presented, with the apology that it has not, to the knowledge of the writer, previously appeared in printed form.

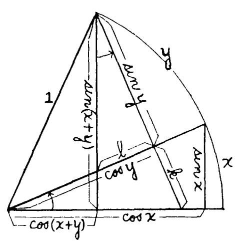


Fig. 1.

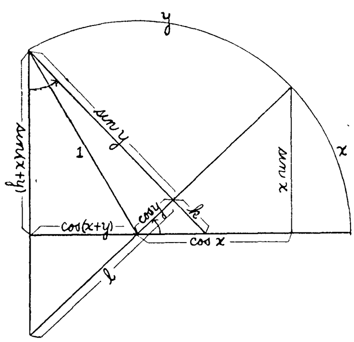


Fig. 2.

The radius of the circle is in each case unity. The notation of the figures is self-explanatory. From similar triangles, two proportions can be set up in each of the cases $\sin(x+y)$ and $\cos(x+y)$. For the case of $\sin(x+y)$,

$$k/\cos y = \sin x/\cos x, \tag{1}$$

[1] This plan follows an outline of the subject by Professor C. O. Gunther, of Stevens Institute of Technology.

and

$$\sin (x + y)/(k + \sin y) = \cos x/1. \tag{2}$$

These are combined into the equation

$$\begin{aligned}\sin (x + y) &= \cos x \left(\frac{\sin x \cos y}{\cos x} + \sin y \right) \\ &= \sin x \cos y + \cos x \sin y.\end{aligned}$$

Also

$$l/\sin y = \sin x/\cos x, \tag{3}$$

and

$$\cos (x + y)/(\cos y - l) = \cos x/1; \tag{4}$$

whence

$$\begin{aligned}\cos (x + y) &= \cos x \left(\cos y - \frac{\sin x \sin y}{\cos x} \right) \\ &= \cos x \cos y - \sin x \sin y.\end{aligned}$$

In figure 2, since $\cos (x + y)$ is negative and $\cos y$ is less than l, the relation (4) still holds.

The development of $\sin (x - y)$ is obtained by substitution as

$$\cos (\pi/2 - \overline{x - y}) = \cos (\overline{\pi/2 - x} + y),$$

and so for $\cos (x - y)$.

It is assumed above that x and y are in all cases positive and less than $\pi/2$. If either or both are greater, the functions can be transformed by substitutions for x of $\pi \pm x'$ or of $2\pi - x'$, or for y of $\pi \pm y'$ or of $2\pi - y'$, obviating the necessity for any geometric proofs other than those given, to make the formulæ general.

These proofs have the advantage of showing graphically the relative values of the various functions involved.

REMARKS BY A. A. BENNETT.

The Greeks had a strong hold of the notion of what the modern physicist has rediscovered under the title of "dimensions." For Euclid, the product of two line segments was necessarily measured in terms of area. The physical character of the measurements was at every stage so conspicuously in the consciousness that purely arithmetical studies fared ill. The discovery of Analytical Geometry shows, perhaps, the originality of genius more obviously in the following feature than in any other,—Descartes was willing to adopt the consequences of permitting the product of two quantities, each of which was laid off as a segment of a line, to be itself so expressed. In short, he was willing to grant a spatial representation to pure numbers. Since that time, mathematicians have learned much that requires mental contortions. Many obvious theorems are now known to be false and many time-honored terms have been generalized beyond all recognition. Fortunately, however, any convention must stand upon its own merits and

vindicate its right to adoption. Are we prepared to accept a convention that makes trigonometry deal with six fundamental quantities, of which sines and cosines are line segments, tangents and cotangents are pure numbers, and secants and cosecants are the reciprocals of line segments—and consequently of dimension minus unity, whatever that may mean? As for the angles on which the whole discussion is conceivably based, why, they do not appear at all; these are all functions of abstract numbers. Is not the relation, $\sin^2 x + \cos^2 x = 1$, worth preserving, when the symbol 1 stands, as it ought, for a pure number? These harsh remarks are, of course, no reflection on the interest offered by the method.

II. The Derivation of Formulæ in the Mathematics of Investment.

By C. N. Reynolds, Jr., West Virginia University.

In teaching the mathematics of investment most of us base the derivation of the formulæ for annuities upon formulæ concerning geometric series which are but faint memories in the minds of most students. Although this links the students' work with their preparatory school work, personal experience in teaching this subject has convinced me that we should make the mathematics of investment an independent mathematical discipline based upon concepts of its own.

In particular I would suggest the adoption of the following fundamental concepts:

(1) The accumulated value of one dollar at the end of n years;
(2) The discounted value of one dollar due at the end of n years;
(3) The present value of a perpetuity of one dollar;

where money is supposed to be worth $100\,i\%$ per annum compounded annually.

I would suggest that the formulæ for these quantities be made to play the rôle of definitions. The definition of the present value of a perpetuity as $1/i$ could be justified by the fact that if a man deposits $1/i$ dollars in a bank which pays $100\,i\%$ interest per annum, he receives in return a perpetuity of one dollar per annum, provided that he and his heirs draw the interest from the bank as soon as it falls due. Similarly the other formulæ could be justified.

Having introduced these definitions, an annuity can now be treated as the difference between two perpetuities. For example, the present value of an annuity of n annual payments of one dollar each will be the present value of a one dollar perpetuity minus the discounted value of a perpetuity beginning after n years, *i.e.*,

$$a_{\overline{n}|} = \frac{1}{i} - \frac{1}{i}(1+i)^{-n} = \frac{1-(1+i)^{-n}}{i}.$$

Similarly all the other formulæ of the mathematics of investment can be derived from the three formulæ mentioned above.

RECENT PUBLICATIONS.

REVIEWS.

Descartes Savant. By GASTON MILHAUD, late professor at the Sorbonne. Paris, Felix Alcan, 1921. 252 pages.

In the death of Gaston Milhaud, in October, 1918, France lost one of her best-known writers on the history of philosophy, particularly as related to the exact sciences. He had contributed to our knowledge of the origins of Greek science, to the history of the philosopher-geometers of Greece, to a study of scientific thought in the classical and the modern periods, and to various other lines having to do with the development of the human spirit. He also wrote, for various reviews, a number of articles upon the life and works of Descartes, and later arranged and amplified these for publication in book form. The present volume, published posthumously, represents, therefore, all of M. Milhaud's various monographs upon this great scholar, together with the lectures relating to him which he gave at the Sorbonne in 1917 and 1918.

Students of the history of mathematics have heretofore had to depend chiefly upon the biographies written by J. Millet (Paris, 1867), Elizabeth S. Haldane (London, 1905), and C. Adam (Paris, 1910). Valuable as these were, they were not written with the philosophic insight that characterizes the work of M. Milhaud, primarily for the reason that the latter was himself interested in philosophy to the same degree as in mathematics.

Perhaps the most striking topic treated by M. Milhaud is the one which forms the introduction—the question of the sincerity of Descartes. Some further idea of the work may be obtained from the titles of the several chapters, as follows: I. The first scientific ventures of Descartes; II. A "crise mystique" of Descartes in 1619; III. Descartes's work during the winter of 1619–1620; IV. The event which recalls the date of November 11, 1620; V. His works on optics (1620–1629); VI. The problem of Pappus and the analytic geometry (1631); *La Geómētrie* (1637); VII. The quarrel between Descartes and Fermat on the subject of tangents; Descartes and the infinitesimal analysis; VIII. Descartes and the concept of work; IX. Descartes the experimenter; X. Descartes and Bacon; XI. The double aspect of the scientific work of Descartes.

It will be observed, by considering these titles, that M. Milhaud realized the value of arousing the curiosity of his readers. There is just enough of concealment in his captions to entice the student to solve the mystery by reading each chapter in turn. The reader who is not already familiar with M. Adam's treatise will wonder what happened on November 11, 1620, what the dispute was between Descartes and Fermat, what question of sincerity could arise as to Descartes's work, and what the "double aspect" really means. This is a species of literary dexterity that is entirely excusable; indeed, when a man has written really something worth reading, it is a laudable method, and no people are more gifted in its use than the French.

The work is a scholarly production and a welcome addition to the biographies

of the giants upon whose shoulders Newton so generously confessed that he had stood in viewing the universe, to the study of the laws of which he so extensively contributed.

DAVID EUGENE SMITH.

Practical Least Squares. By O. M. LELAND. New York, McGraw-Hill Book Company, 1921. 8vo. 14 + 237 pages. Price $3.00.

The following extracts from the preface will indicate the purpose and plan of this textbook on the subject of Least Squares:

"It is designed particularly for use in short courses of instruction and by engineers and scientists in connection with their private practice. It will not replace the more elaborate treatises on the subject but the author hopes that it will introduce the student directly to the simpler methods of solving the ordinary problems in adjustment.

"The plan of the work is essentially practical. . . . By far the greater number of applications of Least Squares do not require a consideration of the precision of the results nor a knowledge of the mean square or probable errors. Moreover the subject of precision is usually the most troublesome part of the work for the student or the beginner to understand. Therefore, the practical methods of adjustment are explained directly and fully, without regard to the probable errors or to the theoretical derivation of the Law of Error."

As an introductory text, this book may be recommended on the basis of the following excellent features: the material is well graded, the text is illustrated by numerous examples, the explanation is always clear, and is mingled with a judicious amount of suggestion and of warning; in fact, a more simple presentation for the use of private study could not be asked. There are other points that might be suggested in possible criticism. The style throughout suggests the well-delivered lecture, and is not primarily designed for ready reference. There are no theorems so labelled, no isolated definitions, and worst of all, no problems for solution. Except for the language employed and for the gradual development, there is little if anything original in the treatment. One would almost suppose that the subject was as fossilized as elementary trigonometry. The mathematical features of theoretical interest are relegated to the Introduction and to the Appendices, and are treated with so little concern for logic that any critical reader might have preferred their entire omission. Since the book is intended to be available for a brief course of sixteen lessons, the student is not likely to suffer much from the shortcomings of the supplementary remarks upon the theory underlying the subject.

The work is not intended as a contribution to the development of science, and many questions that would lead into larger fields are avoided in connections that would seem to justify at least a hint for the encouragement of the ambitious or inquiring student, but the author has been generous in his references to easily available American texts, and to one or two classical treatises in foreign languages.

ALBERT A. BENNETT.

ARTICLES IN CURRENT PERIODICALS.

Bulletin des Sciences Mathématiques, volume 45, November, 1921: "Sur la dynamique de la relativité" (conclusion) by G. Fontené, 320–339; "Discourse d'ouverture de la sixième conférence générale des poids et mesures, prononcé le 27 septembre 1921, au ministère des affaires étrangères, en présence de M. le Ministre du commerce" by E. Picard, 340–344 ["Messieurs, la science est une arme puissante, indifférente au mal comme au bien, on ne l'a que trop vu pendant quelques années. Notre vœu le plus cher, a nous tous qui sommes ici réunis, est que, rendue à ses fins bienfaisantes, elle ne cesse plus d'être cet outil de merveilleux service, dont parlait déjà notre vieux Montaigne, qui contribue à l'amélioration des conditions de la vie et fournit un des signes les moins contestables des progrès de l'humanité, tout en restant la grande parure dont l'idéal reflète la curiosité passionnée et désintéressée qui est l'honneur de l'esprit humain. Dans maintes recherches scientifiques, l'âge héroique est passé, où avec un matériel très simple on pouvait faire de grandes découvertes. Quoique tout reste possible aux hommes de génie qui, de loin en loin, ouvrent des voies nouvelles avec des moyens de fortune, le progrès scientifique résulte le plus souvent aujourd'hui de longs et patients efforts, qu'il s'agisse de laborieux calculs, ou de minutieuses observations et expériences. . . . Nous ne croyons plus guère au dogme de la simplicité des lois de la nature, qui enchantait nos prédécesseurs et qui a rendu tant de services aux sciences naissantes, tout en l'utilisant cependant encore de façon plus ou moins consciente. Nous accumulons approximations sur approximations, mais un des articles de nôtre foi scientifique est que ces approximations successives sont convergentes, comme disent les mathématiciens, et que nous approchons sans cesse d'un petit nombre de vérités toujours plus compréhensives, synthèses des nombreuses vérités partielles peu à peu découvertes. C'est peut-être une chimère, mais elle soutient des générations de chercheurs dans leur labeur jamais terminé."]—December, 1921: "La théorie de la relativité et ses applications à l'astronomie" by E. Picard, 355–372—Volume 46, January, 1922: "Sur les intégrales de Fresnel" by A. Bloch, 34–35; "A propos d'un nouvelle revue mathématique: *Fundamenta Mathematicæ*" by H. Lebesgue, 35–48 (see *1921*, 317).

[Quotation from last article: "En 1919, au milieu des événements que l'on connaît, il s'est trouvé à Varsovie, un petit groupe de mathématiciens assez hardis pour oser créer un périodique mathématique nouveau, *Fundamenta Mathematicæ*, et assez épris de leur science pour trouver le temps de travailler d'une façon si heureuse que les deux tomes, actuellement parus, des *Fundamenta* sont presque entièrement remplis par leurs seuls Mémoires.

"La rédaction s'exprime ainsi: 'Le journal *Fundamenta Mathematicæ* publie les Mémoires (ou Notes) consacrés à la Théorie des ensembles et les questions qui s'y rattachent. (Les applications immédiates de la théorie des ensembles, *Analysis situs*, la logique mathématique, les recherches axiomatiques.)'

"Cette délimitation sera très diversement appréciée. Elle sera approuvée par beaucoup de mathématiciens et de philosophes, s'intéressant presque exclusivement aux questions traitées dans les *Fundamenta*, et qui seraient très heureux de trouver réunis tous les articles qui s'y rapportent. Mais, s'il en était ainsi, ils n'auraient plus jamais cette bonne fortune de prendre un journal pour y lire un article, de parcourir un autre mémoire par désœuvrement et de trouver ainsi des sujets de travaux ou, ce qui est mieux, d'avoir une occasion de s'instruire. . . .

"Je souhaite vivement que la rédaction interprète son programme de la façon la plus large, que, dorénavant, chaque tome contienne plusieurs articles traitant aussi des 'vieilles' mathématiques afin que tout lecteur soit incité à s'instruire dans des directions variées. Il suffirait, pour cela, qu'on admette en principe tout article contenant une application *quelconque* de la théorie des ensembles, et non plus nécessairement une application *immédiate*, comme le demande le programme. . . .

"Cette petite réforme engagerait sans doute les Auteurs à écrire de façon à être compris de tous, ce dont il ne semble pas qu'ils se soient toujours beaucoup préoccupés. . . .

"Les notations utilisées ici, qui ne sont pas expliquées dans l'article, sont en relation avec celles introduites par Janiszewski dans sa Thèse et qui avaient vivement intéressé Poincaré. Je n'en puis faire un plus grand éloge; cependant, il ne faut pas oublier que tout le monde n'excelle pas, comme Poincaré, à manier tous les symboles et, pour ma part, je n'ai jamais pu réussir à calculer à l'aide des équations de Janiszewski qui ne sont donc, pour moi, qu'une notation qu'il me faut traduire. Comme, à l'époque de sa Thèse, je le disais à Janiszewski; ce jeune homme ne sut pas me cacher qu'il me considérait comme une vieille baderne. Il avait peut-être raison; toutefois, je crois qu'il faut s'efforcer d'être compris facilement, même par les vieilles badernes. Et, pour cela, éviter de multiplier les notations; en tout cas, ne jamais oublier de les expliquer. . . ."]

L'ENSEIGNEMENT MATHÉMATIQUE, volume 22, nos. 1–2 (issued January, 1922): "Applications géométriques de la cristallographie" by M. Winants, 5–29 (to be continued); "Sur le rayon de courbure d'une courbe" by B. Niewenglowski, 30–37; "Sur les séries entières, dont la somme est une fonction algébrique" by G. Polya, 38–47; "Sur le nombre e" by M. Petrovitch, 48–50.

MATHEMATICAL GAZETTE, volume 11, January, 1922: "Solutions to missing-figure problems" by W. E. H. Berwick, 8–9 [see *1921*, 37–38, 278]; "Partial fractions associated with quadratic factors" by E. H. Neville, 10–13; "Relativity rhymes, with a mathematical commentary" partly by H. Piaggio, 22–23 ["A. *The Restricted Theory.*

"Einstein's is a wonderful notion
That a rod will contract when in motion,
All the clocks will go slow,
And yet no one will know!
So the matter need cause no commotion.

"B. *The General Theory, applied to Planetary Motion.*

"If the path of a planet you'd trace,
You've Christoffel's weird symbols to face,
For an orbit, you see,
Is as straight as can be
On a surface in quintuple space."

Additional verse, "*The German Jeweller's Complaint.*

"Einschtein hash a schandaloush notion
About how the universe gosh on.
He shaysh I sha'n't know
If my glocksh all go shlow
And my money'sh shrunk up by itsh motion."];

"A mathematical recreation" by G. A. Miller, 23–24 [About the area of a triangle. See *1921*, 256-258]; Review of F. Cajori's *History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse* by J. M. Child, 26–30.

MATHEMATICS TEACHER, volume 14, November, 1921: "Vectors for beginners" by J. B. Reynolds, 355–361; "A great mathematician as a school boy" by D. E. Smith and Vera Sanford, 362–366; "The formula in ninth grade algebra" by J. M. Kinney, 367–380; "Combined mathematics" by W. P. Webber, 381–386; "Teaching pupils how to study mathematics, II" by A. Davis, 387–400 (to be continued); News and notes, Round-table Discussion, and New books, 401-411—December: "Religio mathematici" by D. E. Smith, 413–426; "An elementary exposition of the theorem of Bernoulli with applications to statistics" by H. L. Rietz, 427–434; "The relative emphasis upon mechanical skill and applications of elementary mathematics" by F. Allen, 435–443; "Mathematics as found in society: with curriculum proposals" by P. M. Symonds, 444–450; "A note on the failure of educated persons to understand simple geometrical facts" by E. L. Thorndike, 451–453; "Teaching pupils how to study mathematics" (continued) by A. Davis, 454–468; News and notes, 469–473—Volume 15, January, 1922: "The human worth of rigorous thinking" by C. J. Keyser, 1–5; "The nature of algebraic abilities" by E. L. Thorndike, 6–15; "The next step in method" by W. H. Kilpatrick, 16–25; "The next step in content in junior high school mathematics" by D. E. Smith, 26–27; "The next step for the administrator in junior high school mathematics" by J. K. Van Denburg, 28–29; "Minimum mathematical requirements for agricultural study" by H. B. Roe, 30–42; "A composite course for seventh and eighth grade mathematics," 43–48 [A report by the Articulation Committee for Mathematics of the Lake Shore Division of the Illinois Teachers' Association]; "The teaching of mathematics: The need and the method" by F. B. Williams, 49–56; New books, 57–58; "The Chicago meeting of the National Council of Teachers of Mathematics," 59–61—February: "Experimental courses in secondary school mathematics" by R. Schorling, 63–78; "The nature of algebraic abilities" (continued) by E. L. Thorndike, 79–89; "The volume of a sphere," 90–92; "The psychology of errors in algebra" by P. M. Symonds, 93–104; "Prescribed versus elective mathematics in junior high schools" by D. Snedden, 105–109; "Certain cases of extraneous roots" by A. S. Hegeman, 110–118; "Arithmetic in the high school" by L. G. Dake, 119–125; "Program of the Chicago meeting of the National Council," 126.

MONIST, volume 31, October, 1921: "Relativity and its philosophic implications" by W. B.

Smith, 481–511 [An address delivered at the New Orleans meeting of the Southern Society of Psychology and Philosophy, April, 1920]; "Criticisms and Discussions: The principles of functional calculus" by M. Winter, 608–634 [Authorized translation, abridged, by F. Rothwell from *Revue de Métaphysique et de Morale*, volume 21, July, 1913, 462–510]; Reviews, 635–638: S. Brodetsky's *A First Course in Nomography* (by J. M. Child), C. E. Weatherburn's *Elementary Vector Analysis*, H. T. H. Piaggio's *Elementary Treatise on Differential Equations and Their Applications* (London, G. Bell & Sons); A. N. Whitehead's *Organization of Thought, Educational and Scientific* (London, Williams & Norgate).

NATURE, volume 108, December 1, 1921: "Currents of mathematical thought" by G. B. M(athews), 427–429 [Review of P. Boutroux, *L'Idéal Scientifique des Mathématiciens: Dans l'Antiquité et dans les Temps Modernes*]; "Relativity and materialism" by H. Elliot, 431; "The tendency of elongated bodies to set in the north and south direction" by E. A. Reeves, 433 [Mentioning experiments that indicate a much stronger force than the gravitation force certain other writers have described. See *1922*, 22,75]; "Relativity: Particles starting with the velocity of light" by E. Kasner, 434–435; "Reflection from cylindrical surfaces" by C. O. Bartrum, 436 [see *1922*, 75]—December 8: "Relativity and materialism" by H. W. Carr and E. McClure, 467; "The Rayleigh memorial," 471–474 [Account of the unveiling of a memorial tablet in Westminster Abbey, with the address by Sir J. J. Thomson]—December 22: "The tendency of elongated bodies to set in the east and west direction" by W. D. Lambert, 528—December 29: "Relativity and materialism" by H. Jeffreys, S. V. Ramamurty, and N. R. Campbell, 568–569; "Next year's total solar eclipse (September 21, 1922)" by W. J. S. Lockyer, 570–571—Volume 109, January 5, 1922: "Fermat's last theorem," 4 [Review by G. B. M(athews) of Mordell's *Three Lectures on Fermat's Last Theorem*]—January 12: Review by W. E. H. B. of M. Cashmore, *Fermat's Last Theorem: Proofs by Elementary Algebra*, 39 ["In view of the considerable erroneous literature concerning Fermat's last theorem it may not be out of place to direct attention to two valuable additions to the correct literature which have appeared since the last edition of Mr. Cashmore's book was reviewed in *Nature*. They are: (1) Mr. L. J. Mordell's 'Three Lectures on Fermat's Last Theorem,' and (2) a chapter in vol. 2 of Prof. L. E. Dickson's 'History of the Theory of Numbers.' "]; "Reform of the calendar: mean value of the year" by A. Rose-Innes, 44 ["If we say that 'a century-year shall be a leap-year only if it gives a remainder of 2 or 7 when divided by 9,' we have a rule which is much more approximate than the Gregorian rule, and one which has been followed *de facto* since 1582 (year of the Gregorian reform). The new rule would not differ in its application from the Gregorian rule before the year 2400. The Gregorian year, 365 97/400 days, differs from the true tropical year by 26 seconds; if the above modified rule were introduced the difference would be reduced to 2 seconds."]—January 19: "Congress of philosophy in Paris: Recent developments of relativity theory," 90.

LA NATURE, volume 49_2, October 29, 1921: "La Conférence générale des Poids et Mesures" by E. Picard, 276–278; "Utilisons la 'Houille Bleue'" by H. Lémonon, 278–283 (to be continued) [Description of some machines invented for using the energy of the sea]—November 5: "La conduite des navires à distance par T. S. F." by S. Jourdan, 292; "La lune est-elle un monde vivant?" by E. Touchet, 293–300 [Resumé of W. H. Pickering's theories as to the existence of mechanical and physical movements and of vegetation on the moon]—December 17: "L'eclipse de Soleil du 21 septembre 1922", supplement, 191; "Le plus gros télescope du monde," supplement, 191—Volume 50_1, January 7, 1922: "Einstein et les théories de la relativité" by A. Troller, 11–12; "La théorie de la relativité" by H. Lafond, 12–15 [The first of a series of articles to be published on this subject]—January 21: "Les bases fondamentales de la théorie de la relativité" by H. Lafond, 35–39—January 28: "Le principe de la relativité" by H. Lafond, 57–60—February 4: "La théorie de la relativité: Quelques résultats et quelques expériences" by H. Lafond, 73–78.

OBSERVATORY, volume 44, November, 1921: "The age of the earth" by Lord Rayleigh, J. W. Gregory, and A. S. Eddington, 325–329; "On the origin of comets" by C. D. Perrine, 329–331; "Elliptical halos" by A. Grace Cook, 334–335; "The interpretation of the experiment of Michelson and Morley," 340–341 [. . . "the eminent physicist, the late Prof. Righi, of Bologna, shortly before his death presented to the Royal Institute of Bologna a couple of memoirs purporting to prove that the generally accepted interpretation of Michelson and Morley's experiment is unsound," . . .]; "Repetition of Michelson-Morley experiment on ether drift," 341 ["In view of the importance of this historic experiment which was originally made in 1887, and repeated by Morley and Miller in 1904–5, the work is being repeated this year with all modern conveniences and refinements."]—December: "Meeting for the discussion of geophysical subjects," 366–368 [Discussion of the "Eötvös Gravity Balance"]; "The Einstein tower," 372–373 [with a photograph as frontispiece].

Popular Astronomy, volume 29, December, 1921: "The glacial period and Drayson's hypothesis" by J. Millis, 608–625 [Accounting for the glacial periods in the earth's history by supposing that the equatorial axis has an eccentric motion about the axis of the ecliptic so that the angle between the two axes varies between $23\frac{1}{2}$ and $35\frac{1}{2}$ degrees in the course of a revolution of the former about the latter]; "Note concerning the total solar eclipse of next year," 665.

Proceedings of the Edinburgh Mathematical Society, volume 39, session 1920–21, November, 1921: "Multiply perspective polygons inscribed in a plane cubic curve" by D. G. Taylor, 2–6; "An analytical treatment of the cam problem" by G. D. C. Stokes, 7–12; "Asymptotic expressions for the Bessel functions and the Fourier-Bessel expansions" by T. M. MacRobert, 13–20; "Some extensions of Pincherle's polynomials" by P. Humbert, 21–24; "Taylor's theorem and Bernoulli's theorem: A historical note" by G. A. Gibson, 25–33 ["It is, from our present standpoint, strange to see how near Newton in particular came to Taylor's Theorem and yet did not actually attain to it; the reason for this is not quite easy to understand but we should at least learn the lesson that it is not safe to credit a writer with the possession of recondite theorems unless on plain evidence. The tendency to read our own ideas into the work of previous writers is just as bad as the opposite tendency of crediting to ourselves what was the possession of our predecessors. Above all, the detestable vice of 'nationalism' in science must be studiously shunned; it would be hard to overstate the loss that British mathematics suffered from the baleful controversy on the invention of the Calculus."]; "The vibrations of a particle about a position of equilibrium" by B. B. Baker, 34–57; "On the relation between Pincherle's polynomials and the hypergeometric function" by B. B. Baker, 58–62; "Addition of a third of a period to the argument of the elliptic function" by D. G. Taylor, 63–67; "Mathematical notes," 69–79 [Note on page 74 is a communication from A. D. Russell referring to a proof of the law of tangents to which the editor of the Monthly had called his attention (*1920*, 53–54)].

Revue générale des Sciences, volume 32, November 15, 1921: Review by J. Bosler of the French translation by J. Rossignol of Eddington's *Space, Time, and Gravitation* (*Espace, Temps, Gravitation*), 617–618 ["L'édition anglaise comportait en appendice quelques notes théoriques assez succinctes: un important complément inédit, écrit tout exprès par M. Eddington, permettra cette fois au lecteur géomètre d'approfondir une foule de points délicats que le corps de l'ouvrage laissait à dessein sans démonstration et de dominer ainsi tout le sujet. . . . Il n'y a pas, a-t-on dit jadis, de 'route royale' en géométrie; il n'y en a pas non plus, semble-t-il, pour pénétrer au coeur de la doctrine relativiste. L'ouvrage de M. Eddington constitue du moins, avec ses compléments, sur tous ces problèmes, un traité aussi clair et aussi complet que possible."]—December 30: "Les nébuleuses spirales sont-elles des galaxies?" by H. Grouiller, 726; "Sur la théorie d'Einstein et le mouvement du périhélie de Mercure" by J. Richard, 726–727 ["Le calcul pour parvenir à l'expression de ds^2 est long et pénible. Le facteur γ devant dt^2 s'explique par le fait qu'en l'adoptant on trouve la loi de Newton un peu modifiée. Mais ce qui parait moins naturel, c'est la présence de $1/\gamma$ devant dr^2. J'ai trouvé une explication sans longs calculs, qui n'est peut-être pas absolument satisfaisante, mais qui est simple. . . . Cette théorie me laisse profondément sceptique; elle prête à des objections énormes. Elle n'a de vrai, je crois, que les formules."]—Volume 33, January 15, 1922: "Y a-t-il une erreur dans le premier mémoire d'Einstein?" by E. Guillaume, 5-10—January 30: Review by M. Sauger of three books on relativity (Paris, 1921): L. Fabre, *Les Théories d'Einstein;* G. Moch, *La Relativité des phénomènes;* C. Nordmann, *Einstein et l'Univers*, 56–57.

Revue de Mathématiques Spéciales, volume 32, December, 1921: "Sur une expression de la somme des puissances $p^{\text{ièmes}}$ des n premiers nombres entiers" by G. Beauvais, 49.

Science, new series, volume 54, December 9, 1921: "Lectures by Professor Lorentz at the California Institute of Technology," 572—December 23: "The National Academy of Sciences and the metric system" by C. D. Walcott, 628–629; "The order of nature" by R. D. Carmichael, 631–634 [Review of A. N. Whitehead, *The Principles of Natural Knowledge* (Cambridge University Press, 1919), of L. du Sablon, *L'Unité de la Science* (Paris, 1919), of L. J. Henderson, *The Order of Nature* (Harvard University Press, 1917), and of J. A. Thomson, *The System of Animate Nature* (London, 1920)].

Scientia, volume 15, January 11, 1921: "Euclidean constructions" by Hilda P. Hudson, 346–354; "Critical note: l'oeuvre mathématique de Klein" by F. Enriques, 393–396 [About Klein, *Gesammelte mathematische Abhandlungen*, Bd. 1. Connects Klein's ideas of geometry and groups of transformations with the ideas of Einstein]; Review by G. Loria of Halsted's translation of Saccheri, 397–398 ["Par cette splendide publication, M. Halsted ajoute un nouveau mérite à ceux, déjà nombreux, qu'il s'est acquis en répandant la connaissance des écrits de celui à qui la géométrie noneuclidienne est redevable de son état actuel."]

Scientific American, volume 13, December, 1921: "Eliakim Hastings Moore," 137 ["Following the old rule that the office of President of the American Association for the Advancement of Science should pass from a representative of the natural sciences to one of the physical sciences, this year a prominent entomologist gives place to a distinguished mathematician. The high office of President of the Association is almost without exception conferred upon the foremost representative of some special branch of science in the United States and therefore Professor Moore's election at the Chicago meeting held last December at once indicates that in the opinion of his colleagues he ranks first among American mathematicians, a decision in which there can be no dissenting voice."]

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, volume 52, nos. 9–10, published September 1, 1921: "Ein Schaubild zur Darstellung der Zeit-Raum-Verhältnisse in der speziellen Relativitätstheorie" by F. P. Liesegang, 193–201; "Zur Einführung des Integralbegriffes" by A. Harnack, 201–205; "Das Prinzip der vollständigen Induktion. Seine Geschichte und Anwendung im mathematischen Unterricht" by W. Lorey, 205–209; "Kleine Mitteilungen," 209–219; "Aufgabenrepertorium," 219–224; "Bücherbesprechungen," 235–245.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

2959. Proposed by J. W. M. Wedderburn, Princeton University.

Solve the functional equation, $[g(x)]^2 = -2x + g(x^2)$.

2960. Proposed by E. P. Lane, University of Wisconsin.

When do two cones circumscribing a sphere intersect in two ellipses and when are the planes of the ellipses perpendicular?

2961. Proposed by J. L. Riley, Stephenville, Texas.

Being given a triangle ABC, to determine two points P, P' such that the angles PBC, PCA, PAB are equal; also that the angles $P'CB$, $P'AC$, $P'BA$ are equal. Find also the equation of the circle which passes through the points P, P' and through the center of the circle circumscribing the triangle ABC.

2962. Proposed by R. M. Mathews, Wesleyan University.

To construct a triangle similar to a given triangle with its vertices lying on: (*a*) any three coplanar lines; (*b*) any three lines in space. (See Problem 2895, *1921*, 184.)

2963. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Through a given point to draw a line so that the sum of the squares constructed on the two segments cut off by it on the sides of a given angle should be equivalent to a given square.

2964. Proposed by N. P. Pandya, Amreli (Kathiaward), India.

$ABCD$ is a quadrilateral circumscribed about a circle and an ellipse. AB touches the ellipse at P and the circle at Q. DC touches the same ellipse and the same circle at L and M, respectively. AD touches the ellipse at K. BC touches the circle at E. Find the condition that $PEMK$ may be a parallelogram.

2965. Proposed by C. N. MILLS, Tiffin, Ohio.

If a quadrilateral inscribed in a square has the diagonals a and b, and the area A, show that the area of the square is $\dfrac{a^2b^2 - 4A^2}{a^2 + b^2 - 4A}$.

SOLUTIONS

2824 [1920, 185]. Proposed by G. Y. SOSNOW, Newark, N. J.

If n_1, n_2, n_3, n_4 be the lengths of the four normals and t_1, t_2, t_3 the lengths of the three tangents drawn from any point to the semi-cubical parabola, $ay^2 = x^3$, then will $27n_1n_2n_3n_4 = at_1t_2t_3$ [From *Mathematical Tripos Examination*, Cambridge, England].

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let the parametric equations of the curve be

$$x = au^2 \qquad \text{and} \qquad y = au^3.$$

Then

$$\frac{dy}{dx} = \frac{3u}{2},$$

the equations of the normal and tangent will be

$$3au^4 + 2au^2 - 3yu - 2x = 0 \tag{1}$$

and

$$au^3 - 3xu + 2y = 0, \tag{2}$$

and the lengths of the normal and tangent from (x, y) to the curve will be

$$n = \left(\frac{x - au^2}{3u}\right)\sqrt{9u^2 + 4} \qquad \text{and} \qquad t = \left(\frac{x - au^2}{2}\right)\sqrt{9u^2 + 4}.$$

The polynomial whose roots are the squares of the roots of (1) is easily found[1] to be:

$$9a^2z^4 + 12a^2z^3 + 4a(a - 3x)z^2 - (8ax + 9y^2)z + 4x^2 = 9a^2\Pi(z - u_i^2).$$

Multiply the roots by a, according to the familiar rule, and then replace z by x. This gives

$$\Pi(x - au_i^2) = x(x^3 - ay^2).$$

Also multiplying the roots by 9 and replacing z by -4 we obtain

$$\Pi(4 + 9u_i^2) = \frac{4}{a^2}[729(x^2 + y^2) + 216ax + 16a^2].$$

Hence

$$n_1n_2n_3n_4 = -\frac{x^3 - ay^2}{27}[729(x^2 + y^2) + 216ax + 16a^2]^{1/2}.$$

In a similar manner from equation (2) we get,

$$\Pi(x - au_i^2) = 4(x^3 - ay^2),$$

$$\Pi(4 + 9u_i^2) = \frac{4}{a^2}[729(x^2 + y^2) + 216ax + 16a^2];$$

and hence,

$$t_1t_2t_3 = \frac{x^3 - ay^2}{a}[729(x^2 + y^2) + 216ax + 16a^2]^{1/2}.$$

Therefore

$$at_1t_2t_3 = 27n_1n_2n_3n_4$$

neglecting the sign.

2834 [1920, 273]. Proposed by OTTO DUNKEL, Washington University.

In any triangle ABC let M and N be, respectively, the points in which the median and the bisector of the angle at A meet the side BC, Q and P the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to

[1] See, for example, Todhunter, *An Elementary Treatise on the Theory of Equations*, London, 1880, p. 36; or Salmon, *Lessons Introductory to the Modern Higher Algebra*, Dublin, 1885, p. 350.—EDITORS.

II. Solution by T. M. Blakslee, Ames, Iowa.

Let the coördinates of A be $(0, 0)$, of B, $(c, 0)$, of C, (h, k), and let the lengths of the sides of the triangle be a, b, c. As the slope of BC is $k/(h - c)$, the theorem is proved if we show that the slope of OQ is $(c - h)/k$.

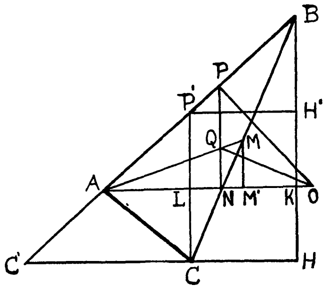
The coördinates of M are $[(c + h)/2, k/2]$, and the equation of AM is $y = kx/(c + h)$. The coördinates of N are $(\lambda b_1, \lambda k)$, where $b_1 = b + h$ and $\lambda = c/(b + c)$, and the equation of AN is $y = kx/b_1$. The equation of NP is $ky + b_1 x = \lambda(k^2 + b_1^2)$, and hence the coördinates of P are $(\lambda s/b_1, 0)$ where $s = b_1^2 + k^2$. Since the x-coördinate for P and O are the same, we have for the coördinates of O, $(\lambda s/b_1, \lambda ks/b_1^2)$. The coördinates of Q are found from the equations of AM and NP to be $[\lambda s(c + h)/(k^2 + b_1 c + b_1 h), \lambda sk/(k^2 + b_1 c + b_1 h)]$. The slope of OQ is now found to be $[k^2 + b_1 c - b_1(b_1 - h)]/(kb_1)$ and, using the relations $b_1 - h = b$ and $k^2 = b_1(b - h)$, this reduces to $(c - h)/k$. The proof for the external bisector may be carried through in the same manner.

III. Solution by the Proposer.

The truth of the theorem is evident if the angles B and C are equal. Suppose then that the angle B is the smaller of the two.

Prolong BA to C' so that $AC' = AC$. Then $C'C$ is parallel to AN. Let BH be the perpendicular from B upon $C'C$ produced to H. Draw CP' parallel to HB, cutting AN in L and AB in P'; also MM' parallel to HB meeting AN produced in M'. Draw $P'H'$ parallel to CH, meeting HB in H'. Then AN produced bisects HH' at K.

Since M is the middle point of BC, $M'M = (KB - KH)/2 = (KB - KH')/2 = H'B/2$ and $AM' = (AL + AK)/2 = C'H/2$. From similar triangles we have



$$\frac{NQ}{AN} = \frac{M'M}{AM'} = \frac{H'B}{C'H}, \quad \text{or} \quad \frac{NQ}{H'B} = \frac{AN}{C'H} = \frac{NP}{HB}.$$

Therefore

$$\frac{NQ}{NP} = \frac{H'B}{HB} = \frac{CH}{C'H}.$$

But

$$\frac{NP}{NO} = \frac{AN}{NP} = \frac{C'H}{HB}.$$

Therefore $NQ/NO = CH/HB$. Hence the right triangles ONQ, CHB are similar, and, since ON is perpendicular to HB, OQ must also be perpendicular to BC.

Also solved by P. J. da Cunha.

2851 [1920, 377]. Proposed by HILLEL PORITSKY, Cornell University.

Does there exist an analytic function satisfying the functional equation $f(z + 1) = e^{f(z)}$?

Solution by A. A. Bennett, University of Texas.

This is a typical problem of iteration and may be answered at once in the affirmative by reference to the usual methods in that theory.

It is necessary to find the fixed points of the iteration of e^x, that is, solutions of $x = e^x$. There are three such points; two proper finite points[1] ($.318 \pm i\, 1.337$, approximately), and the point at infinity which latter satisfies the relation only for certain methods of approach. Let a denote one of the two finite fixed points above mentioned, then $a = e^a$, and the constant function a, is one solution of the problem. A one-parameter family of non-constant solutions including a as a member is readily found by the use of undetermined coefficients as follows:

Write $g(ax - a^2 + a) = e^{g(x)}$ where $g(ax - a^2 + a) = f(z + 1)$ and $g(x) = f(z)$; that is, $x = a^z + a$, and $g(x) = f[\log_a (x - a)]$. Now $g(x)$ may be expanded in the form,

$$g(x) = a + c_1(x - a) + c_2(x - a)^2 + \cdots + c_n(x - a)^n + \cdots,$$

[1] Compare this Monthly, *1921*, 141–142, footnote.

where c_1 is chosen arbitrarily and the remaining coefficients obtained by comparison in the expansion of the two members of the relation[1]

$$g(ax - a^2 + a) = e^{g(x)}.$$

The series will necessarily converge in accordance with the general theory. Since there are two such values, a, there exist two one-parameter families of analytic functions satisfying the problem.

The above solutions are not real functions, so that the existence of a real analytic solution, other than plus infinity, remains to be examined. A smooth curve can be drawn containing one parameter of translation satisfying the problem, except for its possible non-analytic character. Owing to the highly singular character of the point at infinity, in connection with this problem the usual methods cannot be applied to this point. The following method of approach may prove suggestive. Modify the problem so that the fixed point at infinity shall appear at the origin; thus replace f and z by their reciprocals h and t, giving $h(t/(1+t)) = e^{-1/h(t)}$. Choose the parameter of translation so that $h(1) = 1$. Then $h(1/2) = 1/e$, $h(1/3) = 1/e^e \cdots$. At the points where $h(t)$ is thus determined, it approaches zero with extraordinary rapidity as t moves in toward the origin. Use Newton's or some other interpolation formula to obtain a real function $H_1(t)$ coinciding with $h(t)$ at $t = 1, 1/2, 1/3, \cdots, 1/n, \cdots$ and smooth, in the intermediate intervals.

Now H_1 may not satisfy the functional relations but $H_1(t)$ and $-1/\log H_1(t/(1+t))$ both coincide with $h(t)$ at the points $t = 1, 1/2, 1/3, \cdots$. The real solution desired may be expected to lie in most places between them, since if they coincide they will form a solution. Take a mean of $H_1(t)$ and $-1/\log H_1(t/(1+t))$ and call it $H_2(t)$. The particular method of choosing a mean is not significant since only a process of successive approximations is attempted. From $H_2(t)$, form $-1/\log H_2(t/(1+t))$; take $H_3(t)$ as a mean of these and proceed thus indefinitely. Inspection would suggest that the process might be arranged to lead to a determinate real analytic function in the limit, which function $H(t)$ would coincide with $-1/\log H(t/(1+t))$, so that $H(t)$ is a solution, $h(t)$, desired for the modified problem. Thence, the solution, $f(z)$, of the given problem is obtained by taking reciprocals. The above method lends itself to numerical handling, but the proof of the convergence to an analytic function will, of course, involve the usual theoretical complications.

2862 [1920, 428]. Proposed by J. L. RILEY, Stephensville, Texas.

Show that the whole area commanded by a gun on a hillside is an ellipse whose focus is at the gun, whose eccentricity is the sine of the inclination of the hill to the horizon, and whose semi-latus rectum is twice the greatest height to which the gun could send a ball.

SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec.

Let G be the position of the gun, AGA' the line of greatest slope, α the inclination of the hillside, and GL the maximum range in a direction making an angle θ with the line of greatest slope.

Also let β be the angle which GL makes with its projection GN on the horizontal plane through G. Then if u, $\phi + \beta$ represent the muzzle velocity and angle of elevation, respectively, for the maximum range GL we have

$$\begin{aligned} u \cos(\phi + \beta)\cdot t &= GN \qquad (t = \text{time of flight}) \\ &= GL \cos \beta. \qquad (1) \end{aligned}$$

[1] We may use the equation $ag'(ax - a^2 + a) = g'(x)g(ax - a^2 + a)$, g' denoting derivative. We shall find each c equal to a polynomial in the c's with lower suffix; in fact, c_n will be equal to c_1^n times a function of a alone, so that we can put $c_1 = 1$ in these equations. We shall have $2(a-1)c_2 = 1$, and in general

$$(n+1)(a^n - 1)c_{n+1} = a^{n-1}c_n + 2a^{n-2}c_2c_{n-1} + \cdots + (n-1)ac_{n-1}c_2 + nc_n.$$

Now suppose $|c_r| \leqq k^{r-1}$, $r = 2, 3, \cdots n$. Then

$$(n+1)|a^n - 1||c_{n+1}| \leqq [|a|^{n-1} + 2|a|^{n-2} + \cdots + n]k^{n-1} < (n+1)\frac{|a|^n - 1}{|a| - 1}k^{n-1}.$$

Also $|a^n - 1| \geqq |a|^n - 1$, and hence $|c_{n+1}| \leqq k^{n-1}/(|a| - 1)$.

Furthermore, $|c_2| < 1/(|a| - 1)$. Therefore, if we take $k \geqq 1/(|a| - 1)$, we shall have for all values of n $|c_n| \leqq k^{n-1}$.

The radius of convergence of the series for $g(x)$ is at least equal to $1/k|c_1|$ or $(|a| - 1)/|c_1|$.
—EDITOR.

Also resolving perpendicular to GL

$$0 = u \sin \phi \cdot t - \tfrac{1}{2} g \cos \beta \cdot t^2. \tag{2}$$

Hence from (1) and (2) $GL = [2u^2 \cos(\phi + \beta) \sin \phi]/[g \cos^2 \beta]$ when the expression on the right-hand side is a maximum, *i.e.*, $GL = u^2[\sin(2\phi + \beta) - \sin \beta]/[g \cos^2 \beta]$; whence $2\phi + \beta = \pi/2$, and

$$GL = \frac{u^2(1 - \sin \beta)}{g \cos^2 \beta} = \frac{u^2}{g(1 + \sin \beta)}. \tag{3}$$

Again, if LE is drawn perpendicular to GA, and GF is the horizontal projection of GE, we have

$$\sin \beta = \frac{NL}{GL} = \frac{NL}{GE} \cdot \frac{GE}{GL} = \frac{FE}{GE} \cdot \frac{GE}{GL} = \sin \alpha \cdot \cos \theta.$$

Hence, (3) becomes $(u^2/g)/GL = 1 + \sin \alpha \cos \theta$, *i.e.*, L is on an ellipse, focus G, eccentricity $\sin \alpha$, and the semi-latus rectum u^2/g.

REMARK BY OTTO DUNKEL, Washington University—This problem may also be solved by finding the intersection of the envelope of the trajectories (see Granville, *Calculus*, first edition, p. 216) $z = (u^2/2g) - (g/2u^2)(x^2 + y^2)$ and the inclined plane $z = y \tan \alpha$.

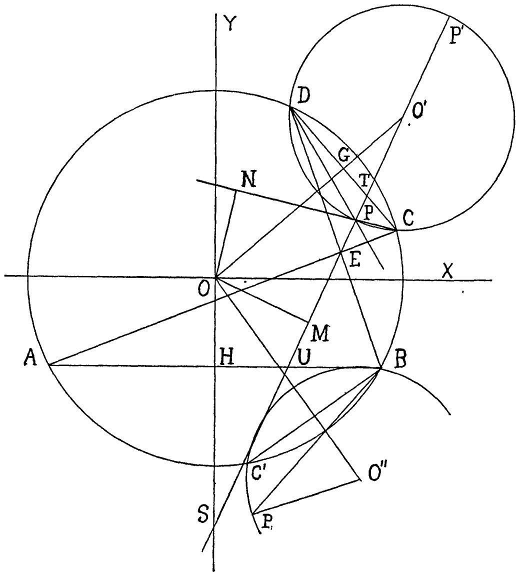
Also solved by AUGUSTUS BOGARD, A. M. HARDING, and WILLIAM HOOVER.

2865 [1920, 482]. Proposed by JOSEPH ROSENBAUM, Milford, Conn.

In a circle, a chord AB is fixed in position and a moving chord CD is constant in length. Find the locus of the intersection of the bisectors of the angles ACD and BDC.

I. SOLUTION BY F. L. WILMER, Omaha, Neb.

A solution of the problem in all its ramifications requires a discussion of a number of distinct cases, segregable into two classes according as the lines connecting the extremities of the two chords do or do not cross within the given circle. One special case of the first class has been selected here for discussion to show a workable method of attack of the problem in the various cases.



Let the radius of the given circle be the unit of length and suppose that the internal bisectors of the angles of the triangle CDE (see figure) meet in P, a point of the locus. Let the center O of the unit circle be the origin of rectangular coördinates with the x-axis parallel to AB. It will be seen at once that the acute angle b which PC makes with DP is constant, so that P, C, D, and P' (the intersection of the two external bisectors of the angles C and D) lie upon a circle of constant radius with center at O'. The central angle $DO'C = 2b$. Denote the lengths of the arcs AB and CD of the fixed circle by c and a, the angle YOO' by φ. When $\varphi = 0$, P lies on the y-axis and the angle $OO'P$ is also zero. When CD takes the position $C'D'$, $D' = B$, $\angle OO'P$ becomes in the new position at O'' $\angle OO''P_1 = \angle OO''C' + \angle C'O''P_1$ $= b + 2\angle C'D'P_1 = b + a/2$. The angle φ is now seen from the figure to be $\angle YOO'' = \angle YOB + a/2 = \pi - c/2 + a/2 = 2b + a$, since $a + c + 4b = 2\pi$. Hence $\angle YOO'' = 2\angle OO''P_1$ and it may be shown that for any position of P this relation is true. Hence $\angle OO'P = \varphi/2$.

We shall obtain the equations of PC and $O'P$ using φ as a parameter. Let ON and OM be the perpendiculars from the center O upon PC and $O'P$, respectively. Since $\angle NOO' = \angle NCD = 1/2 \angle DO'P = b/2 + \varphi/4$, $\angle YON = \varphi - b/2 - \varphi/4 = 3\varphi/4 - b/2$; also $\angle YOM = \pi/2 + \varphi/2$ and $\angle NOC = \angle NOO' + a/2 = (a+b)/2 + \varphi/4$. Then $ON = \cos[(a+b)/2 + \varphi/4]$ and $OM = r \sin(\varphi/2)$, where $r = OO'$. Hence the coördinates of P satisfy the equations

$$x \sin\left(\frac{3}{4}\varphi - \frac{b}{2}\right) + y \cos\left(\frac{3}{4}\varphi - \frac{b}{2}\right) = \cos\left(\frac{a+b}{2} + \frac{\varphi}{4}\right),$$

$$x \cos\frac{\varphi}{2} - y \sin\frac{\varphi}{2} = r \sin\frac{\varphi}{2},$$

and by solving these equations we obtain the parametric equations of the locus of P:

$$x = \sin\frac{\varphi}{2}\left[\cos\left(\frac{a+b}{2} + \frac{\varphi}{4}\right) + r\cos\left(\frac{3}{4}\varphi - \frac{b}{2}\right)\right] \Big/ \cos\left(\frac{1}{4}\varphi - \frac{1}{2}b\right),$$

$$y = \left[\cos\frac{\varphi}{2}\cos\left(\frac{a+b}{2} + \frac{\varphi}{4}\right) - r\sin\frac{\varphi}{2}\sin\left(\frac{3}{4}\varphi - \frac{b}{2}\right)\right] \Big/ \cos\left(\frac{1}{4}\varphi - \frac{1}{2}b\right).$$

II. Solution by Otto Dunkel, Washington University.

Let O be the center of the given circle and let us suppose that, when CD lies on the same side of AB as O, the intersection E of DB and CA lies within the circle, and that P is the intersection of the internal bisectors of the angles of the triangle ECD. Indicate by G and H the middle points of the chords CD and AB, respectively, and by O' and S the points in which the bisector EP meets OG and OH. The external bisectors of the angles D and C of the triangle ECD are perpendicular, respectively, to PD and PC, and meet EP in P', the center of the escribed circle in $\angle DEC$. Hence PP' is a diameter of the circle through P, D, P', C, with O' as center. This circle has a constant radius since $\angle DEC$ is constant in magnitude and hence $\angle DPC$ has also a constant value; also OO' is constant in length. Let EP meet CD and AB in T and U, respectively; the construction of the figure shows that the triangles AUE and DTE are similar, and, therefore, that $\angle HUS = \angle GTO'$, and, finally, that $\angle HSU = \angle GO'T$. The triangle OSO' is thus isosceles and S is a fixed point. This determines an easy construction for the curve as follows: With O as center draw a fixed circle of radius OO' passing through S; draw a variable chord SO' and lay off upon it the constant lengths $O'P$ and $O'P'$; then P is the center of the inscribed circle of ECD (in the position mentioned above) and P' is the center of the escribed circle in $\angle DEC$, both of which are points of the locus. The curve is, therefore, the *limaçon of Pascal*.

When C falls on B, P also falls upon B; when D coincides with A, P also coincides with A. When CD or any part of it is on the side of AB opposite to that of O, E and the two points P and P' are all outside of the given circle, the latter two being escribed centers in $\angle CDE$ and $\angle DCE$, respectively. After one revolution of O' and CD, P and P' are interchanged.

The equation of the locus is easily obtained in polar coördinates, taking S as the pole, SO as the axis, $\angle OSP = \theta$, $SP = \rho$. It will be convenient to take the radius OD of the given circle as unity and to denote the lengths of the arcs CD and AB by a and c, respectively, and the acute angle which PC makes with DP by b. Then $b = 1/2 \angle CEB = \pi/2 - (a+c)/4$. In the triangle ODO', $\angle DO'O = b$ and, by the Law of Sines we have $OO' = \sin(b + a/2)/\sin b = \cos(a/4 - c/4)/\cos(a/4 + c/4)$; $O'D = \sin(a/2)/\sin b = \sin(a/2)/\cos(a/4 + c/4)$. Hence

$$\rho = 2\frac{\cos\dfrac{a-c}{4}}{\cos\dfrac{a+c}{4}}\cos\theta - \frac{\sin\dfrac{a}{2}}{\cos\dfrac{a+c}{4}}.$$

Reversing the order of the points C and D causes E to be outside of the given circle in the initial position. In order to obtain the equation for this case we had merely to replace a by $-a$ in the above equation and in the expression for OS. The remaining cases may be treated in a similar manner.

2869 [1921, 36]. Proposed by the late L. G. WELD.

The successive segments of a broken right line are represented by the successive terms of the harmonic progression, 1, 1/2, 1/3, 1/4, *ad infinitum*. Each segment makes with the preceding a

given angle θ. What is the distance and what is the direction of the limiting point (if there be such) from the initial point of the first segment?

SOLUTION BY P. H. GRAHAM, Washington Square College, New York University.

Take the origin of rectangular coördinates as the initial point and let the first segment make an angle θ with the x-axis. Let X and Y be, respectively, the sums of the projections of the segments on the x-axis and on the y-axis; D the distance of the limiting point from the initial point and α the angle which the radius vector to the limiting point makes with the x-axis. Then

$$X = \sum_1^\infty \frac{\cos k\theta}{k}, \qquad Y = \sum_1^\infty \frac{\sin k\theta}{k}, \qquad D = \sqrt{X^2 + Y^2}, \qquad \alpha = \tan^{-1}\frac{Y}{X}. \tag{1}$$

Setting $z = \cos\theta + i\sin\theta$, we have the known development

$$-\log(1 - z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \qquad z \neq 1. \tag{2}$$

Hence

$$-\log(1 - \cos\theta - i\sin\theta) = \sum_1^\infty \frac{(\cos\theta + i\sin\theta)^k}{k} = \sum_1^\infty \frac{\cos k\theta + i\sin k\theta}{k}. \tag{3}$$

But $-\log(1 - \cos\theta - i\sin\theta) = -\log(2 - 2\cos\theta)^{1/2} + i\tan^{-1}[\sin\theta/(1 - \cos\theta)]$ and, hence, equating the real and imaginary parts of (3), we have

$$X = -\log(2 - 2\cos\theta)^{1/2} = -\log\left(2\sin\frac{\theta}{2}\right), \qquad Y = +\tan^{-1}\frac{\sin\theta}{1 - \cos\theta} = \frac{\pi - \theta}{2}. \tag{4}$$

Therefore

$$D = \sqrt{\log^2\left(2\sin\frac{\theta}{2}\right) + \left(\frac{\pi - \theta}{2}\right)^2}, \qquad \alpha = \tan^{-1}\frac{\theta - \pi}{2\log\left(2\sin\frac{\theta}{2}\right)}, \qquad 0 < \theta < 2\pi.$$

NOTE ON THE ABOVE SOLUTION BY OTTO DUNKEL, Washington University—The angle θ may be taken so that $0 < \theta < 2\pi$, and the angle of $1 - z$, say ψ, may then be taken so that when $\theta = \pi$, $\psi = 0$, $-\pi/2 < \psi < \pi/2$. Inspection of a figure will show at once that $\psi = (\theta - \pi)/2$ and that the absolute value of $1 - z$ is $2\sin(\theta/2)$, so that

$$\log(1 - z) = \log(2\sin\theta/2) + i(\theta - \pi)/2.$$

The development in (2) is valid for all points on the circle of convergence of the series except for the singular point $z = 1$. The proof of this may be found in Goursat-Hedrick, *A Course in Mathematical Analysis*, vol. 2, part 1, page 19, foot-note, where the convergence of the series is proved, while the argument on pages 20, 21 shows that the series converges to the value on the left in (2). See also pages 38, 39 in the same text for a treatment of $\log(1 + z)$ which gives the above results by a simple substitution.

Also solved by AUGUSTUS BOGARD, R. E. JOHNSON, and ELIJAH SWIFT.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Mr. C. C. PHIPPS, of the University of Montana, has been appointed instructor of mathematics at the University of Minnesota.

Miss MINNA SCHICK, instructor of mathematics at the University of Minne-

sota, has been appointed associate professor at the University of the Philippines.

Mrs. Mary W. Newson, associate professor of mathematics at Washburn College, has been appointed head of the department of mathematics at Eureka College, Ill.

At H. Sophie Newcomb Memorial College, New Orleans, Miss Anna M. Howe has been promoted to an assistant professorship of mathematics and Miss Anna Nancy, of Tulane University, has been appointed assistant professor of mathematics.

At Hampden-Sidney College, Va., Mr. B. D. Painter has been appointed assistant professor of mathematics.

At Colorado College, Associate Professor W. V. Lovitt has been promoted to a full professorship. Miss Wilhelmina M. Spingler and Mr. A. R. Wapple have been appointed instructors of mathematics.

The Corporation of Yale University has recently raised the title of H. S. Uhler from the grade of Assistant Professor to Associate Professor. The latter rank was established at Yale only a year or two ago.

At the University of Southern California, Associate Professor H. C. Willet has been promoted to a professorship in mathematics, and Dr. Victor Steed has been appointed an assistant professor.

At Albion College, Mr. Leon Sears has been appointed instructor in applied mathematics and astronomy.

At the California Institute of Technology, Dr. Clyde Wolfe has been promoted to an assistant professorship in mathematics.

Professor W. J. Rusk, Grinnell College, is absent on sick leave for the second half of the present year. He is spending the spring months at Whittier, California.

Mr. A. L. Dixon, fellow and tutor of Merton, has recently been appointed Waynflete Professor of Pure Mathematics in succession to Professor E. B. Elliott, fellow of Magdalen, resigned. Mr. Dixon has also been elected a fellow of Magdalen.

Charles Henry Davis, 2nd, Rear Admiral, retired U. S. Navy, who was twice Superintendent of the Naval Observatory, died at Washington, D. C., December 27, 1921. He was born in Cambridge, Mass., August 28, 1845. His father, Charles Henry Davis, had also been twice Superintendent of the Naval Observatory and had established the Nautical Almanac Office.

Wooster Woodruff Beman, whose services to mathematics in America are known throughout the country, died at his home in Ann Arbor on January 18, 1922. He had been a member of the Faculty of the University of Michigan continuously since 1871, a record of service unequalled in the history of the University. Professor Beman was born in Southington, Connecticut, on May 28, 1850. He graduated from the University of Michigan at the age of twenty and during the following year taught Greek and mathematics at Kalamazoo College. In 1871 he returned to his Alma Mater as instructor of mathematics and in 1873

received the degree of Master of Arts. He became an assistant professor in 1874, an associate professor in 1882, a full professor and head of the department in 1887, a position which he held until his death. For several years before completing his half century of service he was the senior member of the Faculty. In 1908 he received the honorary degree LL.D. from Kalamazoo College. The series of text books that were prepared by Professor Beman in collaboration with Professor D. E. Smith of Columbia University included *Plane and Solid Geometry* (1895), *New Higher Arithmetic* (1897), *New Plane and Solid Geometry* (1899), *Elements of Algebra* (1900), *An Academic Algebra* (1902). The translation of Klein's *Famous Problems of Elementary Geometry* was an outcome of this same collaboration (1897). The *Michigan Alumnus* makes the following personal allusion:

"Professor Beman was a man of strong individuality, with a somewhat precise and even abrupt manner that only served partly to conceal a kindly and considerate disposition. He had an extraordinary memory and a gift for the details of knowledge particularly effective in his field of mathematics. Alumni who have returned to the University after long years of absence have found to their surprise that Professor Beman not only recognized them, but also remembered personal incidents of their college career. His interest in his former students was continuous and never failing, a characteristic which endeared him to everyone who ever sat in his classes."

Marie Ennemond Camille Jordan[1] died January 21, 1922. He was born in Lyon, January 5, 1838, and when he died he was "dans la plénitude de ses facultés." He entered the École Polytechnique in 1855 and obtained his Dr. ès sciences in 1861. He was engineer of mines at Privas, Chalon-sur-Saône, then at Paris (1867). In 1876 he became professor at the École Polytechnique and suppléant at the Collège de France. In 1881 he succeeded Chasles in the section of geometry of the Académie des Sciences. He was vice-president of the Académie in 1915 and president in 1916. In 1920 he was elected a foreign associate of the National Academy of Sciences (America). Since 1885 he has conducted the *Journal de Mathématiques pures et appliquées.*

Jordan lost three of his six sons in the war and the oldest of his grandsons; "tous les quatre dans des circonstances héroïques." His wife also died 1918.

His most important books are *Traité des Substitutions et des Équations Algébriques* (Paris, 1870) and *Cours d'Analyse de l'École Polytechnique*[2] (three volumes, Paris, 1883–1887; 3d edition, 1909–1915). His reputation rests chiefly on his researches in the "Theory of groups" and their application to the solution of algebraic equations. He was also a pioneer in the modern theory of functions of a real variable, having introduced into this portion of analysis the important notion of function with limited variation. His name has been given to those curves which divide the plane into two distinct parts.

[1] There is an obituary notice by Villat, inserted in the first number for 1922 of the *Journal de Mathématiques* and accompanied by a fine portrait of Jordan. M. Villat is the successor of Jordan as editor of this journal. The address of Picard is included in this notice and is also published in the *Revue Scientifique*, February 11, 1922, pp. 95–96. The tribute by M. d'Adhémar is in the *Revue générale des Sciences pures et appliquées*, February 15, 1922, pp. 65–66.

[2] In 1890 at the Johns Hopkins University we learned from Craig to appreciate our "Jordan" (*Cours d'Analyse*) and in the years that followed, when difficulties arose in the "Theory of functions," it was to Jordan that we went for help.—H. P. M.

M. E. Picard, in the Academy of Science, on the 23d of January, gave an address describing briefly the work of Jordan and concluding with the following paragraph:

"Tous les travaux de Jordan dénotent une rare profondeur d'esprit et une extraordinaire puissance d'abstraction. Il se jouait au milieu des discussions les plus subtiles sur des concepts comme ceux de *groupes* ou de *substitutions*, se plaisant à aborder les questions dans toute leur généralité, comme s'il craignait que quelque particularité l'empêchât de voir les vraies raisons des choses. Jordan a été vraiment un grand algébriste; les notions fondamentales qu'il a introduites en Analyse préserveront son nom de l'oubli."

The following incidents given in a tribute from the pen of M. R. d'Adhémar show something of Jordan's character:

"Je parlais, un jour, avec M. Jordan, du travail considérable que demande la publication d'un cours, parce qu'il est impossible d'avoir, sur *toutes* les questions, des vues personnelles.

"Etant, un jour, embarrassé—me répondit-il—j'allai me renseigner auprès d'Henri Poincaré, *notre maître à tous!*

"Je n'oublierai ni ce trait, ni l'impression presque enfantine de bonté et de douceur, que je lisais, à ce moment, sur le visage de M. Jordan. Ce vieillard illustre parlait, avec respect, d'un confrère beaucoup plus jeune que lui! Je suppose qu'Henri Poincaré a aussi, parfois, demandé des renseignements à Camille Jordan.

"Chargé d'honneurs, M. Jordan était infiniment modeste, bienveillant, juste et ferme. Son caractère était remarquablement pondéré; il y avait autant de force que d'équilibre dans cette belle tête! . . .

"Savant génial, M. Jordan a été un homme dont la haute dignité morale était universellement respectée.

"L'homme était aimé et l'oeuvre sera toujours admirée."

M. H. Villat, after writing of the griefs that came to him in the war, adds:

"Dans ses convictions religieuses, auxquelles il était depuis son enfance profondément attaché, il sut trouver un réconfort et un appui; il restait entouré de la chaude affection des siens, s'occupant d'ailleurs beaucoup des études de ses petits-enfants, ce que son érudition, étendue dans tous les domaines, lui rendait facile; ses lectures étaient innombrables, et même les classiques grecs et latins, dans le texte original, n'avaient pas de secrets pour lui.

"Trop brièvement, j'ai dit ce qu'a été la vie admirable de Jordan. Ce que je n'ai pas dit et ce dont tous ceux qui l'ont approché dans l'intimité pourront témoigner, c'est l'extrême délicatesse et l'infinie bonté avec laquelle il savait traiter ses amis. N'attachant pour lui-même qu'un prix médiocre aux honneurs, il ne s'épargnait nulle peine pour les faire obtenir, à leur insu, à ceux qu'il en estimait dignes; et ce trait, entre mille autres, explique la profondeur des affections qui s'étaient multipliées autour de lui."

At the University of Cambridge, England, a special syndicate appointed by the University has reported in favor of the addition of mathematics to the list of subjects for the Natural Sciences Tripos, Part 1. This attempt to facilitate the acquisition of mathematical knowledge by students of the natural sciences will be of interest to teachers of mathematics in our colleges and universities.

At the annual meeting of the Mathematical Association of Great Britain, January 2–3, 1922, Sir T. L. Heath was elected president as successor to Canon J. M. Wilson. The new president is well known through his publications *The Thirteen Books of Euclid's Elements*, 3 volumes, *The Works of Archimedes*, *Diophantos of Alexandria*, etc., all published by the Cambridge University Press.

The University of Nebraska will offer three or four fellowships in mathematics for the year 1922–23. The applicant must have the Bachelor's Degree

from some college or university of recognized standing. Some teaching experience is also desirable. These fellows will be required to teach part time, and the compensation will vary from $500 to $1000.

A National Academy on the model of the Institut de France is being founded in Ireland. There are five sections or academies, one of which is devoted to the mathematical and physical sciences. A foundation meeting was scheduled for May 18, 1922. In the Mathematical and Physical Section twenty-two persons were invited to be foundation members, among these being Professor F. D. Murnaghan of the Johns Hopkins University. The Academy has received the promise of financial support from the Department of Education of the Irish Free State and it will encourage and publish the results of research submitted by its members and associates.

At the annual meeting of the National Academy of Sciences, on April 24, 25 and 26, the following papers were presented: "Some extensions in the mathematics of hydromechanics" by Dr. R. S. Woodward; "Normal coördinates and Einstein space" by Professor G. D. Birkhoff; "Algebraic solutions of Einstein's cosmological equations" by Professor Edward Kasner; and "Geometry of paths" by Professor Oswald Veblen. At this meeting Professor L. P. Eisenhart was elected to membership in the National Academy.

The following abstract from a letter of the Ohio Section will perhaps also express the ideas of the executive committees of other Sections: Our Section can never reach its highest development by the efforts alone of the officers, or any program committee—the most vital ideas come from you members. Each member has an obligation to help promote the usefulness of the Section. You know that merely buying a book does not make it your intellectual property. Neither does merely paying your dues to a scientific society gain you a great part of its benefits. To be the best possible Section our members must not only be "in" it, they must also be "of" it. Now give some thought to your State Association. Let the secretary have the benefit of your suggestions. He will see that they get very careful attention. Let us begin now to plan for next year's program.

The National Council of Mathematics Teachers.

The National Council of Mathematics Teachers held its annual meeting at Chicago on March 1st, 1922, in connection with the meeting of the Department of Superintendence of the National Education Association. At the business session in the morning, matters of policy for the development of the National Council were discussed, and these matters were set forth by President Minnick at the dinner in the evening. It seems logical and desirable that the Council should in a sense continue the work of the National Committee on Mathematical Requirements, which is about to complete its formal work. For this purpose, as well as for many other reasons, it is desirable that the National Council should double, or even quadruple, its present membership of twenty-seven hundred, and a campaign for that purpose will be undertaken at once.

At the afternoon session Dr. J. M. Kinney gave a paper on "The function concept in high school mathematics"; Professor H. E. Slaught discussed the question of "Elective courses in senior high school mathematics," in place of Professor E. R. Hedrick, who was unable to be present; Professor J. W. Young gave a report on "Some phases of the work of the National Committee"; and Mr. Alfred Davis discussed "Unsettled problems concerning the teaching of secondary mathematics." At the evening meeting following the dinner, at which over two hundred members were present, Professor H. E. Slaught presided. The speakers were Mr. W. D. Reeve, who spoke on "The case for general mathematics"; Professor G. W. Myers, on "Reaction versus radicalism in teaching secondary mathematics"; Mr. Raleigh Schorling, on the question "Is the teaching of mathematics responding to modern demands in secondary education?"; and President J. H. Minnick, on "A program for the National Council of Teachers of Mathematics."

It was universally agreed that this was the most successful meeting of the National Council which has been held.

Important Announcements

1. The Seventh Summer Meeting of the Association will be held at the University of Rochester on Wednesday and Thursday, September 6–7, 1922. The first session will be devoted to a symposium on the progress of unified mathematics, comprising four papers, "The problem of organizing freshman college courses" by Professor J. W. Young of Dartmouth College, "Historical consideration of unified mathematics" by Professor L. C. Karpinski of the University of Michigan, "Some aspects of unified mathematics for freshmen" by Professor R. W. Burgess of Brown University, and "Internal reasons for unification" by Professor C. E. Comstock of Bradley Polytechnic Institute, followed by discussion led by Professor F. B. Williams of Clark College and Professor K. D. Swartzel of the University of Pittsburgh.

The second session will consist of papers as follows: "Contradictions in the literature of group theory," presidential retiring address by Professor G. A. Miller of the University of Illinois, "An English text on mathematics written about 1810" by Professor Elizabeth B. Cowley of Vassar College, "Impressions of mathematics and mathematical instruction in Italian universities" by Professor Virgil Snyder of Cornell University, and "The present status of the formal discipline controversy" by Professor N. J. Lennes of the University of Montana.

Thursday morning will occur a meeting at the Research Laboratory of the Eastman Kodak Company by invitation of the Company. At this meeting a paper will be read, "Mathematical puzzles as an introduction to investigation" by Professor W. B. Carver of Cornell University, and a paper by Doctor L. A. Jones of the Eastman Kodak Company, followed by an inspection of the Research Laboratory and a business meeting.

2. There have been elected to membership since the last printed report 88 individual members and 6 institutional members. The names of those elected at the Toronto meeting are given in the Secretary's report in this issue. The names of all new members elected since the last Register was printed will appear in the new Register soon to be published.

3. Amendments to the By-Laws and Articles of Association will be discussed and voted on at the Rochester meeting. Aside from the elimination of certain paragraphs now obsolete, and the simplification of other portions, four explicit changes will be proposed:

(a) To add one new officer—a librarian;

(b) To provide for notification of proposed amendments *by mail* to members as alternative to publication in the MONTHLY and to make this operative in the present case;

(c) To provide for a life membership clause, on which a committee of the Association has already been making a careful study;

(d) To provide for the management of other publications than the official journal, e. g., the Carus Monographs.

It is also proposed to amend the Articles of Association of the corporation so as to increase the number of Trustees from 19 to 20.

4. The report of the committee appointed to nominate an editorial board for the Carus Monographs and to formulate a statement of powers of this board has been presented to the Trustees and approved by their mail vote. A copy of this report has been sent to each member of the Association. It will be presented for formal ratification by the Trustees at the Rochester meeting and will appear as a part of the Secretary's report of that meeting in the MONTHLY.

5. The printer now gives assurance that the issues of the MONTHLY can be handled with something like normal speed, and we hope to catch up on the schedule during the autumn.

W. D. CAIRNS,
Secretary-Treasurer.

Published September 15, 1922.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, A. A. BENNETT, University of Texas, Austin, Texas.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December, 1922

The following are dates of Section meetings of the Association in 1921 (unless otherwise specified):

TEXAS, Dallas, November 25; Houston, December 1–2 1922,
ILLINOIS, Rockford, Ill., April 28–29, 1922
IOWA, Simpson College, Indianola, April 30; Des Moines, November 4
KANSAS, Topeka, January 21, 1922; University of Kentucky, January, 1923
KENTUCKY, University of Kentucky, May 7; Georgetown College, April 8, 1922
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, May 13, 1922; Washington, December, 1922
MINNESOTA, St. Paul, June 4; Macalester College, St. Paul, May 27, 1922.
MISSOURI, Washington University and Soldan High School, St. Louis, November 25–26; Kansas City, November, 1922
OHIO, Columbus, March 25–26; Columbus, April 14–15, 1922
ROCKY MOUNTAIN, Denver, March 25–26; Greeley, Colo., April 14–15, 1922
SOUTHEASTERN, Atlanta, Ga., April 29, 1922
TEXAS, Dallas, November 25; Houston, December 1–2, 1922

DESIRED COPIES OF THE MONTHLY

Through an oversight the Association has not reserved a sufficient number of copies of the Monthly for October, 1920 and for August–September and October, 1921. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

Extra copies or volumes of any dates which members wish to contribute will be used to the best advantage of the Association.

Address all communications to the Secretary, W. D. Cairns, Oberlin, Ohio.

THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

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THE JANUARY MEETING OF THE KANSAS SECTION.

The eighth regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, on January 21, 1922, in connection with a meeting of the Kansas Association of Mathematics Teachers. Two sessions were held, the first of which was a joint session with the Kansas Association. Professor Shirk presided at both sessions.

There were sixty-eight in attendance, including the following twenty-three members of the Association:

C. H. Ashton, L. C. Bagby, H. W. Bailey, Florence Black, R. H. Carpenter, Lucy Dougherty, W. H. Garrett, W. A. Harshbarger, T. B. Henry, W. H. Hill, Emma Hyde, S. Lefschetz, C. F. Lewis, Anna Marm, U. G. Mitchell, H. S. Myers, B. L. Remick, D. H. Richert, J. A. G. Shirk, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler.

The following officers were elected for the coming year: Chairman, Professor WHITE; Vice-Chairman, Miss DOUGHERTY; Secretary, Professor MITCHELL.

The following papers were presented:

(1) "Report of the Kansas Committee coöperating with the National Committee on Mathematical Requirements" by Professor U. G. MITCHELL.

(2) "Plane geometry as a college subject" by Professor J. A. G. SHIRK.

(3) "Does the usual algebra and geometry course fit the needs of the pupil?" by Mrs. H. E. RYNERSON, Lawrence High School (by invitation).

(4) "What does the college expect of the algebra and trigonometry courses?" by Professor W. A. HARSHBARGER.

(5) "History and development of trigonometry" by Professor W. H. HILL.

(6) "Mathematics in Europe" by Professor SOLOMON LEFSCHETZ.

(7) "Mathematics of insurance" by Mr. RICHARD DEAVER, Provident Life and Trust Company (by invitation).

(8) "Mathematical determination of orbits" by Professor L. C. BAGBY.

(9) "Content of a course in college mathematics for students who take no other mathematics courses" by Professor D. H. RICHERT.

Abstracts of the papers follow below, the numbers corresponding to numbers in the list of titles:

1. Professor Mitchell reported concerning the further activities of the National Committee on Mathematical Requirements since his report at the meeting a year ago and concerning the arrangements for the printing and distribution of their final report. He reported that the Kansas Committee had received no further publications from the National Committee and had held no meetings since the last report.

2. Professor Shirk reported that an extensive investigation was made of the mathematical training needed in the various trades and industries of the mining and industrial sections of Kansas. Because many students do not elect geometry

in their high school course, it has seemed advisable to give a freshman college course in this subject in order that these students may be enabled to take further college courses in mathematics. The State Manual Training Normal School has conducted three classes in plane geometry, each of increasing size. The course gives three hours college credit.

3. Mrs. Rynerson pointed out the great use of mathematics and emphasized the need of a careful investigation as to whether the usual algebra and geometry courses cannot be made more valuable to the pupil.

4. Professor Harshbarger presented a topical outline of the subjects that should be given in the fourth semester of high school algebra and in plane trigonometry to students expecting to enter college. Emphasis was placed on the conception of the function, the liberal use of the graph, and the derivation and intelligent use of formulæ in algebra. In trigonometry emphasis was placed on training the student to change readily the form of trigonometric expressions.

5. Professor Hill traced the history of trigonometry from Ahmes down to the present time. He mentioned especially the contributions of Hipparchus, Ptolemæus, Mueller, Rheticus, Napier, Briggs, Speidell and Oughtred.

6. During his recent stay in Europe Professor Lefschetz became very much impressed with the seriousness of the outlook for science and for the men who pursue it. In his paper he endeavored to make this clear by telling of some personal experiences. He also pointed out that for mathematicians in Europe as well as here relativity is the all-absorbing topic. He stated his belief that it is the duty of mathematical people to interpret this important idea to the public at large, and indicated what mathematical reading would form a good preparation to a thorough understanding of it.

7. Mr. Deaver outlined an original method of computing life insurance premiums. The method was illustrated by the calculation of several premiums.

8. Professor Bagby's paper consisted in a discussion of the methods of determining the orbits of asteroids and comets about the sun. The elements of an orbit were defined. The discussion was confined to the determination of the elements from three observations. An outline of the Laplacian method was given.

9. It seems probable that the usual courses in algebra and trigonometry do not contain the material most satisfactory for the student who takes no other college courses in mathematics. Professor Richert outlined a course extending over two semesters for a class meeting three times a week. The course would cover the important parts of algebra, trigonometry, differential calculus and integral calculus. It should probably be confined to juniors and seniors.

E. B. Stouffer, *Secretary-Treasurer.*

THE IMAGINARY POINTS OF GEOMETRY.

By ALBERT A. BENNETT, University of Texas.

1. Infinity and Imaginary Points. Geometry was at one time regarded as describing actual spacial relations in an absolute and incontrovertible manner. Many philosophers supposed that by a process of pure deduction one started from self-obvious logical facts and discovered the inner constitution of space. Popular interest in the discovery of the now classical non-Euclidean geometry has been due largely to the still bewildering novelty of the fact that at certain stages in the development of an axiomatic system, there is a feature of arbitrary choice.

The geometry of real finite points being assumed in accordance with Euclidean standards, certain alternatives are yet open. One of these alternatives is in regard to the conception of points at infinity. As is well known, we may consistently adopt any one of the following conventions and indeed other possibilities remain.

(*a*) All points are finitely accessible. In other words there are no points at infinity.

(*b*) The points at infinity for a space of n dimensions constitute a (projective) space of $n - 1$ dimensions. Thus the Euclidean plane is completed by a line at infinity, the Euclidean three-space by a plane at infinity, and so forth. This is the convention of projective geometry.

(*c*) There is a single point at infinity. This is the convention of the geometry of inversion.

(*d*) The points at infinity fall upon n linear $(n - 1)$-spaces. Thus the Euclidean plane is completed by two intersecting straight lines at infinity, the Euclidean space by three planes at infinity and so forth. This is the convention of what has been called by some writers "function space."

The character of the set of points at infinity having been disposed of, the question of imaginary points is still open. The two common alternatives are as follows.

(*a*) To exclude from the discussion the consideration of imaginary points, or in other words to assume the existence of real points only.

(*b*) To assume the existence of points whose Cartesian coördinates are complex numbers as used in algebra.

So far as the writer is aware no other alternative has been currently suggested.

For problems of analysis situs and for investigations confined to a single net of rationality[1], it is assuredly permissible to assume the non-existence of imaginary points. In enumerative geometry, on the other hand, one has constant use for the so-called "fundamental law of algebra." The theorem that two curves, one of the mth order, and the other of the nth order, intersect in mn points not necessarily distinct, and like theorems, form the basis of this branch of mathe-

[1] Cf., for example, Veblen and Young, *Projective Geometry*, vol. 1, page 84.

matical science. For enumerative geometry, therefore, the convention that imaginary points exist with complex algebraic numbers for coördinates is essential.

But there are still other types of geometrical investigation and such as lead quite as naturally to a different method of extending the set of real points, and the present paper is intended to suggest some thoughts along this line.

Most mathematicians are familiar with the extensive use of complex numbers of the form $a + ib$, where $i^2 = -1$, in the study of electrical alternating current phenomena. A single algebraic variable is made to handle an essentially two-dimensional problem. In a similar manner, the methods of Hamilton are devised to treat a four-dimensional problem by the use of a single symbol, the quaternion, of the form $b = c_0 + ic_1 + jc_2 + kc_3$. The range of variation of two independent quarternions may be spoken of as a *two-space* or *plane*. A point in the plane will be given when particular values are assigned to the coördinates $(x_0 + ix_1 + jx_2 + kx_3,\ y_0 + iy_1 + jy_2 + ky_3)$. Here x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3, are eight real quantities in the ordinary sense.

The analysis in such a case differs somewhat from that for the usual complex numbers, since for quaternions multiplication is not in general commutative. Despite this fact one is occasionally obliged to investigate a problem in two quaternions, and the free use of geometric language serves here as elsewhere to render the results easily intelligible. This corresponds merely to speaking of a quaternion as a generalized number. A point in the plane will be called real, however, only if x_1, x_2, x_3, y_1, y_2, y_3 are simultaneously zero. The set of real points constitutes a real plane in the usual sense. The quaternion $b = c_0 + ic_1 + jc_2 + kc_3$ is from this point of view, real, when $c_1 = c_2 = c_3 = 0$, and "complex," or better, "hypercomplex" in other cases. The quaternions, $b = c_0 + ic_1 + jc_2 + kc_3$ and $\bar{b} = c_0 - ic_1 - jc_2 - kc_3$ are "conjugates," so that $b\bar{b} = \bar{b}b = c_0^2 + c_1^2 + c_2^2 + c_3^2$.

2. Vector Analysis. When we approach geometry from the standpoint of vector analysis, we find certain operations and rules of combination to be of great significance and others to be apparently of but slight importance. We shall illustrate our future remarks by reference to a space of three dimensions although all of our conclusions will be readily extended to linear spaces of any desired dimensionality.

Vectors are interpreted as points by the artifice of introducing a fixed origin, and of regarding any point as the terminal point of a vector originating from this fixed origin. Vectors and hence points are capable of unrestricted combination by addition and subtraction. Two vectors and hence two points, say a and a', determine a scalar called their inner product $a \cdot a'$. Any point may be multiplied by any scalar. Scalars in the ordinary theory are merely numbers and are called scalars to emphasize their distinction from vectors. Two or more points have furthermore an outer product that may be expressed by a determinant, when the given points are referred to any system of Cartesian coördinates having the given origin. The square of the distance between two points a and a' is a scalar expressible as $(a - a') \cdot \overline{(a - a')}$, where the second factor is the "conjugate"

of the first. Hence the point a' lies on the surface of a sphere of center a and radius r, if $(a - a')\cdot\overline{(a - a')} = r^2$. A fundamental property of vectors and hence of points in this system, is the following. If a and a' be any two points, $\sqrt{(a\cdot\bar{a})} + \sqrt{(a'\cdot\bar{a}')} \geqq \sqrt{[(a - a')\cdot\overline{(a - a')}]}$. In other words the sum of the lengths of two sides of a triangle is always greater than the length of the third side. It is to be noted that this inequality serves to give a meaning to certain infinite processes by making available a definition of limits among vectors.[1]

In extending a real system so as to include also imaginary points, a student versed in vector analysis is inclined to insist upon the preservation of such vectorial properties only as have proved most significant. To be sure, simple examples will always arise in which the imaginary points may be dispensed with. But when problems occur analogous to those with which vector analysis usually deals it is always a matter of interest to determine whether the customary methods are applicable or whether special technique must be developed. The outer product of two vectors is not commutative, and to sacrifice the commutative character of the inner product is not out of the question.

In the generalization to be considered, a point in three-space will continue to be designated by an ordered set of three non-homogeneous scalar coördinates (a_1, a_2, a_3). By this means the notation current for real points is preserved in the generalization, and the difficulties are referred a step further back, namely to the notion of scalar. Each coördinate is called a scalar or *generalized number* and is such as to reduce in a special case to a real number. We shall represent each of these scalars, a, in terms of more elementary quantities, b, in the following manner: Each a is a square matrix of order m, having the elements of the principal diagonal equal to each other, and elements below this principal diagonal, throughout equal to zero, while elements above this principal diagonal are left unrestricted. Any two scalars, a_1 and a_2, will be combined for purposes of addition and multiplication in accordance with the usual rules for adding and multiplying matrices, the multiplication for a pair of these square matrices being by rows of the first and columns of the second factor. The product of two generalized numbers is again a generalized number, but this product is not commutative unless further restrictions be imposed. Explicitly, let

$$a \equiv \begin{vmatrix} b_0, & b_{12}, & b_{13}, & \cdots, & b_{1m} \\ 0, & b_0, & b_{23}, & \cdots, & b_{2m} \\ 0, & 0, & b_0, & \cdots, & b_{3m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, & 0, & 0, & \cdots, & b_0 \end{vmatrix}.$$

The scalar, a, will be said to be "real" when and only when all of the elements above the principal diagonal vanish and when furthermore the single remaining independent element, namely the common element in the principal diagonal, is

[1] The importance of an inequality such as here illustrated for the establishment of limits, is well developed by J. Kürschák, "Über Limesbildung und allgemeine Körpertheorie", *Journal für die reine und angewandte Mathematik*, vol. 142, 1913, pp. 211-253.

itself "real," as this term will be later defined. By the *conjugate* of a, i.e., $\bar{a}$, is meant the scalar $b_0\bar{b}_0/a$, and it is defined only for such scalars as have $b_0 \neq 0$.

We may use the term *absolute value*, in connection with a scalar, a, and denote it by the symbol, $|a|$. This we define in terms of simpler elements, putting $|a|$ equal to $|b_0|$, where b_0 is the common diagonal element of the matrix defining the scalar, a, as above. As may be verified presently, the absolute value of any element, b, is so defined that the absolute value of a sum is equal to or less than the sum of the absolute values, and the absolute value of a product is equal to the product of the absolute values. The same propositions, therefore, will hold for the absolute value as applied to scalars, a, in fact $|a|^2 = a\bar{a}$, whenever $\bar{a}$ is defined. The distance from the origin to the point (a_1, a_2, a_3) is defined as the square root of the sum of the squares of the absolute values of the several coordinates. This is also the length of the vector from the origin to the point discussed. The question of signs arises here, in connection with the square root, in precisely the same manner that it does for real points. More generally, the inner product of the two points (a_1, a_2, a_3) and (a_1', a_2', a_3') in this order is defined as the scalar $a_1a_1' + a_2a_2' + a_3a_3'$. The sum of two vectors being defined in the ordinary manner, we shall have as a general theorem that the length of the vector obtained as the sum of two given vectors is equal to or less than the sum of the lengths of the two component vectors.

3. **Quaternions as Elements.** It remains to identify the elements, b, in terms of known quantities. One possibility is to consider the b's as themselves ordinary real numbers. A more interesting possibility has been suggested already, namely, each b may be taken as the quaternion expressed in the usual form, $b = c_0 + ic_1 + jc_2 + kc_3$, where c_0, c_1, c_2, c_3, are all ordinary real numbers, called the "fundamental components" of the quaternion b. If in particular, a subset be considered where the coefficients of j and k are zero for each of the quaternions, b, in question, these quaternions become merely ordinary complex numbers of algebra, or at least may be so regarded. If further, the order, m, of the matrices a, be unity, the scalars, a, will be themselves ordinary complex numbers.

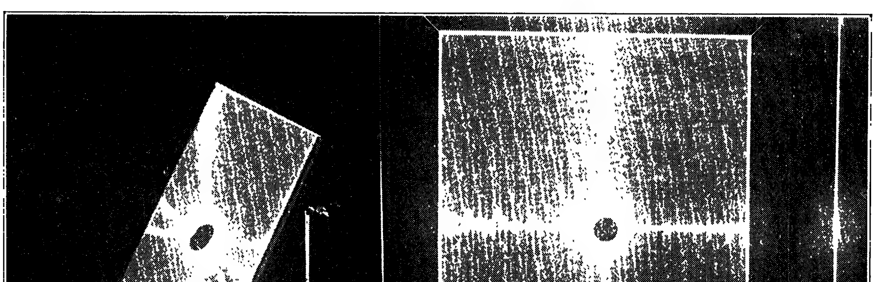
Equations in vector analysis with real coefficients may frequently be interpreted in terms of these generalized numbers and so refer as well to imaginary points of geometry, different from those usually treated. When m, the order of the square matrix used in defining the "scalar," a, is greater than unity, we are confronted with the difficulty that the conjugate of a sum is not in general the sum of the conjugates of the given quantities. Thus the rationalization of even so simple an equation as that of the spheroid whose bipolar equation may be written $\sqrt{[(x-a)\cdot\overline{(x-a)}]} + \sqrt{[(x-a')\cdot\overline{(x-a')}]} = \sqrt{(d\cdot\bar{d})}$ presents difficulties in the general case. Other inconvenient features are likely to impress any one more interested in the simplicity of handling equations than in the generality of the results secured. However, considerations of this sort are of interest, since, on the one hand, they show the possibilities of other interpretations of equations which have long been studied and, on the other hand, indicate the availability

of geometrical language in domains which might, at first, seem unsuited to such phraseology. If the order, m, of the matrices representing generalized numbers be allowed to become infinite in an appropriate manner, these generalized numbers become operators of the form used in the theory of integral equations. While it has not been customary to study integral equations in the domain of quaternions, some interesting results might be obtained in such a case.

LINES OF ILLUMINATION CAUSED BY THE PASSAGE OF LIGHT THROUGH A SCREEN.[1]

By WM. H. ROEVER, Washington University.

1. Description. On viewing a point source of light through a screen one will see on the screen several streamers of light emanating from the point in which the plane of the screen is pierced by the line connecting the eye with the light. The number of streamers which appear depends upon the direction of this line with respect to the plane of the screen. If, in particular, this line is perpendicular to the plane of the screen, there are four streamers, or two lines of light which cut at right angles. This is also the case for some other positions of the light and eye. In general, however, there are six streamers of light. On an old screen, these streamers are short and straight and one gets the impression of seeing a four- or six-pointed star. This is especially noticeable on looking at a source of light through the dust screen of a sleeping-car window. At first, when the source of light is far ahead of the train, the star will appear to have six points, but when the train is opposite the source of light the star appears to have only four points. If the screen is new and is made of polished wires of brass or copper, the streamers



extend to the frame and are well-defined curves. Views of this phenomenon are shown in the accompanying photographic reproductions. The black spot in these

[1] Presented to the American Mathematical Society, November 27, 1915. See *Bulletin* of that Society, vol. 22, p. 218.

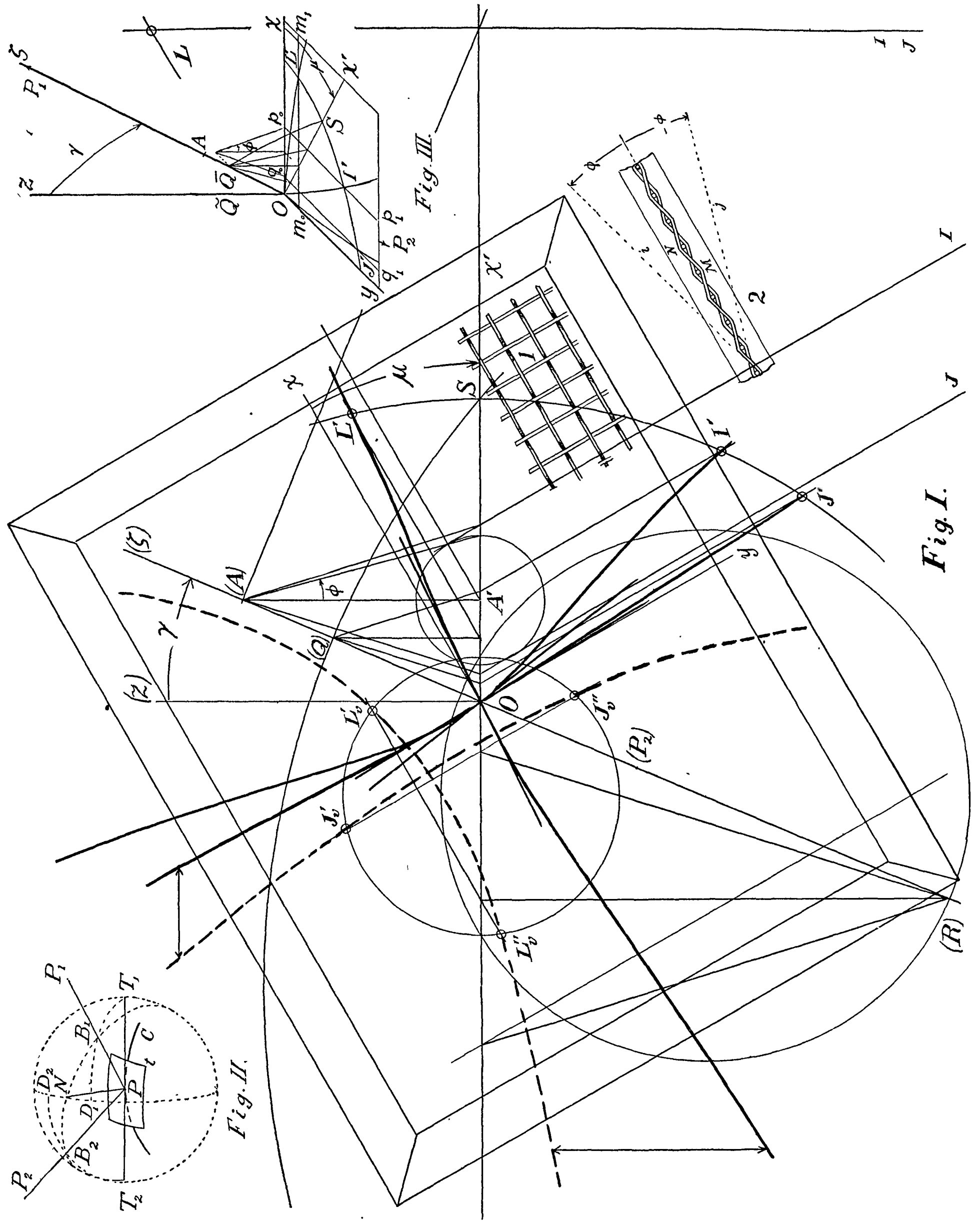


Fig. I. Fig. II. Fig. III.

reproductions is due to the presence of a small disk which was used to obscure the intense direct light from the source.

2. Cause. A close examination of the screen will show that the screening, from which it is made, must have been woven like cloth. The screening consists of a series of longitudinal wires, called the warp, which are interlaced with a series of tranverse wires, called the woof. In the weaving process of raising certain of the longitudinal wires and depressing others so as to form a *shed* for the passage of the transverse wires, the warp wires become crinkly, or wavy, but the woof wires remain unbent. The result of this operation is, that after the screening is attached to the frame, the wires of the woof are practically straight, parallel to each other and lie in the plane of the screen (or rather between two parallel planes which are separated by a distance equal to the diameter of these wires). The wires of the warp, however, are crinkly, or wavy, because they bend in and out around the straight wires of the woof, and each such wire lies in a plane perpendicular to the wires of the woof (or rather, between two such planes which are separated by a distance equal to the diameter of this wire). A clear notion of this construction may be obtained from views 1 and 2 of Fig. I. In view 2 the wires of the woof are represented by their cross sections, which are small circles separated by intervals of constant length, and the wires of the warp are shown as broad wavy lines. In view 1, the wires of the woof are represented by broad unshaded straight lines, while those of the warp are represented by broad straight lines which are shaded so as to represent the bending which takes place in crossing the wires of the woof.

From this description it is clear that the bend points of the wires of the warp, like M and N in view 2 of Fig. I, either receive no light from the source, which is on one side of the screen, or are obscured from the observer who is on the other side. Therefore, of the wires of the warp, only those approximately straight portions, which lie between the bend points, are effective in reflecting the light which comes from a source on one side of the screen to an observer who is situated on the other side. The straight wires of the woof (or, rather, those portions not obscured by the wires of the warp) are also effective in reflecting this light.

As far as its ability to reflect light from a source on one side to an observer on the other side is concerned, the screen may be regarded as composed of three groups of short, needlelike, pieces of wire. Those of one group (the portions of the wires of the woof not obscured by, or in the shadow of, the wires of the warp) lie in the plane of the screen and are parallel to each other. Those of each of the other two groups (composed of the short straight portions of the wires of the warp) are parallel to one another, are inclined to the plane of the screen at a constant angle and are perpendicular to the wires of the woof. View 2, Fig. I, represents the wires of one of these two groups as parallel to the line i and those of the other as parallel to the line j.

The wires of the screen, or rather the groups of short portions just described, reflect the light which falls upon them from the source, and many of the reflected rays reach the eye of the observer (even though it be on the side of the screen

opposite to that of the light). Those points of the wires where the rays are reflected which reach the eye of the observer, are the images of the source of light in the reflecting surface of the wires, and are called brilliant points. On account of the fineness of the mesh of the screen, consecutive brilliant points are close together, and it is the aggregate of these closely packed brilliant points which forms the streamers of light observed.[1]

3. Assumptions, Definitions, Tests. The solution of our problem may be much simplified without any appreciable loss in value, by assuming that the wires composing the screen are mathematical curves. For, the position of a brilliant point of a wire of small cross section differs very slightly from the position of the point which is approached by this brilliant point, as the cross section of the wire is diminished indefinitely. The geometric description of this limit point on the mathematical curve approached by the wire furnishes a definition for the brilliant point of a curve.

From the laws of the reflection of light it follows that the brilliant point of a surface may be described according to

DEFINITION 1. *A point P is said to be a brilliant point of a surface t with respect to a source of light P_1 and an observer's eye P_2, if the internal bisector of the angle P_1PP_2 is the normal to this surface at the point P.* See Fig. II.

If now the reflecting surface is supposed to be tubular[2] (like that of a wire) and the diameter of a cross section of this tube be regarded as infinitesimal (*i.e.*, as approaching zero), the brilliant point approaches a point on the mathematical curve approached by the tube which may be described by

DEFINITION 2. *A point P is said to be a brilliant point of a curve c with respect to a source of light P_1 and an observer's eye P_2, if the internal bisector of the angle P_1PP_2 is a normal to this curve at the point P.* See Fig. II.

It is of course evident that the rays PP_1 and PP_2 do not, in general, lie in the same tangent plane to the curve at the point P. If in definitions 1 and 2 the words "internal bisector" be replaced by the words "external bisector," these definitions become respectively those of the virtual brilliant point of a surface and a curve. Such points are not visible, but they present themselves in the analytic treatment when certain equations are freed from radicals.[3]

The method of treatment used in this paper was selected not only for the purpose of obtaining the equation of the brilliant point locus in question, but also

[1] At the suggestion of a referee, attention is here called to the fact that the spacing of the wires of the screen is so great as to make the phenomenon of diffraction negligible. The phenomenon described in this paper should therefore not be confused with the phenomenon observed in looking at a source of light through the stretched cover of an open umbrella.

[2] A tubular surface may be defined as the envelope of a sphere of constant radius which moves so that its center always lies on a curve (plane or skew), which is called the *axis*. If, in particular, the axis is a straight line, the surface becomes a cylinder of revolution; and if the axis is a circle, the surface is a torus.

[3] In definitions given by the author in the *Transactions of the American Mathematical Society*, vol. 9, pp. 245–279, the term *actual brilliant point* was used to designate the point which is here called simply *brilliant point*, and the term *brilliant point* was used to include both actual and *virtual brilliant points*.

with a view to furnishing a method of constructing this locus by means of circles and straight lines.[1]

In order to understand this method it will be necessary to know the following property of a sphere. If any point P on the surface of a sphere be connected by right line segments with each of a pair of points P_1 and P_2, which are conjugate relative to an inversion with respect to this sphere, and also with the points Q and R in which the diameter containing P_1 and P_2 pierces the sphere, the latter segments bisect internally and externally the angle formed by the former.

Any two points P_1 and P_2 of space constitute such a pair of points for any member of a certain one-parameter family of spheres. If r_1 and r_2 denote the distances of a general point P of space from the points P_1 and P_2, respectively, this family of spheres is represented by the equation

$$\frac{r_1}{r_2} = k, \qquad (1)$$

in which k is a (positive) constant for any particular sphere of the family.

This property of the sphere furnishes the following

TEST. *To determine whether a point P of a curve C is a brilliant point of C with respect to two points P_1 and P_2, construct the normal plane to the curve at the point P and denote by Q the point in which this normal plane cuts the line P_1P_2. If Q lies between P_1 and P_2, pass through it a sphere having P_1 and P_2 as conjugate points. Then the point P is, or is not, a brilliant point of the curve C with respect to P_1 and P_2, according as it does, or does not, lie on the surface of this sphere. If, in particular, the normal plane contains both P_1 and P_2, the point P is a brilliant point.*[2]

The truth of the statement of this test lies in the fact that the normal line PQ to the curve C at the point P does, or does not, bisect internally the angle P_1PP_2 according as P does, or does not, lie on the sphere which passes through Q and has P_1 and P_2 as a pair of conjugate points. If, in particular, the normal plane contains both P_1 and P_2, the bisector of the angle P_1PP_2 is surely a normal to the curve at the point P.

4. Solution of the Problem. In order to find the equations of the streamers of light described in section 1, we will make use of the test just given. To this end we will choose (as in Fig. III) a set of rectangular axes, Ox, Oy, Oz, of which the origin O is the point where the line P_1P_2 (connecting the eye P_2 with the source of light P_1) pierces the plane of the screen, which is taken to be coincident with the plane xOy, and the axis Oy is coincident with the wire of the woof passing through the origin O. We will further choose on the line P_1P_2 an axis $O\zeta$ of which the origin is the point O and the positive sense is from P_2 (taken below

[1] For other methods of treatment the reader is referred to an article by the author entitled "Brilliant point phenomena" published in the *Washington University Studies*, vol. 8, Scientific Series, pp. 131–160, 1921.

[2] To get the corresponding test for a virtual brilliant point, it is only necessary to replace the words "lies between P_1 and P_2" by the words "lies beyond the limits of the segment P_1P_2." If, in particular, the normal plane is parallel to the line P_1P_2 the sphere becomes the perpendicular bisecting plane of the segment P_1P_2.

the screen) to P_1 (above the screen), and we will denote by ρ the (perpendicular) distance of a general point P from this axis. We will denote by A the point halfway between P_1 and P_2, and by h and c the distances OA and AP_1, respectively.

The quantities r_1 and r_2 of equation (1) may now be expressed by the formulas $r_1 = \sqrt{[(\zeta - h - c)^2 + \rho^2]}$, $r_2 = \sqrt{[(\zeta - h + c)^2 + \rho^2]}$ and the equation (1) thus becomes

$$\frac{(\zeta - h)^2 + \rho^2 + c^2}{\zeta - h} = 2l, \tag{2}$$

in which $l = c(1 + k^2)/(1 - k^2)$ is the distance of the center of this sphere from the point A.

To express the coördinates ρ and ζ in terms of x, y, z, we will denote by α, β, γ the direction angles of the axis $O\zeta$ with respect to the set of axes, Ox, Oy, Oz. Then, evidently,

$$\begin{aligned} \rho^2 + \zeta^2 &= x^2 + y^2 + z^2, \\ \zeta &= x \cos \alpha + y \cos \beta + z \cos \gamma. \end{aligned} \tag{3}$$

Making these substitutions, equation (2) becomes

$$\frac{x^2 + y^2 + z^2 - 2h(x \cos \alpha + y \cos \beta + z \cos \gamma) + h^2 + c^2}{x \cos \alpha + y \cos \beta + z \cos \gamma - h} = 2l. \tag{4}$$

By allowing l in this equation to take on all real values, we obtain all the spheres which have P_1 and P_2 as a pair of conjugate points.

We will denote by $\bar{Q}$ (Fig. III) the point in which the line P_1P_2 (*i.e.*, the axis $O\zeta$) is cut by a common normal plane of the wires of the woof. This plane has the equation

$$y = \bar{y} = \text{constant } (= Om_0 \text{ in Fig. III}), \tag{5}$$

and therefore $\bar{Q}$ is determined by the relations $\bar{\zeta} = O\bar{Q} = \bar{y}/\cos \beta$, $\bar{\rho} = 0$. Substituting these values for ζ and ρ in equation (2) we obtain the expression

$$2l = \left[\frac{\bar{y}^2}{\cos^2 \beta} - 2h\frac{\bar{y}}{\cos \beta} + h^2 + c^2\right] \Big/ \left[\frac{\bar{y}}{\cos \beta} - h\right], \tag{6}$$

which gives the value of l for that sphere of the family (4) which passes through $\bar{Q}$. This sphere cuts the plane $zO\zeta$ in the circle $\bar{Q}S$ of Fig. III and the plane (xOy) of the screen in the circle $L'SI'J'$. The latter circle is represented by the equation which is obtained from equation (4) by giving $2l$ the value (6) and putting $z = 0$, *i.e.*, by the equation

$$\frac{x^2 + y^2 - 2h(x \cos \alpha + y \cos \beta) + h^2 + c^2}{x \cos \alpha + y \cos \beta - h} = \frac{\dfrac{\bar{y}^2}{\cos^2 \beta} - 2h\dfrac{\bar{y}}{\cos \beta} + h^2 + c^2}{\dfrac{\bar{y}}{\cos \beta} - h}. \tag{7}$$

On the other hand, the plane (5) cuts the plane (xOy) of the screen in the line m_0m_1 (Fig. III), which is represented by the equations

$$y = \bar{y}, \qquad z = 0. \tag{8}$$

According to the Test (stated in the preceding section), those points only (like L' of Fig. III) of the line (8) [m_0m_1 of Fig. III] are brilliant points of wires of the woof, with respect to P_1 and P_2, in which this line is cut by the circle (7) [$L'SI'J'$ of Fig. III]. Consequently, the locus of brilliant points (actual and virtual) of the wires of the woof is obtained by giving $\bar{y}(= Om_0$ in Fig. III) all possible real values, *i.e.*, by eliminating $\bar{y}$ between equations (7) and (8). The resultant of this elimination may be written in the form:

$$xy^2 \cos\alpha - x^2y\cos\beta + h(x^2\cos^2\beta - y^2\sin^2\beta)$$
$$+ (c^2 - h^2)\cos\beta(x\cos\alpha\cos\beta - y\sin^2\beta) = 0. \tag{9}$$

In order to determine the locus of brilliant points of the short (approximately) straight portions of the wires of the warp, we will denote by Q (Fig. III) the point in which the line P_1P_2 (*i.e.*, the axis $O\zeta$) is cut by a plane perpendicular to one, or other, of the two groups of such short portions. The equation of such a plane may be written in the form

$$\pm\, \epsilon z + x = \bar{x} = \left\{ \begin{matrix} Op_0 \\ Oq_0 \end{matrix} \text{ in Fig. III} \right\}, \tag{10}$$

in which $\bar{x}$ is the distance of the horizontal trace of this plane from the axis Oy, and $\epsilon = \tan\phi$ where ϕ is the angle of inclination of the short straight portions of the wires of the warp to the plane (xOy) of the screen. Then, for the upper sign equation (10) represents a plane perpendicular to the line i of view 2, Fig. I, and for the lower sign it represents a plane perpendicular to the line j. Let us denote the coördinates of $\tilde{Q}$ by $(\tilde{x}, \tilde{y}, \tilde{z})$. Since $\tilde{Q}$ is a point of the plane (10), we have $(\pm\, \epsilon\tilde{z} + \tilde{x})/\tilde{\zeta} = \bar{x}/\tilde{\zeta}$ or $\pm\, \epsilon\cos\gamma + \cos\alpha = \bar{x}/\tilde{\zeta}$, whence $\tilde{\zeta} = O\tilde{Q} = \bar{x}/e$, $\tilde{\rho} = 0$, where

$$e = \cos\alpha \pm \epsilon\cos\gamma. \tag{11}$$

Putting these values for ζ and ρ in equation (2), we obtain the relation

$$2l = \left[\frac{\bar{x}^2}{e^2} - 2h\frac{\bar{x}}{e} + h^2 + c^2\right] \Big/ \left[\frac{\bar{x}}{e} - h\right], \tag{12}$$

which gives the value of l for that sphere of the family (4) which passes through $\tilde{Q}$. This sphere cuts the plane $(z = 0)$ of the screen in the circle

$$\frac{x^2 + y^2 - 2h(x\cos\alpha + y\cos\beta) + h^2 + c^2}{x\cos\alpha + y\cos\beta - h} = \frac{\dfrac{\bar{x}^2}{e^2} - 2h\dfrac{\bar{x}}{e} + h^2 + c^2}{\dfrac{\bar{x}}{e} - h}. \tag{13}$$

It is no loss of generality to assume, as in Fig. III, that $\tilde{Q}$ coincides with $\bar{Q}$. Hence equation (13) is also represented by the circle $L'SI'J'$ in Fig. III. The plane (10) cuts the plane of the screen in the trace whose equations are

$$x = \bar{x}, \qquad z = 0. \tag{14}$$

In Fig. III this trace is represented by the line p_0p_1 or q_0q_1 according as the upper or lower sign of equation (10) is considered. According to the Test, those points only (like I' or J' of Fig. III) of the line (14) are brilliant points, with respect to P_1 and P_2, of wires of the warp (or rather, of the short straight portions thereof) in which this line is cut by the circle (13). Consequently, the locus of brilliant points (actual and virtual) of the short straight portions of the wires of the warp (and hence of the wires of the warp themselves, since, as we have seen, the bend points yield no brilliant points) may be obtained by giving $\bar{x}$ all possible real values, that is, by eliminating $\bar{x}$ between equations (13) and (14). The eliminant thus obtained may be written in the form

$$\pm \epsilon x^3 \cos\gamma - x^2y\cos\beta + exy^2 + h[(1 - e^2)x^2 - e^2y^2] + e(c^2 - h^2)[(1 - e\cos\alpha)x - ey\cos\beta] = 0, \tag{15}$$

in which e stands for the expression (11). For the upper sign this equation represents the brilliant point locus for the short straight portions of the wires of the warp which are parallel to the line i of view 2, Fig. I, and for the lower sign it represents the corresponding locus for the portions parallel to the line j.

The space construction represented in (cavalier) perspective in Fig. III is executed in the drawing plane of Fig. I by means of the Mongean method of descriptive geometry; thus the graphs of the curves (9) and (15) are obtained in Fig. I. The curve $OL'L_v'L_v''$ is the graph of the equation (9), the curve $OJ'J_v'J_v''$ is the graph of equation (15) for the lower sign, and the curve $O'I'$ is the graph of equation (15) for the upper sign. The full portions of these curves are the loci of the (actual) visible brilliant points and the dashed portions are the loci of the invisible virtual brilliant points.

If we equate to zero the first degree terms of equations (9) and (15) we obtain the equations

$$(1 - \cos^2\beta)y - \cos\alpha\cos\beta\cdot x = 0, \tag{16}$$

$$[1 - \cos\alpha(\cos\alpha \pm \epsilon\cos\gamma)]x - \cos\beta(\cos\alpha \pm \epsilon\cos\gamma)y = 0, \tag{17}$$

of the tangents to these curves at the origin. These are, in fact, the equations of the short straight streamers which we observed in viewing a light through a tarnished screen. From equation (17) it is evident that the two lines which this equation represents coalesce when $\beta = 90°$ (*i.e.*, when the line connecting the eye with the light is perpendicular to the wires of the woof). In this case equations (16) and (17) become $y = 0$ and $x = 0$ respectively. Thus for $\beta = 90°$ the six-pointed star becomes a four-pointed star.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

22. Sir David Brewster and the Stereoscope.

To students of mathematics Sir David Brewster (1781–1868) is best known for his *Life of Newton* (1828, greatly enlarged in 1855); to the physicist he is best known for his work in optics; and to the average citizen he is not known at all, although to him is due the kaleidoscope (1816), the major part of the invention of the stereoscope, and probably the first suggestions of the opera glass.

In connection with the invention of the stereoscope (1849), which had been to some slight degree anticipated by Wheatstone (1838), and which was greatly improved by our Dr. Oliver Wendell Holmes, he had a just grievance against his one-time friend, the Abbé Moigno (1804–1884), of Paris, a physicist and mathematician of considerable ability, and the founder of *Cosmos* (1852). Moigno had not given Brewster the credit which the latter felt was his due, and as a result a protest had been made. Moigno's failure to publish the correspondence in full in *Cosmos* resulted in a number of heated communications from Brewster, several (and probably all) of which are now in my collection. One of them, now published I believe for the first time, is given below and shows that the President of the Peace Congress, held five years earlier in London, was prepared for battle when he felt that his rights of discovery were not duly recognized.

The letter is as follows:

Dear Abbé Moigno,

I am very unwilling to do anything disagreeable to you, but *truth must be told* and *justice done* whatever be the cost.

By your delay in publishing a reply to an article in *Cosmos*, and your not having fulfilled the promise you lately made to me in Paris to publish it, you have driven me to write the enclosed pages which are still *unpublished and private*.

I will, of course, strike out the observations *which refer to you personally* if you publish, without comment, the communication I previously sent you.

Mr. Wheatstone, *who knows well that no stereoscope with prisms was ever made for me*, has *allowed you* to continue in the belief which you erroneously cherished, and has thus acted a most dishonest part.

The idea of a *Refracting Stereoscope* was *first* given by me, and *first* published by me in the article from the *Phil. Mag.* which I gave you in 1850.

I call your attention especially to the hidden, or rather palpable meaning of your phrase

Le croirait on?

and trust that as a man of honour you will do me justice. I am,

Ever Most Truly yrs.

D. Brewster

St. Leonards College
St. Andrews, May 8th, 1856.

23. Clifford's Genius Shown as a Boy.

Newton's remark that if Cotes had lived "we might have known something" may also be applied to the case of William Kingdon Clifford, one of the most promising of the British mathematicians of his day. Born at Exeter in 1845,

he died at Madeira, where he had gone in the vain hope of recovering his health, at the age of thirty-four. He was only twenty-six when he was made professor of applied mathematics at University College, London, and only twenty-nine when he was elected a Fellow of the Royal Society. His contributions to the study of the graphic methods of Möbius, to the theory of Riemann's surfaces, and to the development of biquaternions, together with his *Common Sense of the Exact Sciences,* are all well known, as is the edition of his *Mathematical Papers* by Mr. Tucker in 1882.

Among several of his autograph letters now in my collection is one of particular interest, written on St. Giles's Day (September 1) 1863, just before he entered Trinity College, Cambridge. It shows in a very personal manner the nature of his interests at an age when most boys are concerned with pursuits that are quite in contrast with those which he sets forth.

The letter is as follows:

9 Park Place, Hill's Court,
EXETER, S:Giles, 1863.

My dear Sir

I thank you very much indeed for your kindness, and am sorry that I should have given you the trouble to write to me. I have been on a walking tour in the North of Devon, or I should have written long before. By to-morrow I hope to send you something, and will do what I can to follow out your kind suggestions; but I am a very junior reader myself. I have had in my mind almost from the time I began to fly kites (I have not yet left off) the problem of finding the form of a kite-string under the action of the wind. On a rough trial the other day, the intrinsic equation seemed not very difficult to obtain; if I get at any result, I will send it you to-morrow. I have been trying to construct a second interpretation of mechanical equations, similar to that of tangential coordinates, but have failed hitherto. Being a firm believer in the duality of symbols, I should look upon complete failure as a proof that our symbolical system is wrong. You will be amused by my visionary attempt at obtaining a method of inventing problems by the dozen.

With best thanks, believe me to remain,

Yours very sincerely
+ W: K: CLIFFORD.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

A number of questions proposed at various times preceding the present year are reprinted below in the hope that fresh interest may be aroused in them. Some of these have already called forth replies, as indicated in the notes, and in several cases very profitable discussions have arisen; but the questions have still to be completely settled.

At the same time the editor hopes that there will continue to be a lively flow of discussion on new matters. The unfortunate delay in the appearance of recent numbers of the MONTHLY has probably interrupted to some extent the correspondence of this department; and the consequent speeding up in publication which is now looked for calls for a proportionate acceleration in the supply of contributed material. The function of these columns is to afford opportunity for the exchange of views of all possible types; and this end may be furthered by the number, variety, and suggestiveness of questions proposed, and by a liberal supply of answers and discussions on both old and new topics.

REPRINT OF OLD QUESTIONS.

15 [*1914*, 278; *1916*, 353; *1919*, 68; *1920*, 114, 361; *1921*, 124]. In the *Proceedings of the Royal Society of Edinburgh*, vol. 7, p. 144, in some mathematical notes by Professor P. G. Tait, it is stated:

"If $x^3 + y^3 = z^3$, then $(x^3 + z^3)^3y^3 + (x^3 - y^3)^3z^3 = (z^3 + y^3)^3x^3$.

"This furnishes an easy proof of the impossibility of finding two integers the sum of whose cubes is a cube."

How does this "easy proof" follow?

Students are notoriously suspicious of those steps which an author announces as "easy," and are sometimes inclined to believe that the word is used in a humorous sense. The present case almost justifies such an attitude, since so far no reader has furnished a satisfactory explanation, and it is difficult to resist the suspicion that the "proof" is an illusion.

In a previous editorial note (*1920*, 361) a statement was made of the principal types of answer arriving in response to this question. To prevent the repetition of a former misunderstanding, it may be said here that the question is not how the second equation may be deduced from the first, but how the second equation may be used to facilitate the proof that the first is impossible. There are of course proofs in existence that the sum of two cubes cannot be a cube.[1]

21 [*1914*, 341; *1916*, 354; *1919*, 68, 239; *1920*, 114; *1921*, 114]. For the Diophantine equation

$$x^2 - y^3 = 17$$

there are known the following solutions:

$$\begin{array}{rrrrrrrrr} x = & 3, & 4, & 5, & 9, & 23, & 282, & 375, & 378661, \\ y = & -2, & -1, & 2, & 4, & 8, & 43, & 52, & 5234. \end{array}$$

One of our readers, who supplied the foregoing facts, desires to know the answers to the following questions: Are there other solutions of the given Diophantine equation? How may all the solutions of this equation be found by a systematic procedure?

A note by E. B. Escott (*1919*, 239) contained several methods for finding solutions, and references to the literature[2] suggested the probability that an infinite number of integral solutions existed; but no others were given. Hence both parts of the question await a final reply.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460; *1921*, 19, 125]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h)\right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

Although nine replies by six different writers have been published, the problem is by no means exhausted. As to the interpretation of the question, there is no problem worth discussing if h is taken to be constant, so that each correspondent has naturally assumed that the equation is to be true for a variable h over the interval $0 < h < H$. Starting from the well-known fact that the equation is true for polynomials of degree three at most,[3] E. Swift (*1920*, 301) proved that it was

[1] See, for instance, R. D. Carmichael, *Diophantine Analysis*, p. 67.

[2] L. J. Mordell, *Proceedings of London Mathematical Society*, series 2, vol. 13, 1914, pp. 60–80. See also L. E. Dickson, *History of the Theory of Numbers*, vol. 2, pp. 537–9.

[3] The prismatoid formula is that special case of Cotes' formula of approximate integration for which the interval is divided into *two* equal parts, and the integrand assumed to be of degree two. When the method is applied to a polynomial of degree three, the interpolation for the

true for no other analytic functions regular at the origin, and A. A. Bennett (*1920,* 301) obtained a somewhat more comprehensive result. D. C. Gillespie (*1920,* 405) showed that the equation was satisfied by no other function having six continuous derivatives in the interval. On the other hand, A. A. Bennett (*1920,* 460) found analytic solutions of the type $f(x) = x^2 \sin (b \log x - c)$ having essential singularities at the origin; and, in editorial remarks by W. A. Hurwitz (*1920,* 462), it was shown (1) that the value of b may be real, (2) that the function then has, from the real variable point of view, a continuous derivative at the origin. The main question now is: *How many continuous derivatives at most may $f(x)$ possess at the origin, if $f(x)$ satisfies the given equation, but is not of the form $ax^3 + bx^2 + cx + d$?* The answer may be anything from *one* to *five* inclusive.

Throughout the discussion the behavior of the function in the neighborhood of the origin is the point of critical importance, and the problem is fundamentally different when so modified that the identity of the origin is lost. Thus the related question where the equation is to be true after an arbitrary translation of axes was satisfactorily answered by J. P. Ballantine (*1921,* 19), who showed that $ax^3 + bx^2 + cx + d$ was the only *continuous* solution.[1]

36 [*1919,* 69, 291–295], part 1. For what values of n can $\cos 2\pi/n$ be expressed in the form $(a + \sqrt{b})/c$, where a, b, and c are integers?

The other parts of this question, relating to equations of the third and fourth degrees, have been answered (*1919,* 292). In a recent issue (*1921,* 374) R. S. Underwood answered a question very near to this, showing that, of angles commensurable with π, only multiples of $\pi/3$ have *rational* cosines. We know that the answer to the present question includes multiples of $\pi/4$, $\pi/5$ and $\pi/6$. What else does it include?

39 [*1920,* 256; *1921,* 125]. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion it should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may

integrand is inexact, but the value of the definite integral is by accident correct. More generally, if $2n$ equal subdivisions are used, the definite integral will be found to be correctly evaluated, not only for an integrand of degree $2n$, but also for one of degree $2n + 1$. Gauss was perhaps the first to notice this fact (see *Encyclopédie des Sciences Mathématiques,* I 21, pp. 120–2). It is ignored in a number of the standard texts dealing with interpolation; though it is quite evident when central differences are used. In fact, when the interpolation is made from *any* $2n + 1$ values of x, as x_1, x_2, ..., situated *symmetrically about* $h/2$, the error in approximating to a polynomial of degree $2n + 1$ is of the form $C(x - x_1)(x - x_2) \ldots$, which is an odd function of $x - h/2$, so that its integral between 0 and h vanishes.

[1] For further illustration of the importance of the fixed origin in the problem as proposed, the reader may contrast it with the case of the equation

$$\int_{x=-h/2}^{x=h/2} f(x)dx = (h/6)[f(-h/2) + 4f(0) + f(h/2)]$$

for which the most general continuous solution is $f(x) = a + bx^2 +$ (an odd continuous function).

be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an inscribed polygon of greatest area R. For a fixed value of the integer n and for all convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$ the upper limit of Q/P is 2.

3. Let the area of a simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \ldots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see part 1 of this question), always remaining inscribed to K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

The questions contained in no. 39 were proposed by Professor S. Kakeya, and were contributed to this department at the request of the editors. The questions have aroused considerable interest, but up to the present have not been solved.[1]

41 [*1920*, 365; *1921*, 126]. A reader asks for an elementary proof of the following two propositions in number theory, either of which can readily be obtained from the other:

Every positive integer of the form $8n + 3$ *is the sum of three odd squares.*

Every positive integer is the sum of not more than three triangular numbers.

According to Bachmann[2] these results have only been proved by means of the theory of ternary forms.

42 [*1921*, 65, 126]. In connection with the questions of Kakeya, Professor W. B. Ford is led to the following inquiry: A line-segment AB is to be moved in its plane to a new position $A'B'$. How should this be done in order that the area generated may, to the greatest extent possible, be passed over three times?

Professor Ford has proved that, if the generated area is to be passed over, to the greatest possible extent, but *two* times, AB should be moved so that its instantaneous center of rotation is always on its right bisector.[3]

43 [*1921*, 260]. Is any rapid method known for the evaluation of the Sylvester determinant met with so often in elimination by the dialytic method? It would seem that there must be, both on account of its interesting shape, and of its frequent occurrence.

45 [*1921*, 305]. Is every non-trivial solution in integers of the equation $t^3 = x^3 + y^3 + 1$ expressible in the form $x = 9r^4 - 3r$, $y = 9r^3 - 1$, $t = 9r^4$? If there are non-trivial solutions not expressible in this form, can a general solution be found?

This problem arose out of a note by H. C. Bradley (*1921*, 307) in which he obtained the solution quoted by specialization of the formula

$$x = -(a - 3b)(a^2 + 3b^2) + 1, \qquad y = (a + 3b)(a^2 + 3b^2) - 1,$$
$$u = -(a^2 + 3b^2)^2 + (a + 3b), \qquad v = (a^2 + 3b^2)^2 - (a - 3b),$$

which was shown by Euler and Binet to be (except for a common multiplier) the most general rational solution of the equation $x^3 + y^3 = u^3 + v^3$, a and b being any rational numbers.[4] This leads at once to the general *rational* solution of the

[1] See, however, Question **42**, reprinted in this number.

[2] *Niedere Zahlentheorie*, Leipzig and Berlin, 1910, Teil 2, p. 325. See also Dickson, *History*, vol. 2, Chapter 7.

[3] W. B. Ford, "On Kakeya's minimum area problem," *Bulletin of the Amer. Math. Soc.*, vol. 28., 1922, pp. 45-53. Other references are given in this paper.

The editorial note in this MONTHLY (*1921*, 126) quotes Professor Ford's conclusions inaccurately.

[4] Carmichael, *Diophantine Analysis*, p. 65; Dickson, *History*, vol. 2, p. 555.

more special equation, so that the problem awaiting solution is one of *integral* values only.

DISCUSSIONS.

The following note is a reply to that of Professor E. T. Bell (*1920*, 413), in which the use of mathematical induction in elementary teaching was criticized on logical grounds. Professor Bell's position was that the method required the postulation of a special logical principle, and was therefore unsuited to beginners.

On Proofs by Mathematical Induction.

By R. S. Hoar, South Milwaukee, Wis.

The objections to the usual method of stating mathematical induction, raised by Piofessor Bell in the Monthly of November, 1920, seem to call for some further discussion.

When I was a boy, the part that confused *me* was: "We first establish a theorem for $n = 1$; then we show that, if it is true for $n - 1$, it is true for n." But, if n is 1, $n - 1$ must be zero. We have not proved it for zero, and we already know that it is true for n. Of course, such an attitude is absurd, even though natural. But is it absurd to object that we never prove the proposition for $n - 1$ equals anything (but merely for n equals something), and therefore never lay the foundation for the second step?

Why not state the process as follows? "We first show that, if the proposition is true for n, it is true for $n + 1$; then we show that it is true for a particular value of n, namely $n = 1$; then, since any whole number can be reached from any preceding whole number by successively adding 1 a finite number of times, the proposition is true for n equals any whole number."

The false proof that any assemblage contains an infinity of members, cited by Prof. Bell, is not based on mathematical induction in the form objected to by him, but rather is based upon the layman's substitute for mathematical induction; namely, that if a proposition is true for $n = 1$, $n = 2$, $n = 3$, etc., repeated a "reasonable" number of times, then it must be true for any value of n.

The fact that such "proofs" are current among freshmen is probably due to the failure of their teachers to emphasize the necessity of taking the abstract general step from n to $n + 1$; just as the currency of "proofs" that two equals one, is due to the failure of teachers to introduce non-divisibility by zero at the very start, as a fundamental part of the concept of division, instead of dragging it in later as a rather lame exception, in order to defend division from the onslaughts of two-equals-one.

A real example of the non-applicability of mathematical induction would be the series set forth on page 15 of Huntington's *The Continuum* (Cambridge, Mass., 1917):

$$1_1\ 2_1\ 3_1\ \cdots;\qquad 1_2\ 2_2\ 3_2\ \cdots;\qquad \cdots.$$

Perhaps it would be well for teachers to exhibit both this series and the "infinite assemblage," in order that the students may realize the necessity for all three steps in mathematical induction.

REMARKS BY THE EDITOR.

Mathematical induction may be criticized with regard either to its logical basis or to its practical intelligibility. Professor Bell's contribution was of the former kind, and Mr. Hoar's of the latter. Philosophy apart, the difficulty about a proof by induction will be in making "the abstract general step from n to $n + 1$." Certainly the irregular use of n and $n - 1$ to which allusion has been made is a needless source of confusion, though perhaps not very serious in the long run. There are of course many to whom the very idea of the abstract general step is a hidden mystery. Such persons constitute a perennial problem in elementary teaching.

While we have the subject before us, it may be of interest to note that mathematical induction is only one of a number of processes by which theorems are extended over a wider range according to the following general plan:

It is known of a theorem T: (1) that T is true for numbers lying in a given range N_0; (2) that whenever T is true in any range N, T is also true in the range $\Omega(N)$ derived from N by means of the operations forming the set Ω. Then T is true for the numbers of any range which can be reached from N_0 by a finite number of successive applications of Ω.

Special cases are:

(*a*) N_0 consists of the number 0, and Ω consists of the operation which, applied to any set of consecutive integers, gives the same set with the next higher integer added. This is ordinary mathematical induction, proving results for all positive integers.

(*b*) The same except that it is the next lower integer that is added. This proves theorems for all negative integers.

(*c*) N_0 consists of all integers; Ω includes division by any positive integer. This proves theorems for all rational numbers.

(*d*) Cases where Ω includes the operation of adjoining to a set of numbers the limits of the set.

Some of the proofs given for the binomial and exponential theorems for general real exponents are combinations of these four types of argument.

(*e*) N_0 is the interval $(0, x_0)$; Ω consists of the operation which replaces any interval $(0, x)$ by $(0, f(x))$, where f is a positive function. This extends theorems to the interval $(0, u)$, where u is the least upper bound of $(x_0, fx_0, ffx_0, \cdots)$.

The special case of (*e*) in which $f(x) = x + k$, leading to theorems on all positive numbers, was given in detail by Y. R. Chao,[1] and illustrated with a "thin end of the wedge" fallacy. In the illustration x_0 is the measure of a dose of an intoxicant, too small to be harmful. By using a plausibly small constant k it is argued that any amount is harmless. This optimistic conclusion is reached by correctly applying the method to rather questionable assumptions.

Errors perhaps occur more often from incorrect use of (*e*) with data which are reasonable enough. For an extreme example: An athlete is known to be

[1] "A note on 'Continuous Mathematical Induction,'" *Bulletin of Amer. Math. Soc.*, vol. 26, 1919, p. 17.

able to jump any height up to five feet. However high he may jump, he could always have gone a little higher (which is at least as easy to admit as the contrary). Hence he could jump any height. But in reality, $x_0 = 5$ and $(5, f(5), ff(5), \cdots)$ is a *convergent* sequence whose limit is probably less than 7.

RECENT PUBLICATIONS.

REVIEWS.

Dynamic Symmetry, the Greek Vase. By JAY HAMBIDGE. New Haven, Yale University Press, 1920. 161 pages, small folio. Price $6.00.

A mild sensation has been caused among classicists, artists and the philosophers of æsthetics by the doctrine of "dynamic symmetry" as recently enunciated and promulgated by Mr. Jay Hambidge and his converts. Certain ambitious claims have been made which have been greeted, possibly with scepticism and repugnance by some artists and classicists, but certainly with interest and sympathy by many readers, including the Yale School of Fine Arts, and the members of various museum staffs. The exposition has been clothed in a certain amount of numerical terminology, so that many who would, perhaps, desire to approach the subject with an open mind may have been discouraged at the start and led to regard the matter as too technical for the comprehension of any one not deeply versed in the intricacies of higher mathematics.

So far as the reviewer is aware, this work has not received from professional mathematicians the attention that it deserves, particularly since any adequate criticism of the results claimed can come only from one prepared properly to appraise the mathematical background of the argument. The book under review is assuredly not intended as a reference work in geometrical analysis, but in its wider philosophical aspects, its claim for the historical and logical application of geometry to design must merit the consideration of any one who is interested in the place of mathematical discipline among the cultural developments of civilization.

Let us first cite some passages in which Mr. Hambidge enunciates his claims. "Some twenty years ago, the writer, being impressed by the incoherence of modern design and convinced that there must exist in nature some correlating principle which could give artists a control of areas, undertook a comparative study of the bases of all design, both in nature and in art. This labor resulted in the determination of two types of symmetry or proportion, one of which possessed qualities of activity, the other of passivity. For convenience, the active type was termed dynamic symmetry, the other static symmetry. It was found that the passive was the type that was employed most naturally by artists, either consciously or unconsciously; in fact no design which could be recognized as such—unless indeed it were dynamic—would be possible without the use, in some degree, of this passive or static type. . . . A study of the basis of design in art shows that this active symmetry was known to but two peoples, the Egyp-

tians and the Greeks, the latter only having developed its full possibilities for purposes of art. The writer believes that he has now recovered, through study of natural form and shapes in Greek and Egyptian art, this principle for the proportioning of areas. . . . At some time during the Sixth or Seventh Century B. C. the Greeks obtained from Egypt knowledge of this manner of correlating elements of design. In their hands it was highly perfected as a practical geometry, and for about three hundred years it provided the basic principle of design for what the writer considers as the finest art of the Classic period. Euclidean geometry gives us the Greek development of the idea in pure mathematics; but the secret of its artistic application completely disappeared. Its recovery has given us dynamic symmetry—a method of establishing the relationship of areas in design composition. . . . This idea (the application of areas) is quite unknown to modern art but that it is of the utmost importance will be shown in this book by the analysis of Greek vases. . . . Analysis of Greek and Egyptian composition shows that the artist always worked within predetermined areas. The enclosing rectangle was considered the factor which controlled and determined the units of the form. . . . Analysis of any fine Greek design is sure to disclose an arrangement of area which produces the quality of inevitableness, so conspicuously absent in modern art."

Mr. Hambidge when speaking of mathematical notions does not employ in all particulars customary mathematical language. This may be due to a desire to escape from the unnecessarily technical, but in any case it makes the reading more difficult for the mathematician and probably meaningless for the average artist. When he speaks of "proportioning areas," it has no reference to the familiar artistic principle often discussed under the term, space balance, or the balance of areas, which will be referred to later. The "proportioning of areas" means here, in effect, the determination of significant points and horizontal lines within a rectangular diagram by use of geometrical methods.

A condensed summary of the claims made for dynamic symmetry as applied to the Greek vase may be expressed in the following two statements concerning the profile curve of the lateral elevation of the vase, including the handles if such there be. On the negative side—except for the fundamental bilateral symmetry of the design, equipartition of ordinates and of abscissas is avoided, and any regularity of linear division that may exist avoids simple commensurate ratios. On the positive side—while commensurate ratios are avoided, the incommensurate ratios occurring are such that their squares or rational combinations among these squares are simply commensurable. This means incidentally, as every mathematician knows, that the significant points and lines can be located by an elementary use of ruler and compasses.

To what extent has Mr. Hambidge established his theory? As in the case of other enthusiasts there is among his remarks no hint of doubt or failure. He is convinced whole-heartedly of his claims and expects the reader to acknowledge as ample the evidence displayed. Analysis is given in drawings or tables of many examples and types. A count of these scattered throughout the text

shows the following distinct specimens, here collected and listed by type: amphora (15), small cup (1), dinos and stand (1), hydria (3), kalpis (7), kantharos (2), bell krater (3), volute krater (1), kylix (19), lekythos (10), olpe (3), oinochoe (2), pelike (3), perfume vase (1), psykter (1), pyxis (3), skyphos (14), stamnos (2), a total of 91 cases analyzed. While this is of course an inconsiderable proportion of the extant Greek vases, yet results found to hold uniformly in the examination of the first 91 random samples may be fairly assumed to establish the presumption that the thesis is valid throughout.

In emphasizing the beauties of his new theory, Mr. Hambidge appears to have overstressed the lack of principle in current theories of design. The inevitableness and geometrical background that he claims as the characteristic features of the doctrine of dynamic symmetry have not been rediscovered and expounded here for the first time. Let me quote from E. A. Batchelder's conservative *Principles of Design:* "A finished and satisfactory design should never suggest the question, 'Can I remove this feature'? or 'Is that line necessary'? There should be no question about the importance of each line involved" (page 25). Again, "The law of balance in so far as we have gone may be concisely stated as follows: 'Equal measures of equal contrasts will balance at a point midway on a line connecting the centers of the measures. Unequal measures of equal contrasts will balance on a line connecting the centers of the measures at distances which will be in inverse ratio to the measures'" (page 42). "The Indian has not yet acquired sufficient grasp on the principles of his art to allow his imagination to express itself unrestricted by the limitations of geometry. Japanese art, on the other hand, to the casual observer, contains very little of geometry; but the student of the art of Japan knows that the work of that country rests just as firmly upon the fundamental principles of design as the work of any other nation or period" (page 44).

Mr. Hambidge seeks to forestall certain obvious questions. Most of his measurements are made by the members of the staffs of the museums which contain the vases analyzed, and he takes care to remark in some cases that the drawings were not made to fit the measurements expected, but were first executed carefully to scale, and subsequently analyzed. A not unnatural basis for scepticism he disposes of as follows: "The shape of the vessel is determined with great care" and "shows unmistakably that the picture (the red-figured decoration upon the vase) is secondary. . . . The idea that so much care should be taken to proportion such a commonplace article as a clay pot will probably strike the average reader as fanciful. And it would be so if ordinary pottery were under consideration. The vases considered here, however, are Greek, and the Greek vase is unique. . . . Volumes have been written about the pictures found on Greek pottery, but the shape or form of the vase, which is of much greater importance, has been almost entirely neglected. . . . To the ancient Greek, the form of the vase was of vital importance, the vase painting was usually of secondary importance, a fact made clear by the great preponderance of signatures of potters over those of painters."

Mr. Hambidge has assuredly succeeded in exhibiting certain vase forms in which the simple mathematical proportions of the parts are too convincing to be dismissed as accidental. In particular his single example of a small cup shows each of its important points identified by the simplest sort of construction. The fact that many of the vases are "exactly" inscribed in a square, that so often among the examples discussed the width of the foot is "identical" with that of the lip seems hardly likely to be a mere coincidence. However the reviewer at least is far from being convinced as to the general claim.

The question reduces as to whether evidence has been shown that Greek vases were regularly designed with mathematical accuracy in all essential parts to accord with measurements in a diagram constructible with ruler and compasses. The independent incommensurables referred to by Mr. Hambidge are, in particular, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, the so-called "whirling square" ratio being the positive root of the equation, $1/(x - 1) = x$, namely, $\frac{1}{2}(1 + \sqrt{5})$.

I shall first point out what I regard as fatal flaws in the argument for the historical significance of dynamic symmetry. To put the matter briefly, it seems that the reader is at liberty to question both the exactness and the adequacy of the analyses submitted in the book.

The impression is conveyed that most measurements displayed check to a remarkable degree of exactitude with the drawings exhibited. This impression is heightened not only by explicit statements to that effect but also by the careful pointing out and explanation of a few noticeable but not serious departures. Occasional remarks however have slipped into the text that invite a contemplative pause. Thus, "The percentage of error is much smaller in the bronzes than in the pottery." The term "error" is clearly used to denote discrepancy between assumed analysis and measured dimensions. It is enlightening that no hint of recurrent percentage error in the pottery is made until it is desired to emphasize the accord of the bronzes with assumed canons. A detailed explanation is offered as to why the handles of the kylikes should show such random heights in their relations to the bowls. The explanation sounds more ingenious than plausible, but in any case admits an irregularity as to heights that is thereafter ignored. One wonders why out of a list of fifty-four analyzed kylikes, a table of but seventeen is given. One suspects, perhaps unfairly, that the remaining examples fail to bear out the theory in so obvious a manner. Throughout the treatise the reviewer feels that there is a seeming inclination to overlook or condone apparent discrepancies, and little suggestion of an impartial statistical attitude that would invite and compel conviction.

The adequacy of the analysis may be criticized on many counts. The vase profiles are in general interesting curves not identified by means of a few points and tangents. One is led at once to inquire whether the analysis covers the determination of these most important features of the design. The author says, "No mathematical curves have, so far, been found in Greek art. . . . Curves were apparently drawn by tangents in this manner (with assigned tangents at certain points) all through the Greek classical period." Certain portions in a

design which do not lend themselves to convenient analysis are referred to as "loosely rendered." A cursory survey of the analyses submitted is pleasantly satisfying, but a closer inspection frequently reveals what seems little more than random selection. Certain conspicuous points in the design are located in the system, but others quite as important are not included among those present and accounted for. It is particularly striking that of two vases practically identical to the eye, the analysis should in one case include certain points of the vase profile, while for the other a set notably distinct should be chosen for special mention. One feels instinctively that the featured points were signalized *a posteriori* as being those which appeared to fit passably into an inflexible and inadequate mould. In several cases the drawings exhibit the actual asymmetry of the material vase, and lines determining intersections for the right half of the vase would fail for the left, leaving the actual significance of the points plotted entirely in doubt.

But there are much stronger reasons for challenging the principal argument of the book. One may be prepared to agree with Mr. Hambidge in believing that Greek architecture and architectural design employed dimensions accurately laid out by ruler and compasses in incommensurable as well as in commensurable ratios. An undertaking in which the designer and the artisan are distinct, such as building a temple, gave ample scope for the use of working drawings, laid out carefully by such geometrical drafting instruments as were available. Temples were not essentially commercial and were not multiplied daily. There is however ample evidence that the Greek vases, except only for some of very large dimensions, were turned on a potter's wheel; that the designer was in many of the finest examples also the executor, and could therefore judge with his eye as to esthetic excellence as easily as measure the wet clay against a scaled drawing. That no two Greek vases are known of identical form, despite the rigid commercial types into which they fell, is almost proof positive that no exact measurements were intended. Vases were manufactured by the hundreds and thousands and for strictly utilitarian ends. Their beauty and their charm were merely reflections of the maker's pride in careful and honest workmanship. All authorities agree that the finest Greek vases were never intended as mere decorations. Their design as well as their execution was in the hands of the metics, a skilled but usually poorly educated class of artisans. It is next to inconceivable to me that a complicated working drawing was made afresh for each vase, that differences scarcely noticeable to the eye were purposely secured at great labor by extensive and thoroughgoing alterations in the pattern of construction lines. Yet this is what Mr. Hambidge would have us believe. Granted that the curves themselves were not constructed by mathematical methods, what right have we to suppose that the critical points of the design need have been so chosen?

It remains to dispose of the immediately pertinent inquiry as to how one can account for the remarkable analyses submitted by Mr. Hambidge. My answer is a simple one, perhaps too simple to appeal to the fancies of any who have caught that distinguished illustrator's enthusiasm for dynamic symmetry. The

author uses the square roots of 2, 3, and 5, and all that can be obtained from these as linear expressions with rational coefficients. He reserves also the privilege of utilizing other quadratic surds in the same manner but appears to find these sufficient. Among the infinite set of numbers between zero and unity that can be so obtained, I have noted 53 distinct ones which the author mentions explicitly in the course of the text. Two of these, namely, $2\sqrt{5} - 4$, which he writes as .472, and $\sqrt{2/3}$ which he writes as .4714 (the text employs decimal notation throughout), differ by approximately one part in fourteen hundred. Thus by using this simple difference alone we can obtain something like fourteen hundred intervals between zero and unity. The reply to the query is then this obvious one, namely, that Mr. Hambidge has provided himself with so bountiful a list of incommensurables that any observed measurement may be expressed by the aid of these to a degree of precision beyond the most fastidious ambitions of the potter. There is a single table in the book exhibiting certain relative dimensions of several vases. Without being here concerned with the particular significance of the data selected, we shall write down for comparison two possible sets. The first row is selected from corresponding entries in the table which are supposed to illustrate, by their numerical form, "root two figures." These are to be contrasted with those of the second row which has been selected as illustrating figures which are not "root two" in theme. The numbers have been chosen from the columns given by the author, although it is but fair to remark that the numbers here given do not all happen to occur in a common row in the book. These rows are as follows:

$$3.4142, \quad 2.4714, \quad .9393, \quad .2929, \quad .7071, \quad .3535;$$

and

$$3.382, \quad 2.472, \quad .927, \quad .309, \quad .691, \quad .354.$$

These dimensions would be regarded by the author as characteristic of vases of markedly different type. It is beyond the present credibility of the reviewer to conceive as impossible that the analyst in measuring two vases, one with the ratios of the first row, and the other with the ratios of the second row, might not utilize his assumption of percentage errors and fall into the "confusion" of mistaking these two shapes as negligible variants from a common design. Mr. Hambidge even remarks in one connection that the analysis may not be unique, although perhaps he there refers to the geometrical steps followed and not to the numerical data assigned.

Unlike most books on design, this text explicitly warns the inquiring artist against applying purely esthetic criteria, informing him that to emulate Greek perfection, it is essential and apparently sufficient to acquire facility in manipulating the elementary arithmetical incommensurables. The author does not hesitate to praise one vase above another, both of which are supposed to comply equally with the canons of numerical exactitude. How to design a *good* vase as distinguished from an exact but ill-conceived one is not discussed. And is not this the most important question suggested by the whole discussion?

For its probable future influence, its suggestive analyses, its rich lore of illuminating and interesting mathematical and archeological references, and not least for its exquisite half-tones of rarely charming vases, this book is recommended for the attention and study but not the blind acceptance of such as appreciate science and art.

ALBERT A. BENNETT.

Calculus and Graphs—Simplified for a First Brief Course. By L. M. PASSANO. New York, The Macmillan Company, 1921. 12mo. 8+167 pp. Price $1.75.

As every one is aware, who has had the pleasure of teaching the subject of calculus, many features of the subject present no essential difficulties even to students who have shown no special aptitude along mathematical lines. The mere technique of differentiation can be readily acquired by students who cannot learn facility in the manipulation of trigonometric identities. The chief difficulty in solving elementary problems seems to lie in the lack of ability displayed by the student in converting a verbally formulated problem into tractable equations. This is a difficulty that is earlier presented by the subject of algebra, and has nothing to do with any one particular branch of mathematics. As long as verbal problems are offered for analysis, there will be students who find the subject difficult. In the present book there has been no effort to spare the student in this direction. Since the development of clear reasoning is one of the primary purposes in any mathematical course, this book assuredly fulfils its principal aim.

Of the other difficulties which the subject sometimes presents may be mentioned the extensiveness of the field of applications, and questions of rigor connected with the notion of limit. In the text under discussion, the material is wisely restricted so as to make possible an emphasis upon essentials without which the student can receive no benefit from the course. The author gives us the following warning in his preface: "The aim has been to make the student understand the subject; not to write a book that would satisfy meticulous mathematical pedantry. In so doing the author may have in places sacrificed logical detail to simplicity of presentation, but never, he hopes, accuracy of statement. In the opinion of the writer a too rigidly logical proof with its paraphernalia of subscripted Greek letters is out of place in an elementary first course in calculus, for the reason that the student never understands such a proof. Or if by arduous effort he does grasp its meaning, it is at the expense—in time and labor—of other things that are more important and far more useful." In view of this announcement the reader need not be surprised to see infinite series used where necessary without a word about convergence; but some other features are not so readily explained. On page 15, we read, "We do not know whether $x + 2$ is plus or minus zero; that is, whether $x + 2$ is an exceedingly small positive or an exceedingly small negative number." On page 19, and repeatedly throughout the book, "The increment of the function can be made smaller than any number we may assign, however small, by making the increment of the independent variable small enough . . .," where the word "number" is apparently expected to mean

"positive number." On page 80, the phrase "any number of parts" is to be understood as meaning "any number of equal parts." The book is almost free of definitions. A search through the text has failed to reveal any definition or even hint as to what may be meant by "element" until on page 93, we read, "To make use of this new tool [the definite integral] all that is necessary is to form the element of the thing to be summed; to get a differential expression representing, in terms of some independent variable, any one of the terms to be summed. This element being found and formed, the remainder of the process consists in integrating, and substituting the limits as shown in (40)," while on page 122, we read "of another little element." The problems considered on maxima and minima are of the usual sort, and, as is obvious, examples may readily be constructed in which an extremal is obtained at the end of the interval of definition. That no such examples should happen to be incorporated in the particular collection here given does not justify the statement: "For the function to reach a maximum or a minimum value, must $dy/dx = 0$. This, as we say, is a *necessary* condition for a maximum or minimum value of the function." One need hardly state that the notion of an interval of definition is not considered. The student is asked to *prove* that a locus is a circle, by plotting its points (p. 45, Ex. 26).

A more practical criticism, from the viewpoint of the text, is one of little logical import and refers to the form of the discussion. Answers obtained in the illustrative examples are not so labeled. The discussion consists in most cases of a mere sequence of equations with no hint as to what is assumed and what derived and whence. Many of the discussions continue uninterruptedly for too many pages for any normally impatient student to desire to understand them. This is particularly pertinent to section 20. What student will care to follow the advice: "Formula (63) can best be learned in words. Thus: The integral of the product of two factors, one of which can be integrated separately, *equals* the product of the integral of that one factor times the other; *minus* a new integral consisting of the same integral of the one factor times the derivative of the other"? The use made of units in a book intended for engineers, or rather the abuse of them, is a surprise—for example, "200 π radians minute," without even the apology of an abbreviation, stands for a velocity in radians per minute, and similarly throughout the book. The misprints are few. Only one bad case was noted, where a letter of the wrong font was used on page 124, 5th line. Something like half of the formulas appear with no terminal punctuation, so that this unusual procedure appears to be intentional, although not observed consistently.

Despite all of these incidental objections and many more that are trivial, the general arrangement of the material and the choice of exercises are such as to suggest that the book might prove a very satisfactory text in the increasing number of institutions that are giving first year students a taste of one of the richest treats afforded by mathematics, the calculus.

ALBERT A. BENNETT.

NOTES.

A comprehensive catalogue of books on arithmetic published before 1501 A. D. is in active preparation by K. A. PEDDIE, who has published other scholarly bibliographical works. This catalogue will supplement D. E. SMITH's well-known *Rara Arithmetica* (Ginn & Company) and DE MORGAN's *Arithmetical Books from the Invention of Printing.* Subscriptions at 1 £ 1 s. are desired to insure publication; address Grafton and Co., Coptic House 7 and 8, London W. C. 1.

In *Science*, September 30, 1921, attention is called to a bibliography of the theory of relativity that is being compiled by F. E. BRASCH, now chief librarian at the J. J. Hill Reference Library, St. Paul, Minn. The aim is to make the bibliography as extensive and complete as possible, each entry to include not only author, title, source, and date, but also a short abstract, note, or review indicating the principal idea the author has conveyed. It would seem desirable to have this bibliography published, but at the time of the notice in *Science* no provision had been made for this.

Father J. G. HAGEN, the gracious director of the Observatory of the Vatican, has almost ready the manuscript for the fourth volume of his *Synopsis der höheren Mathematik* of which volumes 1–3 were published in Berlin (Dames), 1891–1905. The older members of the American Mathematical Society will recall with pleasure Father Hagen's presence at their meetings.

Four of the seven volumes of PAUL TANNERY'S *Mémoires Scientifiques* have already appeared (Toulouse, E. Privat), the fourth volume, *Sciences exactes chez les Byzantins,* in 1920. Professors ZEUTHEN and HEIBERG have been the editors. Since Zeuthen's death Professor GINO LORIA has acceded to Madame Tannery's request to take his place as an editor.

Professor LORIA is preparing the manuscript for a third volume of his work on algebraic curves, *Spezielle algebraische und tranzendente ebene Kurven, Theorie und Geschichte* (Teubner, Berlin, 1910–1911). It is to deal with gauche curves.

We have already made note of several institutions which have collected mathematical papers, published by members of their faculty (*1921,* 177). In 1920 was published the first volume of *Publications de L'Institut de Mathématiques de l'Université de Strasbourg.* It contained four memoirs by M. FRÉCHET, H. VILLAT, A. VÉRONNET, and G. VALIRON. The first of these memoirs was originally published in *Transactions of the American Mathematical Society.* The second volume of the *Publications* was for 1921, and was published in May, 1922. Only 25 copies each of volumes 1 and 2 were published.

With the first number of the *Journal de Mathématiques pures et appliquées* (Liouville) for 1922 a new series (the ninth) commences under the direction of Jordan's successor, H. VILLAT, professor at the University of Strasbourg. The number contains a portrait of Jordan and an obituary notice by Villat (see *1922,* 139), who, it will be remembered, was the editor of the large volume of the proceedings of the Strasbourg Mathematical Congress. Villat is also to be director of the *Nouvelles Annales de Mathématiques* which is to start again in

October, 1922. The second number of the *Journal*, published in May, is about double the size of the first number and is entirely devoted to papers on relativity.

That interest in the applications of mathematics to commercial and professional work has been further stimulated in France by the war is evidenced by the number of recent books dealing with such subjects. For example, one lesser publisher (Dunod, Quai des Grands-Augustins 47 et 49, Paris) alone lists, among others, recent books with the following titles: *La culture générale des jeunes gens se destinant à l'industrie; Les mathématiques de l'élève ingénieur*, Part I (1920); *Effets gyroscopiques et méthodes vectorielles* (1920); *Les mathématiques de l'ouvrier moderne*, 2 volumes; *Trigonometrie rectiligne* (1920); *Dessin industriel. Cours pratique de projections* (1919); *Traité de dessin géométrique* (1919); *Cours de comptabilité* (1920); *Notions sommaires de comptabilité industrielle* (1920), *Cours élémentaire de méchanique industrielle* (three volumes, 1919–1920); *Comptabilité commerciale* (two volumes, 1918–1920); *Les calculs commerciaux et financiers* (in press). Few of these would be of practical service in this country and for this reason no mention of author, paging, or prices need be made.

ARTICLES IN CURRENT PERIODICALS.

Annali delle Università Toscane, new series, volume 4, fasc. 2, 1919: "Sulle curve piane algebriche che ammettono una curva data come prima polare" by G. Albanese, 33–41—Volume 5, fasc. 8, 1920: "Sulla moltiplicazione delle serie" by U. Dini, 253–274.

Isis, volume 4, no. 1, 1921: "Different types of mathematical history" by G. A. Miller, 5–12; "Note on the Fahrenheit scale" by F. Cajori, 17–22; "Introduction to the history and philosophy of science (preliminary note)" by G. Sarton, 23–31; "The principle of symmetry and its applications to science and to art" by G. Sarton, 32–38 [Discussion of ideas of F. M. Jaeger, D'A. W. Thompson, T. A. Cook, J. Hambidge, D. W. Ross]; "*Encyclopédie des Sciences mathématiques pures et appliquées*" by G. Sarton, 39–40 [Protest against the slowness of this publication]; Reviews of: J. Ruska, "Zur ältesten arabischen Algebra und Rechenkunst" (*Sitzungsberichte der Heidelberger Akademie der Wissenschaften, Philosophisch-historische Klasse*, pp. 1–125, 1917) by L. C. Karpinski, 67–70; P. Boutroux, *Les Principes de l'Analyse Mathématique*, tome second (Paris, 1919) by J. M. Child, 96–107; L. E. Dickson, *History of the Theory of Numbers*, volume 2, by J. M. Child, 107–108; H. Wieleitner, *Geschichte der Mathematik* (II. Teil, II. Hälfte) by D. E. Smith, 109.

Journal of the Indian Mathematical Society, volume 13, October, 1921: "Some pitfalls in the theory of groups" by G. A. Miller, 161–165—December: "On a certain theorem in geometry" by F. H. V. Gulasekharam, 210–218; "Mathematics in India, then and now" by V. Sankaran, 219–224 (to be continued) ["It is a characteristic of the present times in India that we feel as if we have been shut out from all knowledge of the glories of this ancient land in the palmy days of yore. The cry is now 'Look back into your past history and you will learn more than what you can by merely studying what you are taught in schools.' In every department of knowledge this looking back into the past is necessary. We have begun to realize how advanced our fore-fathers were in the study of the Humanities, Political and Social Sciences, Arts and Philosophy. It is therefore, but natural to expect that in other departments of knowledge also, they should have reached a tolerably high standard of advancement and efficiency."]; "Interference method in astronomy" by T. P. Bhaskara Sastri, 230–231.

Journal de Mathematiques pures et appliquées, 8th series, volume 4, fasc. 4, 1921: Statement by C. Jordan (who has since died), on retiring from the active management of this Journal after 35 years of service; announcement of memoirs to be published in the near future, insert 1–4; "Integral solutions of the equation $\xi^2 + \eta^2 = \zeta^2$ in the quadratic realms of rationality" by H. Hancock, 327–341—9th series, volume 1, fasc. 1, 1922: "Camille Jordan," insert I–IV (see *1922*, 138).

Mathematics Teacher, volume 15, March, 1922: "The psychology of the equation" by E. L. Thorndike, 127–136; "Reaction vs. radicalism in the teaching of mathematics" by G. W.

Myers, 137–146; "The definition of similarity" by G. W. Evans, 147–151; "The place of elementary calculus in senior high school mathematics" by N. B. Rosenberger, 152–156; "Experimental geometry" by G. A. Harper, 157–163; "How can I bring the soul of mathematics to my pupils" by A. H. Huntington, 164–171; "Papers by pupils of the plane geometry classes of Fullerton Union High School" by Lena E. Reynold, 172–181; "Discussion" by L. F. Babcock and P. Huffman, Jr., 182–184; New books, 185–188; News and Notes, 189–190.

NATURE, volume 109, February 23, 1922: "The pioneer of non-euclidean geometry," 232–233 [Review of G. B. Halsted's *Girolamo Saccheri's "Euclides Vindicatus."*]—March 2: "The directive tendency of elongated bodies" by W. D. Lambert, 271–272; "Mathematics in Japan" by G. B. M(athews), 287–288—March 16: "Greek mathematics" by D'A. W. Thompson, 330–334 [Review of Heath's *History of Greek Mathematics*, Oxford, 1921. "Were this book only for the mathematician it would be no book for me; but it is a great deal more. It is for all who care for the historical aspect of science; it is for all lovers of Greek, for mathematics is a true 'Legacy of Greece,' and is interwoven through and through with Greek thought and philosophy. ...There is a vast deal of information in Heath's book, clearly set forth and orderly arranged; we have nothing to compare with it in English, and Gino Loria's 'Scienze esatte' is its only serious rival abroad. I am inclined to think that Loria paints history with a broader brush, while Heath excels in his account of individual mathematicians; but I cannot help thinking that Heath, who has attained such complete and acknowledged success in his editions of Euclid and the rest, must have found that in this history he had struck a harder task than any he had tried before. We may know more of the history of mathematics than of any other science, but the lacunæ are immense, and tradition is poor material for the historian. Moreover the historical aspect is somewhat uncongenial to the mathematician, if only because (as Eva Sachs says) history deals with *das Werden*, and mathematics with *das Sein*! When Sir Thomas Heath deals with Euclid, Apollonius, Archimedes, Diophantus, Hero or Pappus, he gives us in a few pages all we could expect by way of epitome of the trend, the method and the results of their labours. But his book pursues its steady, instructive course with little digression, allusion or anecdote, and with curiously little bibliographical information such as he puts abundantly into his other books. Surely one of the objects of a text-book is to guide the student to what it does not and cannot contain! Some of us, I think, would have liked a little more digression or even gossip. . . . The simple fact is that Sir Thomas Heath has given us so much, and it is all so good, that he makes us ask for more."]—March 23: "The theory of relativity in relation to scientific method" by Dorothy Wrinch, 381–382 [Paper read before the Congress of Philosophy in Paris on December 29, 1921]—April 8: "Mathematical analysis" by G. H. Hardy, 435–438 [Review of Hobson's *Functions of a Real Variable*, volume 1, 2nd ed., (Cambridge, 1921); Carslaw's *Introduction to the Theory of Fourier's Series*, volume 1 (London, 1921); Edward's *Integral Calculus*, volume 1 (London, 1921)]; "Dr. G. B. Mathews" by W. E. H. B., 450–451—April 15: "The fourth dimension" by S. Brodetsky, 474–475 [Review of *The Fourth Dimension Simply Explained.* A Collection of Essays selected from those submitted in *The Scientific American's* Competition (New York, 1913, London, 1921)]; "Pythagoras's theorem as a repeating pattern" by P. A. MacMahon, 479—April 22: "Mathematics and public opinion" by G. B. M(athews), 520–521.

LA NATURE, volume 50_1, February 11, 1922: "L'Univers stellaire" by É. Belot, 87–91—March 18: "Le calendrier perpétuel vivant" by E. Singer, supplement 86–87.

OBSERVATORY, volume 45, March, 1922: "Meeting of the Royal Astronomical Society, February 10, 1922," 65–69 [Address of the President, A. S. Eddington, in presenting the gold medal of the society to J. H. Jeans]; "An eclipse of the moon as seen from the moon" by L. Richardson, 69–71 [Paper read at the meeting of the British Astronomical Association].

PERIODICO DI MATEMATICHE, series 4, volume 1, November, 1921: "Che cosa contiene la Géométrie di Cartesio?" by E. Bompiani, 313–325; "La prospettiva e lo sviluppo dell'idea dei punti all'infinito" by U. Cassina, 326–337; "Il teorema Descartes-Eulero relativo ai poliedri" by A. Maroni, 337–346; "La questione dei nove valori nella risoluzione della cubica" by C. Candido, 347–351; "Il concetto delle due divisioni nell'insegnamento elementare e normale" by G. Furlani, 351–354; "Polemica logico-matematica C. Burali-Forti e F. Enriques," 354–365—Volume 2, January, 1922: "Sui meccanismi articolati nelle costruzioni di Geometria elementare" by E. G. Togliatti, 41–49; "L'area di una superficie curva nella definizione di Minkowski e nell'insegnamento della Geometria elementare" by L. Brusotti, 49–55; "Una rapida visione geometrica del teorema fondamentale dell' Algebra" by O. Chisini, 55–59; "Sui massimi e minimi e sulla variazione di una funzione reale" by E. Maccaferri, 60–67; "Su la configurazione dei circoli dei nove punta dei triangoli determinati da un gruppo di punti situati su una medesima circonferenza"

by D. Montesano, 68–70; "Einstein e l'interpretazione subiettiva della scienza," 77–80 [Address of F. Enriques to the public of Bologna on the occasion of the conference which Einstein held there in October, 1921]; "Atti della Società Italiana di Matematiche 'Mathesis,' Ottobre 1921," 90–116.

Popular Astronomy, volume 30, March, 1922: "Suggestions regarding gravitation" by W. H. Pickering, 155–156; "Note on the velocity of light" by H. Shapley, 192–193 [By an extremely delicate test of the relative velocity of light of different colors it is shown that blue and yellow light can travel through space for 40,000 years without a difference of more than one or two minutes, if indeed there is any difference whatever]—April: "Henrietta Swan Leavitt" by S. I. Bailey, 197–199 [With photograph. "Miss Leavitt graduated from Radcliffe College in 1892. Later she spent several years as advanced student and volunteer research assistant at the Harvard Observatory, in travel, and in teaching. She became a permanent member of the Observatory staff in 1902, and thereafter was closely associated for the remainder of her life with Prof. Pickering, in the solution of problems especially connected with the determination of the photographic magnitudes of stars. Beginning work as an ordinary assistant in the measurement of variable stars on photographic plates, by her exceptional ability and intense application she soon became the head of the department of photographic stellar photometry. . . . Miss Leavitt was of an especially quiet and retiring nature, and absorbed in her work to an unusual degree. She had the highest esteem of all her associates at the Harvard Observatory, where her loss is keenly felt."]; "Shall we accept relativity?" by W. H. Pickering, 199–203 ["Three years have now elapsed since the Principle of Relativity was first announced to a startled public, as perhaps the greatest scientific discovery since the time of Newton. Little attention had been paid to it before that, outside of the ranks of professional physicists. Now that the glamour has somewhat subsided, it may be a good time to take account of stock, to see what evidence has meanwhile accumulated in its favor, and how far its supporters have increased among scientific men. Regarding the latter, the physicists it is believed support it as before. Among the astronomers its stronghold is undoubtedly in England. In America opinion is divided. Which way the majority vote would be cast it is impossible to say. On the continent of Europe I have found the Latins generally skeptical, while I understand that very few of the Germans take it seriously at the present time. Three years is of course too brief a time for any important discovery to win universal acceptance in science. Perhaps a generation may be taken as a fair interval, sometimes a little less, sometimes a little more. Therefore if its adherents have not increased in the past three years, or have even fallen off, that can hardly be counted very seriously against it. Let us now see what evidence has accumulated with regard to it in the meantime. . . . Summarizing our conclusions, we may say that in the first test, . . . the theory of relativity fails to give results in accord with the facts. Regarding the second test, the only result so far obtained is that it has now been clearly and satisfactorily demonstrated what not to do. The evidence so far regarding the third test is as we all know, and have seen, distinctly adverse."]; "Cosmic clouds" by W. F. Rigge, 203–205; "How to know the heavens" by G. B. Chase, 210–213; "The duration of sunrise and sunset" by W. J. Fisher, 213–216; "A letter of John Kepler" by F. Cajori, 217–219; "Planetary configurations" by F. R. Honey, 219–222; "Effect of the sun-spots on the terrestrial temperature" by P. G. Searles, 222–225; "Hypothesis on the formation of new stars" by A. Verronet, 225–227—May: "Eratosthenes—III" by W. H. Pickering, 257–262; "What about Mars?" by J. G. Porter, 268–271; "A suggestion regarding gravitation, II" by W. H. Pickering, 272; "'Astronomy improved'" by C. Rufus, 272–280; "The secular comets and the movement of the sun through space" by G. Armellini, 280–286; "The present position of the island universe theory of the spiral nebulæ" by D. B. McLaughlin, 286–295 (to be continued).

Revue Générale des Sciences, volume 33, February 15, 1922: "Camille Jordan" by R. d'Adhémar, 65–66 [see *1922*, 139]—February 28: Review by M. Solovine of J. M. Child's *Early Mathematical Manuscripts of Leibniz*, 121.

Revue de Mathématiques Spéciales, volume 32, January, 1922: "Sur les conditions nécessaires et suffisantes pour que l'hyperboloïde défini par trois droites soit de révolution" by M. Chenevier, 73–76—February: "Sur les sommes des puissances semblables des nombres entiers" by M. Weber, 97–100—March: "Note sur la balance" by M. Colin, 121–123—April: "Note sur la transformation conforme" by M. Lapointe, 145–146.

Revue Scientifique, volume 59, November 12, 1921: "Le mois mathématique à l'Académie des Sciences (juillet et août 1921)" by R. Garnier, 620–621 [G. Bertrand suggests a formula that conforms to the law of Einstein for the displacement of the perihelion of a planet; E. Borel announces explicitly four fundamental hypotheses which connect the general principles of physics with those of geometry in the theory of relativity; K. Ogara gives an explanation for

the element of integral around a center in the theory of gravitation of Einstein]; "La mécanique classique et la théorie de la relativité" by P. Painlevé, 621–622; "Quelques remarques sur la théorie de la relativité" by E. Picard, 622–623—Volume 60, January 14, 1922: "La durée et la conception Einsteinienne du temps" by R. Paucot, 7–17—January 28: "A propos d'un projet de Calendrier mensuel fixe" by G. Galleano, 55–56—February 11: "Camille Jordan" by R. S., 95–96 [contains the address of E. Picard at the Academy of Sciences on January 23 (see *1922*, 139)] —March 25: "L'histoire de la nomographie" by S. R., 197–198 [Extract from *Comptes Rendus de l'Académie des Sciences* for January 9, page 82.]

SCHOOL SCIENCE AND MATHEMATICS, volume 22, January, 1922: "Class exercise types in high school mathematics, with norms for judging them. II" by G. W. Myers, 7–15; "Required mathematics in the four year high school" by R. D. Shouse, 15–19—February: "Square root of a line without use of the circle" by J. B. Wood, 111–113; "An analysis of an experiment in teaching first year mathematics" by Ina E. Holroyd, 114–121; "Learning fractions" by Myrtie Collier, 121–127; "Problem department" by J. A. Nyberg, 176–180; "Annual meeting of the Central Association of Science and Mathematics Teachers" by G. Warner, 190, 192, 194—March: "The fourth dimension: An explanation the geometry class can follow" by J. V. Collins, 226-231; "The law of exponents" by R. Morris, 232–236; "New mathematical periodicals" by G. A. Miller, 276–280; "Problem department" by J. A. Nyberg, 282–284, 286.

SCIENCE, new series, volume 55, January 6, 1922: "Henry Turner Eddy" by J. J. F., 12–13 —February 10: "Subsidy funds for mathematical projects" by H. E. Slaught, 146–148—February 17: "Relativity and star diameters" by R. A. Fessenden, 180—February 24: "A mechanical analogy in the theory of equations" by D. R. Curtiss, 189–194—March 17: "Doctorates conferred in the sciences by American universities in 1921" by Callie Hull and C. J. West, 271–279.

SPHINX-ŒDIPE, volume 16, November, 1921: "Notice sur Charles Ange Laisant" (continued) by H. Brocard, 161–166; "Sur les racines primitives relatives au mod. P^n dans un corps algébrique quelconque" (concluded) by G. Métrod, 166–168; Questions and Responses, 169–176; "Divers," 176 ["M. R. C. Archibald visitera dans 6 mois des universités européennes. Meilleurs voeux en attendant sa venue parmi nous."]—December: "Notice sur Charles Ange Laisant" (concluded) by H. Brocard, 177–182; Questions and Responses, 183–189; "Table des matières de 1921," 190–192—Volume 17, January, 1922: "Triangles et tétraèdres" by V. Thébault, 2–3; "Histoire des sciences: Correspondance de J. J. Sylvester—Ed. Lucas" translated by L. Chanzy, 4–7 (to be continued); "Solution de l'equation $4\alpha^4 - 4\alpha^3\beta + 13\alpha^2\beta^2 - 36\alpha\beta^3 + 24\beta^4 = \delta^2$" by L. Aubry, 7–10; Questions and Responses, 10–16—February: "Sur trois triangles homothétiques" by V. Thébault, 17–19; "Note sur un problème de la théorie des nombres d'Ed. Lucas" by L. Aubry, 20; Questions and Responses, 21–32.

TÔHOKU MATHEMATICAL JOURNAL, volume 20, nos. 1–2, December, 1921: "On the roots of an algebraic equation $f + k_1f' + k_2f'' + \cdots + k_nf^{(n)} = 0$" by K. Ôishi, 1–17; "Über die Gültigkeitsbedingung der Interpolationsformeln von Gauss" by M. Fujiwara, 18–21; "Sur les différences de deux ensembles fermés" by C. Kuratowski and W. Sierpínski, 22–25; "Ueber die Coefficienten in der Taylorschen Entwicklung rationaler Functionen" by C. Siegel, 26–31; "Tautochronous motion" by P. P. Upadhyaya, 32–36; "Criteria for the irreducibility of a polynomial" by K. Ôishi, 37–40; "Sur la configuration des circonférences des neuf points des triangles determinés par un groupe de points d'une même circonférence" by M. D. Montesano, 41–43 [The printers have somewhat disarranged the parts of this article. If the reader will divide the article into four nearly equal parts and read the third part before he reads the second part, he will get the correct arrangement of paragraphs as may be seen by comparison with essentially the same article printed in Italian in *Periodico di Mathematica* for January (see above).]; "Some theorems on infinite products" by C. Kien-Kwong, 44–47; "Ein Problem in der Wahrscheinlichkeitsrechnung" by M. Fujiwara, 48–50; "Über Stützgeradefunktion der konvexen geschlossenen Kurven" by M. Fujiwara, 51–59; "Sur deux théorèmes de Casey" by V. Thébault, 60–63; "On the representations of functions by the formulas of interpolation" by Y. Okada, 64–99; "On the definite integral $\int_a^\infty \varphi(t)e^{-xt}dt$" by K. Kurosu, 100–106; "On the integrals

$$\int_0^\pi \exp\left(x \begin{matrix}\cos\\ \\ \sin\end{matrix} p\theta\right) \begin{matrix}\cos\\ \\ \sin\end{matrix} q\theta d\theta"$$

by T. Hayashi, 107–114.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

Agnesi Club of Agnes Scott College, Decatur, Ga.

The Agnesi Club was formed at Agnes Scott College, November 28, 1921, at the suggestion of the Departments of Mathematics and Physics. Students in the elective courses of mathematics, astronomy, and physics are eligible for membership. The Club has twenty members. Meetings are held monthly. The officers elected at the first meeting were: President, Mary Barton '22; vice-president, Elizabeth Hoke '23; secretary, Emmie Ficklen '24.

December 12, 1921: "Radium and the X-ray" by Professor Housen; "Agnesi" by Elizabeth Hoke '23. Adoption of the Constitution.

(Report by Miss Ficklen.)

Mathematics Club of Bucknell University, Lewisburg, Pa.

The Mathematics Club of Bucknell University was organized January 28, 1920, through the efforts of Professor H. S. Everett. Active members must have passed all required mathematics, also analytical geometry and one other subject in pure or applied mathematics. Associate membership is open to anyone interested in mathematics. The following officers were elected: President, Sydney Peale '20; vice-president, Elizabeth Spyker Gr.; secretary-treasurer, Grace Poust '22. The following papers were presented:

January 28, 1920: "Non-Euclidean world" by Professor W. C. Bartol.

February 11: "How x came to be used as an unknown" by Lillian Russell '23; "Two applications of the spiral of Archimedes" by Robert Haberstroh '23.

February 25: "Trisection of an angle" by Irvin Holmes '20.

March 10: "Trisection of an angle—continued" by Irvin Holmes '20; "The quadrature of the circle" by Clara Casner '21; "The duplication of the cube" by Elizabeth Weidner '21.

April 7: "Famous mathematicians" by Professor Everett.

April 26: "Probabilities" by Sydney Peale '20.

For the year 1920–21, the following officers were elected: President, Elizabeth Weidner '21; vice-president, Leonard Worthington '21; secretary-treasurer, Elsie Watson '21; critic, Professor J. S. Gold. Papers were presented during the year as follows:

November 1, 1920: "Magic squares" by Professor C. A. Lindeman, of the Department of Engineering.

November 15: "Paper folding" by Kathryn Kimble '23; "Something about zero" by Grace Fry '22.

December 6: "Number puzzles" by Professor Gold.

January 17, 1921: "The planimeter" by Chester Derck '22; "Mechanism for solving a cubic" by Paul Mallay '23.

February 14: "Diophantine analysis" by George Lowry '20.

February 28: "Numerical results for diverse systems of breeding" by Professor F. N. Davis, of the Department of Biology.

March 14: "Use of logarithmic paper" by Leonard Worthington '21.

April 11: "History of codes and ciphers" by Clara Casner '21; "Non-Euclidean geometry" by Professor Everett.

April 18: "Codes and ciphers" by Elsie Watson '21; "A play—The Flatlanders"; "The fourth dimension" by Professor Everett.

April 25: "Einstein's theory of relativity" by Professor Everett.

For the year 1921–22, the following officers were elected: President, Chester Derck '22; vice-president, Grace Poust '22; secretary-treasurer, Kathryn Kimble '23. Programs were presented as follows:

October 10, 1921: "Sir Isaac Newton" by Professor Bartol.

November 14: "Euclid's life and works" by Grace Poust '22; "Archimedes' life and work" by Chester Derck '22; "Life and work of Apollonius" by Ethylwynne Smith '23.

December 12: "Mathematics in colonial times" by Leona Dickrager '22; "Alice in the Wonderland of mathematics" by Nina Grace Smith '23; "Numerical curiosities" by Philip Campbell '22.
January 16, 1922: "Cardan's solution of the cubic" by Professor Lindeman.
February 13: "Life and works of Descartes" by Ruth King '22; "Life and works of Newton" by Nellie Balliet '22.
March 13: "Projective geometry" by Professor Gold.

The number of members each year has been about 40; and the average attendance about 25. In the fall and in the spring also there is a social party.

(Report by Miss Kimble.)

The Mathematics Club of Goucher College, Baltimore, Md. [*1918*, 357; *1919*, 365; *1921*, 274; *1922*, 26.]

Meetings during the second half of 1920–21 were held as follows:
March 3: "Human calculating prodigies" by Grace McCaulley '23; "The theory of Amsler's planimeter" by Katherine Wisner '21.
April 7: "A new theorem concerning three circles" by Henrietta Morris '23; "Proof that an angle cannot be trisected by ruler and compass" by Eva Lazarus '22.
April 24: "History of the theory of numbers, with illustrative problems" by Dorothy Biscoe '22; "The fallacy in a recently proposed method of trisecting an angle by ruler and compass" by Mildred Brown '21; "Rules for checking addition, with proofs" by Mildred Trueheart '22.

The annual picnic of the Club took place in May, and was attended by many of its alumnæ members.

(Reported by Professor Lewis.)

The Mathematics Club of Vassar College, Poughkeepsie, N. Y. [*1918*, 136, 456; *1919*, 264; *1920*, 78, 427.]

October 21, 1920: A tea was given, to which were invited all who were eligible to membership. The following officers were elected for the first semester: President, Mary West '21; vice-president, Theresa McMakin '22; secretary-treasurer, Alta Teeple '22; executive committee, Professor Elizabeth B. Cowley, Elizabeth Larsen '21.
November 16: "Women in mathematics" by Elinor Hollis '21.
December 9: "History, development, and use of logarithms" by Grace Fenstermaker '21, Mathematical tricks were exhibited by Elsie Stuart '22.
February 24, 1921: The following officers were elected: President, Amy Davison '21; vice-president, Theresa McMakin '22; secretary-treasurer, Alta Teeple '22; executive committee, Professor Cowley, Elizabeth Larsen '21.
March 17: "The Einstein theory" by Elizabeth Larsen '21.
April 28: "Codes and ciphers" by Dorothy Laws '22; "The fourth dimension" by Edith Morrison '22.
May 19: A picnic supper. Charades were played with such words as hypothenuse, perpendicular, and algebra.

(Reported by Miss Teeple.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible

sources will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2966. Proposed by OTTO DUNKEL, Washington University.

When a curve produces a caustic by the reflection of the rays proceeding from a fixed point, δ and δ' the lengths of the incident and reflected rays, R the radius of curvature of the curve, and ω the angle of incidence, satisfy the equation (see *1920*, 225)

$$\frac{1}{\delta}+\frac{1}{\delta'}=\frac{2}{R\cos\omega}.$$

Give a geometrical proof of this relation using the harmonic properties of the figure and obtain in this manner a simpler and different geometrical derivation from that given in Humbert's *Cours d'Analyse*, volume 1, page 77.

2967. Proposed by ELIJAH SWIFT, University of Vermont.

A plane revolves about one of two non-coplanar lines as an axis. Find the locus, *in the plane*, of the intersection of the plane and the other line.

2968. Proposed by MALCOLM FOSTER, Yale University.

A curve C is the directrix of a ruled surface, and g is any ruling. Relative to the trihedral of C at the point of intersection with g, the direction-cosines of g are α, β, γ, expressed as functions of the arc of C. Prove that the distance t along g from C to the line of striction is given by

$$t=\frac{\dfrac{\beta}{\rho}-\alpha'}{\Sigma\alpha'^2+\dfrac{2}{\rho}(\alpha\beta'-\alpha'\beta)+\dfrac{2}{\tau}(\beta'\gamma-\beta\gamma')+\dfrac{1}{\rho^2}(1-\gamma^2)+\dfrac{1}{\tau^2}(1-\alpha^2)+\dfrac{2}{\rho\tau}\alpha\gamma},$$

where ρ and τ are the radii of first and second curvature of C.

2969. Proposed by J. N. VEDDER, Union College, Schenectady, N. Y.

The following integral equations arise in connection with electrical theory:

$$f(x)=c-\lambda\int_{-a}^{+a}\log\left|\frac{a-u}{x-u}\right|f(u)du \tag{1}$$

and

$$f(x,y)=c-\lambda\int_{-a}^{+a}\int_{-b}^{+b}\log\sqrt{\frac{(a-u)^2+(b-v)^2}{(x-u)^2+(y-v)^2}}f(u,v)du\,dv. \tag{2}$$

The value of $\int_{-a}^{+a}f(x)dx$ is required from (1) and $\int_{-a}^{+a}\int_{-b}^{+b}f(x,y)dxdy$ from (2) in a form adapted to arithmetical calculation. In each case c is a given constant.

2970. Proposed by E. T. BELL, University of Washington.

For what values of the positive rational integers a, b, c, d, e, f are $a\cos^b(2\pi/c)$, $d\sin^e(2\pi/f)$ quadratic integers?

2971. Proposed by C. E. HORNE, The University of Porto Rico, Mayaguez, P. R.

Prove that the two tangents drawn to an ellipse from any external point subtend equal angles at a focus. (This problem is found in some text-books, but the Proposer is anxious to see an analytic solution of it.)

SOLUTIONS.

2828 [1920, 186]. Proposed by T. M. BLAKSLEE, Ames, Iowa.

On page 72 of R. B. Hayward's "*The Algebra of Coplanar Vectors and Trigonometry*" occurs the sentence: "It will be a good exercise for the student to show that $\cos(90°/7)=\frac{1}{2}\sqrt{x_1}$ where x_1 is the greatest root of the equation

$$x^3-7x^2+14x-7=0.$$

(1) Do not merely verify but deduce the equation and find x_1. (2) Deduce the x-equation (x_1, x_2, x_3, x_4 the roots) such that its greatest root x_1 gives $\cos (90°/9) = \cos 10° = \frac{1}{2}\sqrt{x_1}$. (3) Of what angles are $\frac{1}{2}\sqrt{x_1}, \cdots \frac{1}{2}\sqrt{x_4}$ in (2) the cosines? Develop a method of writing out at once $\cos nv$ in terms of powers of $\cos v$ if these are given for $(n-1)v$ and $(n-2)v$. The same for $\sin nv$. (4) Use the results of (2) and (3) to find the number of degrees in a radian. Hence, find π from radian instead of radian from π as is usual.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

From the formula

$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x+y) \cos \tfrac{1}{2}(x-y),$$

letting $x = nv$ and $y = (n-2)v$, we have[1]

$$\cos nv = 2 \cos v \cos (n-1)v - \cos (n-2)v.$$

So also $\sin nv = 2 \cos v \sin (n-1)v - \sin (n-2)v$.
Putting $n = 2, 3, \cdots 9$, we readily obtain

$$\cos 7v = 64 \cos^7 v - 112 \cos^5 v + 56 \cos^3 v - 7 \cos v,$$

and

$$\cos 9v = 256 \cos^9 v - 576 \cos^7 v + 432 \cos^5 v - 120 \cos^3 v + 9 \cos v.$$

Replacing in these $\cos v = \frac{1}{2}\sqrt{x}$, we get

$$\cos 7v = \frac{x^{1/2}}{2}(x^3 - 7x^2 + 14x - 7), \tag{1}$$

$$\cos 9v = \frac{x^{1/2}}{2}(x^4 - 9x^3 + 27x^2 - 30x + 9). \tag{2}$$

If in (1), we let $\cos 7v = 0$ or $7v = (n + \frac{1}{2})\pi$, we have $v = \pi/14 = 90°/7$ for the greatest root $4 \cos^2 v$, of

$$x^3 - 7x^2 + 14x - 7 = 0. \tag{3}$$

Again, if we let $\cos 9v = 0$ or $9v = (n + \frac{1}{2})\pi$, we have $v = (2n+1)\pi/18$ for the roots of

$$x^{1/2}(x^4 - 9x^3 + 27x^2 - 30x + 9) = 0; \tag{4}$$

and if we call them $x_1, x_2, x_3, x_4, 0$, for $n = 0, 1, 2, 3, 4$, then

$x_1 = 4 \cos^2 \pi/18,\quad x_2 = 4 \cos^2 \pi/6,\quad x_3 = \cos^2 5\pi/18,\quad x_4 = 4 \cos^2 7\pi/18,\quad x_5 = 4 \cos^2 \pi/2 = 0.$

Since $4 \cos^2 \pi/6 = 3$, we may factor (4) into

$$x^{1/2}(x-3)(x^3 - 6x^2 + 9x - 3) = 0. \tag{5}$$

We may solve (3) and (5) by Horner's method to get any of the roots to the desired degree of accuracy, since the roots are all real. In this manner, we find from (5) $x_1 = 3.87938$, $x_3 = 1.65271$, and since $x_1 + x_3 + x_4 = 6$, $x_4 = .46791$, $x_2 = 3$. Now we have $\cos 70° = \frac{1}{2}\sqrt{x_4}$ or

$$70° = \cos^{-1}(\tfrac{1}{2}\sqrt{x_4}) = 90° - \sin^{-1}(\tfrac{1}{2}\sqrt{x_4}),$$

giving

$$20° \text{ in radians} = \sin^{-1}\left(\frac{x_4^{1/2}}{2}\right) = \frac{x_4^{1/2}}{2}\left\{1 + \frac{x_4}{24} + \frac{3x_4^2}{640} + \frac{5x_4^3}{64\cdot 112} + \cdots\right\}$$

$$= .349058 \text{ radians};$$

whence, 1 radian $= 57.297°$ or $\pi/9 = .349058$, giving $\pi = 3.14152$.

2830 [1920, 227]. Proposed by WILLIAM HOOVER, Columbus, Ohio.

An elastic string connects a pair of opposite vertices of a square whose sides are four equal rods freely jointed, each of length $2a$. The system is suspended vertically from a vertex attached to the string and is at rest. If the string be cut, and θ is the acute angle any side makes with the vertical at any moment during the motion, determine the angular velocity of any rod.

[1] Compare Note 3, in this MONTHLY, *1921*, 38–39.

SOLUTION BY THE PROPOSER.

Change the word square in the problem to rhombus. Take the horizontal line through the fixed vertex for the x-coordinate axis, and the vertical diagonal of the rhombus formed of the rods for the y-axis. Let x_1, y_1 be the numerical values of the coordinates of each of the centers of the upper pair of rods; x_2, y_2 of each of the lower pair; m the mass of each rod; and α the value of θ upon the cutting of the string.

We have $x_1 = a \sin \theta$, $y_1 = a \cos \theta$; $x_2 = a \sin \theta$, $y_2 = 2a \cos \theta + a \cos \theta = 3a \cos \theta$.

The energy equation for the motion is

$$2 \cdot \tfrac{1}{2} m \left(\dot{x}_1^2 + \dot{y}_1^2 + \frac{a^2}{3} \dot{\theta}^2 \right) + 2 \cdot \tfrac{1}{2} m \left(\dot{x}_2^2 + \dot{y}_2^2 + \frac{a^2}{3} \dot{\theta}^2 \right) = 2mgy_1 + 2mgy_2 + C;$$

but $\dot{x}_1 = \dot{x}_2 = a \cos \theta \cdot \dot{\theta}$, $\dot{y}_1 = -a \sin \theta \cdot \dot{\theta}$, $\dot{y}_2 = -3a \sin \theta \cdot \dot{\theta}$; then

$$m \frac{4a^2}{3} \dot{\theta}^2 + ma^2 \left(\frac{4}{3} + 8 \sin^2 \theta \right) \dot{\theta}^2 = 8mag \cos \theta + C.$$

But $\theta = \alpha$ when $\dot{\theta} = 0$; then $C = -8mag \cos \alpha$, and the required angular velocity is given by

$$a(1 + 3 \sin^2 \theta)\dot{\theta}^2 = 3g(\cos \theta - \cos \alpha) \qquad \text{or,} \qquad \dot{\theta} = \sqrt{\frac{3g(\cos \theta - \cos \alpha)}{a(1 + 3 \sin^2 \theta)}}.$$

Also solved by J. B. REYNOLDS and F. L. WILMER.

2836 [1920, 273]. Proposed by W. V. N. GARRETSON, Rutgers College.

A ladder 40 feet long rests with one end on the ground against the foot of a building and the other end against the side of a second building directly across the street from the first. A second ladder 25 feet long inclines in a similar manner from the foot of the second building against the side of the first building, the two ladders crossing at a point 15 feet above the ground. How wide is the street?

SOLUTION BY H. S. UHLER, Yale University.

Let x and y denote respectively the width of the street and the distance from the foot of the first building to the lowest point of the 15 ft. vertical.

From similar right triangles, we obtain immediately

$$\frac{15}{\sqrt{1600 - x^2}} = \frac{y}{x}$$

and

$$\frac{15}{\sqrt{625 - x^2}} = \frac{x - y}{x} = 1 - \frac{y}{x}.$$

Therefore, by addition,

$$\frac{15}{\sqrt{1600 - x^2}} + \frac{15}{\sqrt{625 - x^2}} - 1 = 0. \tag{1}$$

The solution of equation (1) may be effected advantageously in the following manner. Let the second fraction be represented by $1/u$ so that

$$x = 5 \sqrt{25 - 9u^2}, \tag{2}$$

and equation (1) transforms into

$$\frac{\sqrt{3}}{\sqrt{13 + 3u^2}} = \frac{u - 1}{u},$$

or

$$3u^4 - 6u^3 + 13u^2 - 26u + 13 = 0.$$

Solving the last equation by Horner's method of approximation we obtain $u = 1.612347811$. The required result is now found by substituting in relation (2). It is $x = 6.33050319$ ft.

Also solved by T. M. BLAKSLEE, E. B. ESCOTT, LAURA GUGGENBUHL, W. W. JOHNSON, G. A. KNAPP, R. M. MATHEWS, H. L. OLSON, ARTHUR PELLETIER, J. B. REYNOLDS, and C. C. WYLIE.

2839 [1920, 274].

By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5}-1)+\tfrac{1}{4}i\sqrt{10+2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

I. Solution by H. L. Olson, University of Michigan.

Consider a circle with center at the origin, O, and unit radius, intersecting the positive x-axis at A and the negative y-axis at D. With center at the mid-point, E, of OD, draw a circular arc passing through A and intersecting the positive y-axis at F. The radius of this circle is $\frac{1}{2}\sqrt{5}$, and hence $\overline{OF} \doteq \frac{1}{2}(\sqrt{5}-1)$. With center A and radius AF, draw a circular arc intersecting the circle with center O at G and H. AG and AH are two sides of a regular pentagon inscribed in the circle O, and $\overline{OG}$ and $\overline{OH}$ are the Argand representations of two of the fifth roots of unity.

Since $AF^2 = 1 + (\frac{1}{2}\sqrt{5}-\frac{1}{2})^2$, the equations of the circles O and A are

$$x^2+y^2=1,$$
$$(x-1)^2+y^4 = 1+(\tfrac{1}{2}\sqrt{5}-\tfrac{1}{2})^2.$$

Hence, the coördinates of G and H are

$$x=\tfrac{1}{4}(\sqrt{5}-1),\qquad y=\pm\tfrac{1}{4}\sqrt{10+2\sqrt{5}};$$

and two of the fifths roots of unity are $\frac{1}{4}[(\sqrt{5}-1)\pm i\sqrt{10+2\sqrt{5}}]$.

II. Solution by Otto Dunkel, Washington University.

In texts on geometry a pentagon is usually constructed by dividing the radius OA, here taken as of unit length, at M in extreme and mean ratio. The length OM is laid off twice on the circle as chords giving the points A, K, G; then AK is the side of a regular decagon and AG is the side of a regular pentagon. From the definition of extreme and mean ratio it follows that $OM^2 = OA\cdot MA = 1-OM$, and hence $OM = \frac{1}{2}(\sqrt{5}-1)$. It easily follows that $MG=OG=1$ and hence the coördinates of G are $x=\frac{1}{4}(\sqrt{5}-1)$, $y=\sqrt{1-x^2}=\frac{1}{4}\sqrt{10+2\sqrt{5}}$. Hence the complex number represented by G is as stated in the problem.

Also solved by T. M. Blakslee, Arthur Pelletier, A. V. Richardson, C. H. Richardson, and F. L. Wilmer.

2853 [1920, 377]. Proposed by J. S. BROWN, Southwest Texas State Normal College, San Marcos, Texas.

Find the side and apothem of a regular pentagon inscribed in a circle without the use of extreme and mean ratio.

Three Solutions by T. M. Blakslee, Ames, Iowa.

I. Let $x+iy$ be the point on the unit circle whose angle is 36°. Then the side of a regular inscribed pentagon will be $p=2y$. In the equation $(x+iy)^5=-1$, the coefficient of i is

$$y(5x^4-10x^2y^2+y^4)=0.$$

We wish the smaller of the two positive values of y. Therefore we can remove the factor y and if we substitute $1-y^2$ for x^2, our equation reduces to

$$16y^4-20y^2+5=0.$$

Whence $y^2=(10-2\sqrt{5})/16$, $p=\frac{1}{2}\sqrt{10-2\sqrt{5}}$, and the apothem is $a=\frac{1}{4}(\sqrt{5}+1)$.

II. $p = 2 \sin 36° = 4s\sqrt{1 - s^2}$, where $s = \sin 18° = \cos 72°$. But $72°$ is four times $18°$; therefore s is a root of the equation

$$8s^4 - 8s^2 - s + 1 = 0.$$

The factors $s - 1$ and $2s + 1$ correspond to the angles $90°$ and $-30°$. Thus we have $4s^2 + 2s - 1 = 0$, and the positive root is $s = \frac{1}{4}(\sqrt{5} - 1)$, from which we get p and a.

III. Let d be the side of the regular decagon inscribed in the same circle; then d being a chord of an arc in a unit circle and p that of twice the same arc,

$$d^2 = 2 - \sqrt{4 - p^2}.$$

Let O be the center of the circle, AB and AC chords equal to p and d, F the foot of the perpendicular from A upon OB, and G the middle point of AB. The four points O, F, G and A are concyclic and $BA \cdot BG = BO \cdot BF$.

Now the triangle AOF is congruent to the right triangle formed by the radius and apothem of the decagon, having the same angles and hypothenuse equal to 1. Therefore $OF = d/2$ and our equation may be written

$$p \cdot \frac{p}{2} = 1\left(1 - \frac{d}{2}\right) \text{ or } p^2 = 2 - d.$$

Eliminating d and removing the factor $p^2 - 3$, we have

$$p^4 - 5p^2 + 5 = 0$$

or, since p is only a very little more than 1, $p^2 = \frac{1}{2}(5 - \sqrt{5})$. This gives p and then a as above.

Also solved by MICHAEL GOLDBERG, R. M. MARSHALL, J. Q. MCNATT, ARTHUR PELLETIER, A. V. RICHARDSON, and L. S. SHIVELY. Some have interpreted the problem as calling for a geometrical construction rather than an algebraic solution.

2859 [1920, 428]. Proposed by L. S. DEDERICK, U. S. Naval Academy.

Derive an expression for the limit of error in evaluating a definite integral by Simpson's Rule.

I. SOLUTION BY A. A. BENNETT, University of Texas.

Let the desired definite integral be $\int_{a-h}^{a+h} f(x)dx$, where $h > 0$. If the function $f(x)$ be supposed capable of a Taylor expansion through the term in $(x - a)^3$, with a remainder, a simple solution for the problem is obtained by employing this expansion, the form of the result depending upon the form of the remainder adopted.

The problem as stated should not depend upon the existence of derivatives at any point of the interval. A solution not involving such derivatives is the following:

$$\left| \int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a + h) + 4f(a) + f(a - h)\} \right| \leqq \text{Max}\,\{h|x - a|^2|D(x) - D(a + h)|\}, \quad (1)$$

where $(x - a)^2D(x) \equiv f(x) - 2f(a) + f(2a - x)$.

If $f(x)$ has a second derivative at $x = a$, this is of course the value approached by $D(x)$ as x approaches a. To prove (1), write

$$g(x) \equiv h(x - a)^2\{D(x) - D(a + h)\}.$$

Expanding and integrating we have

$$\int_{a-h}^{a+h} g(x)dx = 2h\left[\int_{a-h}^{a+h} f(x)dx - \frac{h}{3}\{f(a + h) + 4f(a) + f(a - h)\}\right].$$

Since $\left|\int_{a-h}^{a+h} g(x)dx\right| \leqq 2h \text{ Max } |g(x)|$, (1) is proved.

It is to be noted that $\text{Max}\,\{h|x - a|^2|D(x) - D(a + h)|\} = 0$, for polynomials of less than the fourth order, so that the formula given by Simpson's rule is exact in these cases.

II. Note on Professor Bennett's solution by H. P. Manning, Providence, R. I., and Otto Dunkel, Washington University.

If $f(x)$ can be expanded to four terms with the Lagrange remainder, the substitution of this expansion will show that $\max |g(x)| \leqq \frac{h^5}{3!} \max |f^{iv}(x)|$, but substitution in the integral $\int_{a-h}^{a+h} g(x)dx$ gives for the error the smaller limit $\frac{2}{45} h^5 \max |f^{iv}(x)|$, the same that is given by the substitution for $f(x)$, and for $f(a+h)$ and $f(a-h)$ in Simpson's formula itself. This limit is given in E. B. Wilson's *Advanced Calculus*, 1912, pages 76–77, exercises 23 and 24. However, a limit still smaller, namely, $\frac{h^5}{90} \max |f^{iv}(x)|$, has been found and is given by P. J. Daniell in this Monthly, *1917*, 110, and also by C. J. de la Vallée-Poussin, *Cours d'Analyse Infinitésimale*, volume 1, third edition, 1914, page 396. The method employed by the latter can be applied to the integral $\int_{a-h}^{a+h} g(x)dx$ and leads to the same result. Thus Professor Bennett's expression, obtained without assuming any expansion, leads to the results already found for functions capable of expansion to four terms and a remainder.

In 1874 Chevilliet (*Comptes Rendus de l'Académie des Sciences*, volume 78, page 1841), by taking the infinite expansion of $f(x)$, shows that the first term of the error is $-\frac{h^5}{90} f^{iv}(a)$, and when h is sufficiently small this approximates to the limit given by de la Vallée-Poussin. This result is obtained in the same way in Kiepert's *Grundriss der Differential- und Integralrechnung*, Teil 2, seventh edition, 1900, pages 335–336.

In Heine's *Handbuch der Kugelfunktionen*, Band 2, Theil 1, 1881, there is an exhaustive discussion of mechanical quadrature, and expressions are obtained for the errors in the Newton-Cotes method and in the method of Gauss. In particular, the fraction $-1/90$ can be obtained by multiplying $1/4!$ by the $-4/15$ given in the table on page 9.

William Hoover gave the reference to Wilson's *Calculus*, and H. E. Jordan to Kiepert's work.

2868 [1920, 482]. Proposed by H. S. UHLER, Yale University.

Let the evolute of a given curve be called the evolute of the first order, let the evolute of the first evolute be called the evolute of the second order, etc. Then, being given the following parametric equations in which a is a constant and γ is the parameter, namely,

$$x = (1 + 2\sin^2\gamma)\cos\gamma - a\sin 2\gamma, \qquad y = 2\sin^3\gamma + a\cos 2\gamma,$$

find: (*a*) the parametric equations of the evolute of order n, both for n even and for n odd;
(*b*) a formula for the total length of the nth evolute;
(*c*) a formula for the total area of the nth evolute;
(*d*) the sum of the lengths of all the evolutes from $n = 1$ to $n = \infty$; and
(*e*) the sum of the areas of all the evolutes from $n = 1$ to $n = \infty$.

Note. The original equations represent the envelope required in problem 2819 (*1920*, 134).

I. Solution by F. L. Wilmer, Omaha, Neb., and H. P. Manning, Providence, R. I.

One may note that the equation of the normal to the given curve can be put into the p-form with p a simple function of the parameter. Then differentiation with respect to the latter must give the equation of a perpendicular meeting this line in the corresponding point of the evolute, and so normal to the latter. In this way are obtained the parametric equations of the evolute, and by repetition those of the nth evolute.

We can write the given equations

$$\begin{aligned} x &= 3\cos\gamma - 2\cos^3\gamma - a\sin 2\gamma, \\ y &= \phantom{3\cos\gamma - {}} 2\sin^3\gamma + a\cos 2\gamma; \end{aligned}$$

and from these it follows that $dy/dx = \tan 2\gamma$.

Now the equation of the normal will be

$$X \cos 2\gamma + Y \sin 2\gamma = x \cos 2\gamma + y \sin 2\gamma = \cos \gamma.$$

Thus for the first evolute we have the equations

$$x_1 \cos 2\gamma + y_1 \sin 2\gamma = \cos \gamma, \qquad x_1 \sin 2\gamma - y_1 \cos 2\gamma = (\sin \gamma)/2;$$

or

$$x_1 = \cos^3 \gamma, \qquad 2y_1 = 3 \sin \gamma - 2 \sin^3 \gamma.$$

Similarly, starting with these equations, we get for the second evolute[1]

$$2(x_2 \sin 2\gamma - y_2 \cos 2\gamma) = \sin \gamma, \qquad 2(x_2 \cos 2\gamma + y_2 \sin 2\gamma) = (\cos \gamma)/2;$$

or

$$2^2\, x_2 = 3 \cos \gamma - 2 \cos^2 \gamma, \qquad 2y_2 = \sin^3 \gamma.$$

If we let x_0 and y_0 be what x and y become when a is zero, then $2^2 x_2 = x_0$ and $2^2 y_2 = y_0$. Suppose

$$2^{n-1}x_n = \cos^3 \gamma, \qquad 2^n y_n = 3 \sin \gamma - 2 \sin^3 \gamma; \tag{1}$$

so that $2^{n-1}x_n = x_1$ and $2^{n-1}y_n = y_1$. Then we shall have

$$2^{n+1}x_{n+1} = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2^n y_{n+1} = \sin^3 \gamma,$$

or $2^{n+1}x_{n+1} = x_0$ and $2^{n+1}y_{n+1} = y_0$, getting these equations in the same way that the equations for x_2 and y_2 were derived from those for x_1 and y_1.

Again, suppose

$$2^n x_n = 3 \cos \gamma - 2 \cos^3 \gamma, \qquad 2^{n-1}y_n = \sin^3 \gamma; \tag{2}$$

so that $2^n x_n = x_0$ and $2^n y_n = y_0$. Then $2^n x_{n+1} = \cos^3 \gamma$, and $2^{n+1}y_{n+1} = 3 \sin \gamma - 2 \sin^3 \gamma$, or $2^n x_{n+1} = x_1$ and $2^n y_{n+1} = y_1$.

Thus we prove by induction that (2) holds for n even and (1) for n odd.

For lengths we have $ds_0/d\gamma = 3 \sin \gamma$, $2ds_1/d\gamma = 3 \cos \gamma$; hence for n even $2^n ds_n/d\gamma = ds_0/d\gamma = 3 \sin \gamma$, and for n odd $2^n ds_n/d\gamma = 2ds_1/d\gamma = 3 \cos \gamma$. Integrating through 90° and multiplying by 4, we have in all cases for the complete length of the nth evolute $3/2^{n-2}$.

The length of the first evolute is 6 and the sum of the lengths of all of the evolutes is

$$6 + \sum_{n=2}^{\infty} 3/2^{n-2} = 12.$$

When n is even $ds_n/d\gamma$ becomes zero and changes sign for $\gamma = 0$ or π; therefore each of these evolutes has two cusps on the x-axis.

When n is odd $ds_n/d\gamma$ becomes zero and changes sign for $\gamma = \pm\, \pi/2$, and each of these evolutes has two cusps on the y-axis.

Areas can be obtained by the formula $A = 4 \int_0^1 y dx = -4 \int_0^{\pi/2} y \dfrac{dx}{d\gamma} d\gamma.$

Now $y_0 dx_0/d\gamma = -2 \sin^3 \gamma(3 \sin \gamma - 6 \cos^2 \gamma \sin \gamma)$; therefore $A_0 = 3\pi$.

Also $2y_1 dx_1/d\gamma = (3 \sin \gamma - 2 \sin^3 \gamma)(-3 \cos^2 \gamma \sin \gamma)$; therefore $A_1 = 3\pi/4$.

Then when n is even $2^{2n}A_n = A_0$, and when n is odd $2^{2(n-1)}A_n = A_1$. Thus $A_n = 3\pi/2^{2n}$ for all values of n.

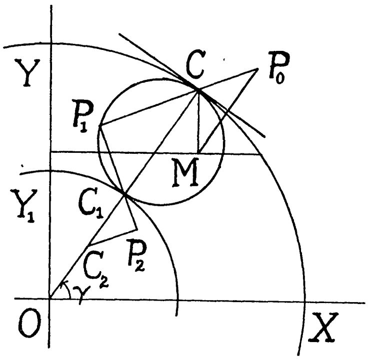
The sum of the areas of all of the evolutes is $\sum_{n=1}^{\infty} 3\pi/2^{2n} = \pi$.

Note. The area between two successive evolutes is swept over by the radius of curvature of the larger as γ varies through 360°. Thus we can write $A_n - A_{n+1} = \int_0^{2\pi} \dfrac{\rho_n^2}{2} \cdot 2d\gamma$, where ρ_n is the radius of curvature of the nth evolute. But then $\rho_n = s_{n+1} = (1/2^{n+1}) \cdot 3 \sin \gamma$ (or $3 \cos \gamma$ when n is odd). Thus this integral becomes $\dfrac{9}{2^{2(n+1)}} \int_0^{2\pi} \sin^2 \gamma$ (or $\cos^2 \gamma$) $d\gamma = \dfrac{9\pi}{2^{2(n+1)}}$.

[1] It is not necessary to go through the process of deriving the first of these two equations, since it is the same as the second of the two equations for x_1 and y_1, the normal to the first evolute being the tangent to the second.

II. Solution by Otto Dunkel, Washington University.

The solution of this problem, like that of 2819 (*1921*, 190) is simplified by the geometry of the curves. These may be constructed as follows: A unit circle is drawn with the origin as center and a chord parallel to the x-axis at a units above it. A radius OC is drawn with the inclination γ to the x-axis and the point C is projected upon the chord in the point M. Then the point P_0 which is symmetrical to M with respect to the tangent at C is a point on the given curve, P_0C is a normal, and the envelope of this normal, the first evolute of the present problem, is the caustic of the unit circle produced by vertical rays. If P_1 is the point of contact with the caustic, $CP_1 = (\sin\gamma)/2$ (formula (1), *1920*, 225, ω of the formula being equal to $\pi/2 - \gamma$ and R to 1), and this point is obtained by projecting the middle point C_1 of OC upon P_0C. P_1P_0 has the inclination $2\gamma - \pi/2$ and length $\rho_0 = (3/2)\sin\gamma - a$. Angle P_1CC_1 = angle YOC_1; therefore the arc P_1C_1 on the circle of diameter C_1C is equal to the arc Y_1C_1, where Y_1 is the point where the y-axis is cut by the circle with center O and radius $OC_1 = 1/2$. It follows that the locus of P_1 is the curve traced by this point when the former circle rolls on the latter circle; it is an epicycloid of two cusps, one at Y_1 and the other at the diametrically opposite point. This is our first evolute and P_1C_1 is its normal.



If $\rho_1 = P_1P_2$ is the radius of curvature of the first evolute, then $\rho_1 = (1/2)d\rho_0/d\gamma = 3(\cos\gamma)/4$, $C_1P_2 = (1/2)\,P_1C_1$, and P_2 is the projection upon this line of C_2 the middle point of OC_1. Now we prove as above for P_1, that the locus of P_2, our second evolute, is a two-cusped epicycloid, this time traced by rolling the circle of diameter C_2C_1 upon the fixed circle of radius OC_2, and having its cusps at the point where the latter circle cuts the x-axis and at the diametrically opposite point.

The same reasoning may be repeated again and again, giving us for the nth evolute an epicycloid with cusps on the x-axis when n is even and on the y-axis when n is odd. The equations in the former case are $x_n = (1/2^{n+1})(3\cos\gamma - \cos 3\gamma)$, $y_n = (1/2^{n+1})(3\sin\gamma - \sin 3\gamma)$, while the minus signs are changed to plus signs for the latter case.

The length of an arc of any evolute after the first measured from the nearest cusp of the preceding evolute is equal to the radius of curvature of the preceding evolute. Now $\rho_{n-1} = 3(\sin\gamma)/2^n$ or $3(\cos\gamma)/2^n$, neglecting signs; hence the complete length of the nth evolute can be obtained from one or the other of these expressions by putting the sine or cosine equal to 1 and multiplying by 4. That is, it is $3/2^{n-2}$ (it is 6 for $n = 1$).

For an evolute whose cusps lie on the y-axis the element of area generated by that portion of the radius of curvature which lies outside of the corresponding fixed circle is equal to twice the element xdy for the circle. For the first evolute, for example, it is $(1/2)\cos^2\gamma d\gamma$. A similar relation, the axes being interchanged, holds for an evolute whose cusps lie on the x-axis. Therefore the area of any evolute of the system is 3 times the area of the corresponding circle; namely, for the nth evolute it is $3\pi/2^{2n}$.

Since the lengths and areas form geometrical progressions it is easy to find their sums.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to H. P. Manning, Brown University, Providence, R. I.

Charles Leonard Bouton died at Cambridge, Mass., February 20, 1922. He was born at St. Louis, April 25, 1869. He graduated at the Washington University with the degree of M.Sc. in 1891, took the degree of A.M. at

Harvard in 1896, and the degree of Ph.D. at Leipzig in 1898. He was instructor at Smith Academy, St. Louis, 1891–1894, and at Washington University, 1893–1894. He went to Harvard as instructor in 1898, became assistant professor in 1904 and associate professor in 1914. He was one of the editors of the *Bulletin of the American Mathematical Society*, 1900–1902, and of the *Transactions*, 1902–1910. He contributed several articles to the *Annals of Mathematics* and to other periodicals, the most important being "Invariants of the general linear differential equation and their relation to the theory of continuous groups," *American Journal of Mathematics*, volume 21, 1899, pages 25–84, and "Examples of the construction of Riemann's surfaces for the inverse of rational functions, by the method of conformal representation," *Annals of Mathematics*, volume 12, 1898–1899, pages 1–26. The former was his dissertation under Lie at Leipzig.

George Bruce Halsted died in New York, March 19, 1922. Professor Halsted was born in Newark, N. J., November 25, 1853. He received the degree of A.B. from Princeton in 1875, and in 1879 the degree of Ph.D. from Johns Hopkins, where he was fellow, 1876–1878. He was instructor at Princeton, 1878–1881, and professor at the University of Texas, 1884–1903, at St. Johns College, Md., 1903, at Kenyon College, Ohio, 1903–1906, and at the Colorado State Teachers College, 1906–1912.

His first book was *Mensuration*, Boston, 1881. In New York, appeared *Elements of Geometry*, 1885; *Elementary Synthetic Geometry*, 1892; "Projective Geometry," Chapter II in *Higher Mathematics*, edited by Mansfield Merriman and R. S. Woodward, 2d ed., 1898, published separately as No. 2 of *Mathematical Monographs*, 1906; *Rational Geometry*, 1904, revised 1907 (translated into French by P. Barbarin, Paris, 1911).

His most important work was the translation of writings on non-Euclidean geometry. Lobachevski's *Researches on the Theory of Parallels* and Bolyai's *Science Absolute of Space* were first published at Austin, Texas, in 1891, as parts of the "Neomonic Series." They are now published in Chicago. His translation of Saccheri, *Euclides Vindicatus*, as *Euclid freed from every Fleck*, first appeared (all but the last thirteen pages) in this Monthly, *1894–1898*. He also translated some of Poincaré's writings on the foundations of science; namely, *Science and Hypothesis*, *The Value of Science*, and *Science and Method*, published in a single volume in 1913, New York and Garrison, N. Y.

Professor Halsted wrote numerous articles, biographical sketches, criticisms, etc., which are scattered through this Monthly, *Science* and other periodicals. Sommerville gives a list of ninety titles in his *Bibliography of Non-Euclidean Geometry* (London, 1911).

The honorary degree of doctor of science has been conferred on Sir Thomas Muir by the University of Cape Town, in recognition of his researches in mathematics and the history of mathematics.

Professor E. I. FREDHOLM, of the University of Stockholm, has been elected correspondent of the Paris Academy of Sciences in the section of geometry, as successor to the late Professor H. A. Schwarz.

A colloquium on the fundamental concepts of electrodynamics and of the electron theory of matter was held at the University of Wisconsin on March 30, 31, and April 1, 1922. The particular occasion for this meeting was the presence of Professor H. A. LORENTZ, the founder of the electron theory. The majority of those present were from the universities and colleges of the middle west, although both the Atlantic and Pacific coasts were represented. During the week preceding the colloquium proper, Dr. Lorentz gave four lectures on the general subject of light and the constitution of matter. These lectures, attended by a large and enthusiastic group of students and physicists, began with the basic concepts of the electromagnetic field, and traced briefly the developments which have led to the modern viewpoint. Professor Lorentz considered the successes and logical difficulties of the Bohr-atom theory, as extended by Sommerfeld and others, and discussed at some length the Michelson-Morley experiment and restricted relativity. In the last lecture a quantum-theory explanation of the Zeemann effect was given, to replace the older theory, based on classical electrodynamics.

During the colloquium itself the following lectures were given: "The experimental basis for the laws of electrodynamic action" by W. F. G. SWANN; "Deduction of the laws of electrodynamics from the relativity principle" by LEIGH PAGE; "Analytic formulation of electromagnetic theory through the field concept" by MAX MASON; "The structure of the electron" by D. L. WEBSTER; "The rotating earth as a reference system for light propagation" by L. SILBERSTEIN; "Application of statistical mechanics to electron theory" by A. C. LUNN; "Scattering of light and resonance radiation in relation to optical theories" by R. W. WOOD; "Thermal radiation—A discussion of recent experimental results" by C. E. MENDENHALL; "Electron theory of metals, volume phenomena" by P. W. BRIDGEMAN; "Electron theory of metals, surface phenomena" by P. T. COMPTON. At the conclusion of each paper the discussion was opened by Dr. Lorentz. The searching keenness, kindly interest, and revealing inspiration of his criticism will undoubtedly stimulate the scientific activities of all those who were present.

Published October 15, 1922.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

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I hereby give [1] to the Board of Trustees of the Mathematical Association of America the sum of .... Dollars,

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Witness: Signature ....

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[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF, A. A. BENNETT, University of Texas, Austin, Texas.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December 28, 29, 1922

The following are dates of Section meetings of the Association in 1921 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29, 1922
IOWA, Simpson College, Indianola, April 30; Des Moines, November 4
KANSAS, Topeka, January 21, 1922
KENTUCKY, Georgetown College, April 8, 1922; University of Kentucky, April, 1923
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, May 13, 1922; Washington, December, 1922
MINNESOTA, St. Paul, June 4; Macalester College, St. Paul, May 27, 1922.
MISSOURI, St. Louis, November 25–26; Kansas City Junior College, November 18, 1922
OHIO, Columbus, March 25–26; Columbus, April 14–15, 1922
ROCKY MOUNTAIN, Greeley, Colo., April 14–15, 1922; University of Colorado, April. 1923
SOUTHEASTERN, Atlanta, Ga., April 29, 1922
TEXAS, Dallas, November 25; Houston, December 1–2, 1922

DESIRED COPIES OF THE MONTHLY

Through an oversight the Association has not reserved a sufficient number of copies of the Monthly for October, 1920 and for August–September and October, 1921. Cash, or credit toward future dues, will be given by the Secretary for such copies at the rate of forty-five cents per copy, up to a limited number of copies.

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THE APRIL MEETING OF THE KENTUCKY SECTION.

The sixth regular annnal meeting of the Kentucky Section was held in the Physics Building, Georgetown College, Georgetown, Kentucky, on April 8, 1922. Professor C. H. Richardson, chairman, presided at both the morning and the afternoon sessions.

There were nineteen in attendance, including the following eight members of the Association:

R. V. Blair, P. P. Boyd, W. J. Brezler, J. M. Davis, H. H. Downing, Elizabeth LeStourgeon, C. H. Richardson, G. A. Seubert.

Those attending the meeting were entertained at luncheon by Professor C. H. Richardson. At the business meeting which followed the presentation of papers, Professor ELIZABETH LESTOURGEON was elected chairman and Professor C. H. RICHARDSON, secretary-treasurer. Upon the invitation of Dean Boyd, it was voted to hold the next meeting at the University of Kentucky.

The following papers were presented:

(1) "Invariants in the singular case of quadrics" by Professor H. H. DOWNING;

(2) "Infinite series in the theory of potential" by Professor ELIZABETH LESTOURGEON;

(3) "Some proofs of Pascal's theorem" by Mr. R. V. BLAIR;

(4) "The mystic hexagram configuration" by Dean P. P. BOYD;

(5) "Kramp's faculty function" by Professor C. H. RICHARDSON.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles.

1. In this paper Professor Downing considered two quadratic forms $\varphi = \Sigma_1^4 a_{ij}x_ix_j$ and $\psi = \Sigma_1^4 b_{ij}x_ix_j$, each of which is singular, the pencil of the pair, $\varphi - \lambda\psi$, having a determinant identically zero in λ. It was shown that the forms are completely classified by means of their ranks and the elementary divisors of the pencil. According to the ranks there are nine cases. These divide into thirty sub-cases according to the elementary divisors. The elementary divisors were found for each of the thirty sub-cases.

2. In the theory of logarithmic potential the distribution of mass along the boundary of a circle has been studied, and an expression for the density, in the form of a Fourier Series, has been obtained by a limiting process out of the series for the potential. Professor LeStourgeon made a similar study in the case of certain other analytic curves in the plane, both for the logarithmic potential and for the conjugate potential function.

3. After stating Pascal's theorem, Mr. Blair gave two proofs from Steiner (*Vorlesungen über Synthetische Geometrie*, volume 1, Leipzig, 1887, p. 16), one from Salmon (*A Treatise on Conic Sections*, London, 1879, p. 245), and the ordinary proof of projective geometry (cf. O. Veblen and J. W. Young, *Projective*

Geometry, volume 1, Boston, 1910, p. 111). It is not known how Pascal proved the theorem, but probably his method was one of the methods used by Steiner.

4. Dean Boyd traced the development of theorems leading up to and supplementing Pascal's theorem, mentioning the work of Euclid, Pappus, Desargues and others. Most of the time was given to proofs of the theorems concerning the Pascal lines, the Steiner points, the Cayley-Salmon lines, the Kirkman points, the Steiner-Plücker lines, and the Salmon points. Carefully drawn figures illustrated the work.

5. Professor Richardson gave a brief introduction to the calculus of finite differences, and showed how this calculus when applied to Kramp's faculty function leads to simple methods for the summation of many series. The following types of Kramp's faculty were discussed and illustrated:

$$(a + bx)^{m/b}; \qquad \frac{1}{(a + bx)^{m/b}}; \qquad \frac{a^{x/b}}{c^{x/b}}.$$

Reference was made to Chrystal's *Algebra*, part II, chapter 31.

ELIZABETH LESTOURGEON, *Secretary-Treasurer.*

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The tenth regular meeting of the Maryland-Virginia-District of Columbia Section of the association was held at Johns Hopkins University, Baltimore, Maryland, on December 10, 1921. Mr. O. S. Adams, chairman of the Section, presided at both morning and afternoon sessions. There were forty-four in attendance, including the following thirty members of the Association:

O. S. Adams, J. J. Arnaud, R. N. Ashmun, Clara Bacon, Sarah Beall, G. A. Bingley, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, A. Dillingham, H. English, J. B. Eppes, H. H. Gaver, W. M. Hamilton, W. E. Heal, L. S. Hulburt, W. D. Lambert, A. E. Landry, Florence P. Lewis, E. S. Mayer, F. Morley, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, C. H. Rawlins, H. M. Robert, Jr., R. E. Root, W. F. Shenton, C. A. Shook.

An amendment to the constitution of the Section was adopted, increasing the membership of the Executive Committee from three to four in order that Washington, Baltimore, Annapolis, and the state of Virginia might each have a representative on this committee.

The following resolution was introduced by Mr. W. D. Lambert, and adopted by the Section:[1]

[1] While the arrangement suggested by this resolution is clearly impractical as a working basis for the annual meetings of the Association, since all feeling of continuity or balance in the program would be sacrificed, there seems no reason why the section meetings should not respond as directly as possible to the several interests of the attendant members. The secretary of the Maryland-Virginia-District of Columbia Section refers to the Section as "nearly the banner section so far as attendance is concerned." In reference to this resolution, he remarks, "The

Resolved: (1) That the Section welcomes papers from any of its members, whether personally known to the officers of the Section or not, and that all members should feel free to present papers at the meetings.

(2) That it is intended to continue the present practice of holding meetings in May and December, and that any member desiring to present a paper should not wait for an invitation but should send an abstract several weeks in advance of the time for the meeting either to the officer of the Section representing that portion of the territory where the member lives, or directly to the Secretary-Treasurer, it being understood that the presentation of a paper may be postponed if the number of papers offered should be large, or if the Executive Committee has limited the topics for discussion at a particular meeting.

Professor A. Cohen reported for the Committee of the Section coöperating with the National Committee on Mathematical Requirements.

The value and pleasure of the meeting were enhanced by the fact that all those in attendance had lunch together, as guests of the members of the Department of Mathematics of Johns Hopkins University. The appreciation of the Section for the hospitality of these gentlemen throughout the day was expressed by a cordial vote of thanks.

The following papers were presented:

(1) "On a remarkable property of $(a + b\sqrt{-n})^n$, where n is of the form $4m + 3$" by Mr. WM. E. HEAL;

(2) "An arithmetical pyramid" by Dr. J. R. MUSSELMAN;

(3) "Multiple-valued solutions of Laplace's equations" by Mr. C. A. SHOOK;

(4) "Recent progress in hydrodynamics" by Professor F. D. MURNAGHAN;

(5) "An extension of Ptolemy's theorem" by Professor FRANK MORLEY;

(6) "Variation of latitude" by Mr. W. D. LAMBERT;

(7) "The fluctuating attitude toward mathematics" by Mr. HARRY ENGLISH;

(8) "Minimal curves" by Dr. L. M. KELLS, U. S. Naval Academy (Introduced by Professor G. R. Clements).

Abstracts of seven of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Writing $(a + b\sqrt{-n})^n = P + Q\sqrt{-n}$, Mr. Heal inquired whether P and Q are each separable into three rational factors. It has been known since the time of Euler that this is true for $n = 3$. In fact $P = a(a + 3b)(a - 3b)$, $Q = 3b(a + b)(a - b)$. Mr. Heal has established the theorem for all prime values of n less than 100, and believes that it is true in general.

2. A three-dimensional series of numbers was built up by Dr. Musselman, following a given law, which cut by transverse planes form an arithmetical pyramid. The numbers in these planes can be used to give a solution of the Problem of Points for three players who are equally skillful.

3. In his paper Mr. Shook showed how to construct the Green's func-

general interest of the meetings could be increased, and the labors of the Executive Committee lightened, if the various members would feel more free to offer discussions on topics of either a scientific or pedagogic nature which they have found interesting.—EDITOR.

tion, G, for a Riemann space of two sheets. This is done as follows: Let R denote the distance from the fixed point (r', φ', z') to any point (r, φ, z). Let R' denote the value of R, when φ is replaced by the complex variable α. Let $f(\alpha)$ denote any analytic function of α which (a) has a simple pole with unit residue at the point $\alpha = \varphi'$ and (b) is periodic with period 4π in both α and φ'. Then form the integral with respect to α having for integrand $1/R' \cdot f(\alpha)$, the path being so chosen that it includes the point $\alpha = \varphi'$, but no other singularity of the integrand. This integral multiplied by $2\pi i$ is the desired function G, with pole at (r', φ', z'). This function is single-valued in the Riemann space, but in ordinary space it is double-valued. Let G' denote the Green's function with pole at $(r', -\varphi', z')$, namely the image of the pole of G in the branch membrane. Then the difference $G - G'$ turns out to be the potential due to a semi-infinite plane under the influence of a unit charge of electricity at the point (r', φ', z'). In this difference the value in the first sheet of space is the one taken. The pole of G', being in the second sheet, does not cause an infinity to occur at that point in the first sheet. In that position it is, so to speak, harmless. Several possible extensions were indicated, such as curvilinear branch line and closed branch line.

4. Dr. Murnaghan spoke on the recent advances in the theory of resistance of fluids which have been made in two different directions: (*a*) by the Italian and French mathematicians, following the lead of T. Levi-Civita, and H. Villat, and (*b*) by the German workers in applied mathematics, following the lead of Prandtl and von Karman. The results obtained by the Germans have been up to the present the more useful to the practical men who work on the design of aëroplanes.

6. Mr. Lambert's paper dealt with the latitude observations of the International Latitude Service and particularly with the observations at Ukiah, Calif. These show on their face a progressive increase of latitude there of almost $0''.01$ a year, an increase which has been interpreted by a well-known geologist as a northward creep of the surface crust at Ukiah over the underlying strata. Mr. Lambert showed that the other stations of the Latitude Service showed changes of latitude of the same order of magnitude, some having greater changes than Ukiah, and that all of these changes could be accounted for within a reasonable margin of error by assuming: (a) a purely fictitious change in latitude common to all stations and due to a cumulative error in the star places used; and (b) an actual change of latitude due to a displacement of the North Pole towards the American continent along the meridian of 77° West at the rate of about $0''.0050$ a year. These deductions are limited to the years 1900–1917 inclusive. It was found that the latitude observations were accurate enough to give some information regarding the figure of the earth. They confirm in a general way Helmert's results from gravity observations that if the Earth is considered as an ellipsoid of three unequal axes, the shortest equatorial radius is 230 meters less than the longest one and lies in longitude 107° West. Attention was also called to Problem 2872, proposed by the author in the MONTHLY for January, 1921;[1] the

[1] Solved in this number of the MONTHLY, 227.

results there stated were found useful in discussing the latitude observations. The detailed investigation will appear in print as a special publication of the U. S. Coast and Geodetic Survey (no. 80); a shorter account appeared in the *Journal of the Washington Academy of Sciences*, January 19, 1922.

7. Mr. English said that the general trend of all opinion in the world seems to be a fluctuating one. The attitude towards mathematics is no exception, as to importance of methods of study and content of courses. The western part of the United States seems now to be dominating the eastern part, though it is not any too sure of itself. The reorganization which it is zealously pushing has many good features, though seemingly only one year of high school mathematics is required and that is somewhat polyglot in its nature. It may well apply to a junior high school course if that is a school of completion, but if it is to be the universal preparation for all senior high school mathematics, there is real danger along the line through the high schools and up into the colleges. There is naturally a haziness about the upper years, which may result fatally, especially in the hands of the majority of teachers, as definite details must be given if no harm is to result to pupils going from one teacher to another. It is a case of making haste slowly, giving careful consideration to all factors involved.

8. Dr. L. M. Kells first reviewed the well-known theory for minimal curves in the plane and in space, and then showed that the general equations of minimal curves in 4-space are

$$\begin{aligned} x_1 &= \tfrac{1}{2}[(\lambda^2 - 1)F_1''(\lambda) - 2\lambda F_1'(\lambda) + 2F_1(\lambda) \\ &\qquad + i\{(\lambda^2 + 1)F_2''(\lambda) - 2\lambda F_2'(\lambda) + 2F_2(\lambda)\}], \\ x_2 &= -\tfrac{1}{2}i[(\lambda^2 + 1)F_1''(\lambda) - 2\lambda F_1'(\lambda) + 2F_1(\lambda) \\ &\qquad + i\{(\lambda^2 - 1)F_2''(\lambda) - 2\lambda F_2'(\lambda) + 2F_2(\lambda)\}], \\ x_3 &= \lambda F_1''(\lambda) - F_1'(\lambda), \\ x_4 &= \lambda F_2''(\lambda) - F_2'(\lambda), \end{aligned}$$

where $F_1(\lambda)$ and $F_2(\lambda)$ are arbitrary functions of λ and the primes indicate derivatives with respect to λ. Properties in 4-space analogous to those for ordinary space were considered.

G. R. CLEMENTS, *Secretary-Treasurer.*

THE APRIL MEETING OF THE OHIO SECTION.

The seventh regular meeting of the Ohio Section was held in the Physics Building, Ohio State University, Columbus, on April 14–15, 1922, in connection with the meetings of the Ohio College Association and allied societies. An afternoon and an evening meeting were held on Friday. On Saturday a joint meeting of two sessions with the Ohio members of the Society for the Promotion of Engineering Education was held. Chairman B. F. Yanney presided, being relieved by Professor K. D. Swartzel for an interval.

There were fifty-three persons in attendance, including the following thirty-three members of the Association:

R. B. Allen, W. E. Anderson, G. N. Armstrong, C. L. Arnold, C. B. Austin, Grace M. Bareis, R. D. Bohannan, E. H. Clarke, H. L. Coar, O. L. Dustheimer, T. M. Focke, Emma L. Konantz, H. W. Kuhn, W. C. McCoy, E. S. Manson, Jr., C. C. Morris, Amy F. Preston, S. E. Rasor, P. L. Rea, Hortense Rickard, Bernice Sanders, R. A. Sheets, Mary E. Sinclair, S. A. Singer, K. D. Swartzel, T. E. Trott, J. H. Weaver, C. E. White, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, B. F. Yanney, and Antioch College represented by Dean P. C. Nash.

At the business session, the secretary reported a membership of seventy-eight and nine institutional members as against sixty-eight and eight, respectively, last year. Officers elected for this year are: Chairman, Professor H. L. COAR, Marietta College; Secretary-Treasurer, Professor G. N. ARMSTRONG, Ohio Wesleyan University; Third member of the executive committee, Professor J. H. WEAVER, Ohio State University. A committee of seven was elected with power to appoint and organize six or more sub-committees, to investigate the mathematics situation in Ohio with respect to the following six aspects at least: the state requirements and privileges, elementary school courses, high school courses, college entrance requirements, college courses, and teacher-training. This committee is to bring in a report, with recommendations, at the next annual meeting of the Section.

A majority of those in attendance joined with the Ohio College Association in the evening dinner at the Ohio Union, which was followed by an address by Dr. Thomas S. Moran, Purdue University.

The following papers were presented at the three sessions:

(1) Chairman's Address: "Some aspects of the mathematical situation in Ohio" by Professor B. F. YANNEY;

(2) Discussion: Professor E. H. CLARKE and Professor F. E. LANDSITTEL, Supervisor of High Schools, Ohio State University;

(3) "Chinese algebra, 'the precious mirror of the four elements'" by Professor EMMA L. KONANTZ;

(4) "The high school-college problem" by Miss MARIE GUGLE, Assistant Superintendent of Schools, Columbus (by invitation);

(5) "Simplification of the proofs of two statistical formulæ by the use of the calculus: (*a*) the moment formula, (*b*) the line of regression" by Professor C. C. MORRIS;

(6) "A theorem in geometrical optics" by Professor J. H. WEAVER;

(7) "The mathematical justification of a fundamental postulate of the theory of relativity" by Mr. DIO L. HOLL, Ohio State University (by invitation);

(8) Report on the Toronto meeting with special reference to "Research problems for college teachers" by Professor MARY E. SINCLAIR;

(9) "Structure and style of mathematical text books" by Professor H. L. COAR.

(Joint meeting with the Ohio members of the S. P. E. E.)

(10) "First lessons in calculus for engineering students" by Professor E. F. CODDINGTON, Ohio State University;

(11) "Mathematics and engineering in the Antioch program" by Dean P. C. NASH;

(12) "An experiment in freshman mathematics for engineers" by Professor K. D. SWARTZEL.

(Additional papers by members of the S. P. E. E.)

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Chairman Yanney, in his opening address, considered some aspects of the mathematical situation in Ohio, the immediate occasion of the address being the recent ruling of the State Department of Education to the effect that the unit of mathematics, hitherto required as one of the constants in every high school curriculum, is no longer required. The whole situation was reviewed, and it was suggested that a committee of seven be appointed to investigate the situation and to bring in a report with recommendations at the next annual meeting of the Section.

2. (*a*) In his discussion, Professor Clarke called especial attention to a study reported by Marsh in *Educational Administration and Supervision* for November, 1921. Marsh's results seem to indicate conclusively that one year of algebra produces a marked beneficial effect upon a pupil's later work.

(*b*) Professor Landsittel presented the position of the state educational authorities with reference to mathematics in high schools. The elimination of the one-unit requirement in mathematics from the list of constant studies prescribed by the state for high schools is not to be understood as an act in contravention of entrance requirements of the colleges. The indispensability of due preparation in mathematics on the part of entrants to standard college courses is fully recognized. The subject as a requirement is to be waived in general only in the cases of students markedly deficient in mathematical ability but possessing sufficient capacity in other lines to make possible creditable graduation from the high school. In the judgment of the speaker these students are exceedingly limited in number.

3. Professor Konantz discussed the Chinese Algebra of Chu Shih Chieh published in 1303. A short resumé of the foundations upon which Chu had to build and an explanation of the old Chinese methods of computation were followed by a discussion of the four elements and the process of finding the roots of higher degree equations. This process was shown to be precisely the same as Horner's method of root extraction. Chu's arithmetical triangle was shown to be the same as Pascal's triangle. The greatest work of the Chinese mathematicians was in arithmetic and algebra and Chu's "Precious Mirror" shows the height to which they attained.

4. The high school-college problem in mathematics teaching was discussed by Miss Gugle. One difficulty is that there are usually great gaps in the continuity of the subject matter. Another is that instructors consider themselves teachers of algebra or of geometry instead of teachers of mathematics. Also by giving material not suitable to the pupil's needs, the teacher develops in him a

distaste for mathematics so that he loses faith in his own ability. There are too many failures, expensive both to the pupil and to the public. We as teachers must find a way or make way for others who will; for pupils must be trained to succeed, not to fail. Public schools demand that teachers be trained in the art and science of teaching. Colleges, too, must realize that the strongest and best teachers are needed to teach college freshmen. Much can be done when the representatives of high school and college confer and in that way bring about a better mutual understanding.

5. Professor Morris emphasized the importance of a knowledge of the calculus on the part of the student of statistics. He exhibited calculus proofs of certain statistical formulæ, the algebraic proofs of which are complicated and difficult to follow.

6. Professor Weaver, following a suggestion in Pappus's comments on Euclid's *Optics*, showed that if rays of light emanate from two sources in a plane P, such that the rays from the sources form at the eye a constant angle which revolves about an axis parallel to P, then the path of either of the sources is a hyperbola provided the rays are not refracted. If, however, the rays are refracted at a second plane Q, the path of a source in plane P is a curve C of degree 8. He also pointed out some properties of C.

7. Mr. Holl showed that the Newtonian transformation for the uniform rectilinear motion of a light emitting source gives unequal light velocities in opposite directions, a moving point source now generating an elongated spheroid light volume, but in postulating constant light velocity in all directions we have the Lorentz form, out of which arises the invariant D'Alembert form of differential equation for light propagation, and a sphere light volume for a moving source. This puts in mathematical language the second postulate of Special Relativity in which light velocities are invariant. The transformation shows the correction for Newtonian mechanics for uniform rectilinear motion as well as the interdependence of time and space.

8. Professor Sinclair reported the discussions given at the Toronto meeting of the Association by Professors Oswald Veblen and G. A. Bliss. The former centered his discussion about foundations of geometry. He urged that more should be done in developing the assumptions, much of it elementary in character but demanding reflection and imagination. He also questioned whether the study already made might not react more directly on the teaching of geometry. Professor Bliss emphasized the fact that research is the discovery of a problem on which to work. One needs enthusiasm, imagination, and somewhat of mathematical maturity. He spoke of the fertile field of the application of the calculus of variations to problems, many of them calling for methods peculiar to the problem, but often only a moderate amount of the theory. Recent research has applied itself to new problems and also to extensions of the solutions of problems by Newton and his contemporaries. The reviewer concurred with Professor Bliss in recommending the attractiveness of this field, and sketched some problems as typical. (See *1922*, 101–103).

9. Professor Coar spoke on mathematical text books, especially those on elementary subjects. He mentioned that too little emphasis is placed in our books on the use of mathematics as a language and upon its interpretation in terms of everyday life. Elementary texts should be more condensed and not be reference books. In any subject only a few well-defined principles are needed. These should be mastered before statement in the form of rules. Analytic geometry should emphasize analysis. Thus far most texts have failed along this line. In writing elementary texts, we should keep in mind the student rather than the subject-matter.

10. Professor Coddington explained his methods of illustrating the general rate meaning, not time-rate meaning, of a differentiation by means of a few simple numerical problems, before attempting to define differentiation. This is done because it is thought that the student may thus acquire a better appreciation of the meaning of a derivative than from the formal definition in terms of concepts such as function, limit, variable, etc., which he does not understand.

11. In the Antioch experiment the amount of actual engineering taught is reduced from that of the conventional technical school to allow, first, more time for a broad cultural education, and second, more time for general business courses needed by the engineer. The coöperative plan is used to allow the student to get actual experience in his vocation. Outlines of the courses of instruction in various branches of engineering were presented by Dean Nash. One year of mathematics is required of all students. Students are segregated into groups of about equal mathematical ability, and those not competent to handle elementary algebra are required to make up the deficiency, without credit, in the academy. The other students take two weeks for a study of two or three principles of advanced algebra, devoting the rest of the semester to trigonometry. The second semester is spent in a combined study of analytical geometry and business graphs and statistics. These are combined, first, to give something in the actual experience of the student which relates the abstract ideas of analytical geometry, and, second, to give him the power of analyzing business graphs and statistics from a theoretical point of view.

12. In describing an experiment in segregation of freshmen in mathematics in the College of Engineering of The Ohio State University, Professor Swartzel spoke of (*a*) the reasons for making the experiment; (*b*) the ten-lesson review in sub-freshman mathematics preceding the segregation; (*c*) the uniform test and other considerations upon which the segregating classification was based; (*d*) the semester records of the two thirds retained in the freshman mathematics and of the one third reviewing the more advanced portions of sub-freshman mathematics; (*e*) a comparison with the records of previous years showing a decrease in failures and withdrawals from 32 per cent. to 10 per cent. in the case of freshman mathematics and from 32 per cent. to 17 per cent. in the case of the freshman and sub-freshman mathematics combined. There were corresponding and material increases in the numbers passing with higher semester standing.

G. N. Armstrong, *Secretary-Treasurer.*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION.

The sixth regular meeting of the Rocky Mountain Section was held at the State Teachers College, Greeley, Colorado, on April 14–15. Sessions were held on Friday afternoon and Saturday morning. The presiding officer was Professor G. W. Finley of the State Teachers College.

There were thirty in attendance, including the following eleven members of the Association:

I. M. DeLong, B. F. Dostal, A. R. Fehn, G. W. Finley, Philip Fitch, J. C. Fitterer, H. A. Howe, Claribel Kendall, G. H. Light, J. Q. McNatt, H. E. Russell.

Those in attendance were royally entertained at a six-thirty dinner in the Club House by Professor and Mrs. Finley. Professor LIGHT was elected chairman and Professor FEHN, vice-chairman, for the meeting to be held at the University of Colorado next year. A committee, consisting of Professor DeLong and Mr. Fitch, was appointed to draw up suitable resolutions upon the death of Dr. G. B. Halsted. It was also voted to extend an invitation to the national Association to hold its next meeting at the University of Colorado in September of this year or as soon thereafter as possible.

The following papers were presented:

(1) "Certain congruences determined by a given surface" by Dr. CLARIBEL KENDALL;

(2) "Kepler's problem for high planetary eccentricities" by Dean H. A. HOWE;

(3) "On the parametric equations of straight lines of which certain polar curves are envelopes" by Mr. PHILIP FITCH;

(4) "Games in mathematics teaching" by Mrs. LAURA C. GRAVES (by invitation);

(5) "A problem in mensuration" by Mr. J. Q. McNATT;

(6) "The application of hyperbolic functions to transmission line problems in engineering" by Mr. B. F. DOSTAL;

(7) "The content of a course in plane geometry" by Mr. FITCH;

(8) "Population curves" by Professor J. C. FITTERER.

Abstracts of papers follow below, the numbers corresponding to the numbers in the list of titles.

1. Miss Kendall gave formulas which she had developed for obtaining the curves on a surface which would give the developables of any congruence associated with the surface. A line of the congruence was given for every point on the surface. She also gave the formula for obtaining the focal points on any such line. These results were applied to several special congruences and some interesting relations among the lines determining these congruences were obtained. Several of these lines are lines of the osculating quadric at the point and are in harmonic relation to one another.

2. Dean Howe gave a method for determining to a very close approximation the position of asteroids, except in the cases where the eccentricity is very great.

3. Mr. Fitch demonstrated a short method for finding the equations of lines traced by reflected rays of light, applying the same to known caustic curves.

4. Mrs. Graves suggested that all elementary mathematics should be taught after the fashion of the old spelling bee.

5. The problem considered by Mr. McNatt was to find the radius of the sphere which displaces the maximum amount of water contained in a conical vessel.

6. Mr. Dostal reviewed recent developments in the applications of the hyperbolic functions to loaded and balanced telephone lines and cables, and to power transmission lines.

7. This paper dealt with the subject matter, fundamental concepts and elementary principles of plane geometry. Mr. Fitch pointed out the relation of these concepts to those of higher mathematics and their use in other subjects.

8. Professor Fitterer showed that the hyper-tan curve, $y = a \tanh bx + c$, closely graphs population data. Its use in municipal and state problems involving probable future growth constituted an important application.

G. H. LIGHT, *Secretary-Treasurer.*

ORGANIZATION MEETING OF THE SOUTHEASTERN SECTION.

On April 29, 1922, mathematicians of the Southeastern States met in the Main Building of Georgia School of Technology, Atlanta, Georgia. There were sixty-three present at the meeting, of which number the following fifteen are members of the Association:

D. F. Barrow, J. B. Coleman, T. R. Eagles, Floyd Field, Tomlinson Fort, Miss Leslie Gaylord, J. F. Messick, A. B. Morton, M. T. Peed, W. W. Rankin, Jr., H. A. Robinson, Douglas Rumble, W. V. Skiles, D. M. Smith, R. P. Stephens.

At the business meeting it was decided to present a petition to the Trustees of the Association asking permission to form a Southeastern Section of the Association, to include the following states: Alabama, Florida, Georgia, North Carolina, South Carolina, and Tennessee. After the program all present were entertained at lunch by Georgia School of Technology.

The officers elected are Professor FLOYD FIELD, Georgia School of Technology, Chairman; Professor R. P. STEPHENS, University of Georgia, Vice-Chairman; Professor W. W. RANKIN, JR., Agnes Scott College, Secretary-Treasurer. The Program Committee is composed of Professor W. W. Rankin, Jr., Chairman, Professor J. B. Coleman, University of South Carolina, and Professor Tomlinson Fort, University of Alabama.

The following program was carried out, abstracts being given with numbers to correspond to those of the program:

(1) "Some possibilities of the slide rule" by Professor D. M. SMITH;

(2) "Marking systems at the University of Georgia" by Professor D. F. BARROW;

(3) "Zero and infinity in elementary mathematics" by Professor J. F. MESSICK;

(4) "History of mathematics" (illustrated with slides) by Professor W. W. RANKIN, JR;

(5) "Einstein theory of relativity" by Professor W. S. NELMS, Emory University (by invitation).

1. In the first section of Professor Smith's paper it was shown that the theory of all fundamental operations of the slide rule flows directly from the definition of the logarithmic scale and from the fact that three units of length are used. The second section discussed the formulation of working rules for the solution of difficult problems when the student is unfamiliar with logarithms. The final section was devoted to certain advanced topics, including the solution of triangles, a type of engineering problem in maxima and minima, and solutions of the quadratic and cubic.

2. Dr. Barrow outlined the history of the marking systems used at the University of Georgia. He then exhibited some data, collected with the help of Mr. A. H. Stevens, which showed the variability of individual instructors in their distribution of high and low grades, and compared them with the distribution for the whole faculty.

3. Professor Messick's paper dealt primarily with the expressions $0/n$, $n/0$, ∞/n, n/∞, and the difficulties of teaching these to freshmen, with their meager knowledge of limits, and the subsequent troubles with the trigonometric functions. It attempted to show that the algebras could and should, by elementary methods, include a discussion of the infinite roots of the quadratic equation along with that of the zero roots. It was pointed out that otherwise some of the problems of analytics, particularly that of finding the asymptotes of the hyperbola, could not be intelligently comprehended by the student.

4. Professor Rankin pointed out that in some peculiar way the mathematicians have allowed an acquaintance with Cæsar, Homer, and Shakespear to count for culture and at the same time have permitted a knowledge of the work of Pythagoras, Euclid, Archimedes, and Newton to be regarded as sordid and commonplace. The failure of the mathematician to appreciate his forefathers has lost almost completely some of the outstanding men, such as Archimedes and Newton, both of whom are generally thought of as physicists, whereas they were also the greatest mathematicians of their day. The mathematician who forgets Thales, Pythagoras, Euclid, Archimedes, Diophantus, Apollonius, Cavalieri, Pascal, Fermat, Descartes, Napier, Newton, Leibnitz, the Bernoullis, Cauchy, and Gauss is no more worthy of the heritage that they have left him than the American who forgets Washington, Jefferson, Lee, Roosevelt, and Wilson.

5. Professor Nelms discussed Einstein's Relativity from the standpoint of a system of mathematics rather than from the idea of a theory of the physical universe. Several postulates were stated, which may be taken as the fundamentals or axioms of the system. An outline was given showing how these postulates would lead to the well-known deductions from the theory: world lines, variable space and time units, etc. Why the velocity of light is to be considered the maximum velocity of the physical universe was discussed from the physical and the Lorentzian transformation equation standpoints.

W. W. RANKIN, JR., *Secretary-Treasurer.*

SPANISH AND PORTUGUESE SYMBOLS FOR "THOUSANDS."

By FLORIAN CAJORI, University of California.

In old Spanish and Portuguese numeral notations there are some strange and curious symbols. In a contract written in Mexico City in 1649 the symbols "7U291*e*" and "VIIUCCXCI*ps*" each represent 7291 pesos. The U, which here resembles an O that is open at the top, stands for "thousands."[1] I. B. Richman has seen Spanish manuscripts ranging from 1587 to about 1700 and Mexican manuscripts from 1768 to 1855, all containing symbols for "thousands" resembling U or D, often crossed by one or two horizontal or vertical bars. The writer has observed that after 1600 this U is used both with Arabic and with Roman numerals; *before 1600 he has seen it used with Roman numerals only.* As the Roman notation does not involve the principle of local value, U played in it a somewhat larger rôle than merely to afford greater facility in the reading of numbers. Thus VIUCXV equals $6 \times 1000 + 115$. This use is shown in MSS. from Peru of 1549 and 1543,[2] in MSS. from Spain of 1480[3] and 1429.[4]

We have seen the corresponding *type symbol* for 1000 in a business blank of Mexico City, 1802, in eighteenth century books printed in Madrid,[5] and in modern reprints of seventeenth century documents.[6] In these publications the printed symbol resembles the Greek sampi ϡ for 900, but it has no known connection with it. In books printed in Madrid[7] in 1655 and 1646, the symbol is a closer imitation of the written U, and is curiously made up of the two small printed letters, "*l f*," each turned half way around. The two inverted letters touch each other below, thus "\ʃ." Printed symbols representing a distorted U have been found also in some Spanish arithmetics of the sixteenth century. The Spaniards call this symbol and also the sampi-like symbol a *calderón*.[8] A recent writer is in error when he states that these symbols were called *cuento*; *cuento* signifies, not one thousand, but one million.

The present writer has been able to follow the trail of this curious symbol U from Spain to northwestern Italy. In A. Cappelli's *Lexicon*[9] is found the follow-

[1] F. Cajori, "On the Spanish symbol U for 'thousands,'" *Bibliotheca Mathematica*, Bd. XII, 1912, p. 133.

[2] *Cartas de Indias publícalas por primera vez el Ministerio de Fomento*, Madrid, 1877, pp. 502, 543, facsimiles X and Y.

[3] José Gonzalo de las Casas, *Anales de la Paleografia Española*, Madrid, 1857, laminas, 87, 92, 109, 110, 113, 137.

[4] Liciniano Saez, *Demostracion Histórica del verdadero valor de todas las monedas que corrian en Castilla durante el Reynado del Señor Don Enrique III*, Madrid, 1796, p. 447.

[5] Liciniano Saez, *op. cit.*

[6] Manuel Danvila in *Boletin de la Real Academia de la Historia*, tomo XII, Madrid, 1888, p. 53.

[7] *Teatro Eclesiastico de la primitiva Iglesia de las Indias Occidentales* . . . el M. Gil Gonzalez Davila, su Coronista Mayor de las Indias, y de los Reynos de las dos Castillas. Tomo segvndo. En Madrid. 1655.

Memorial, y Noticias Sacras, y reales del Imperio de las Indias Occidentales . . . Escriuiale por el año de 1646. Juan Diez de la Calle, Oficial Segundo de la Misma Secretarìa.

[8] In Joseph Aladern's *Diccionari popular de la Llengua Catalana*, Barcelona, 1905, we read under "Caldero," "Among ancient copyists a sign (\ʃ) denoted a thousand."

[9] Adriano Cappelli, *Lexicon Abbreviaturarum*, Leipzig, 1901, p. L.

ing: "In the liguric documents of the second half of the fifteenth century we found in frequent use, to indicate the multiplication by 1000, in place of M, an O crossed by a horizontal line." This closely resembles some forms of our Spanish symbol U. Cappelli gives two facsimile reproductions[1] in which the sign in question is small and is placed in the position of an exponent to the letters XL, to represent the number 40,000. This corresponds to the use of a small *c* which has been found written to the right of and above the letters XI, to signify 1100. It follows, therefore, that the modified U was in use during the fifteenth century in Italy, as well as in Spain, though it is not known which country had the priority.

What is the origin of this *calderón*? Our studies along this line make it almost certain that it is a modification of one of the Roman symbols for 1000. Besides M, the Romans used for 1000 the symbols CIↃ, T, ∞ and ↑. These symbols are found also in Spanish MSS. It is easy to see how in the hands of successive generations of amanuenses, some of these might assume the forms of the calderón. If the lower parts of the parentheses in the forms CIↃ or CIIↃ are united, we have a close imitation of the U, crossed by one or by two bars.

Allied to the distorted Spanish U is the Portuguese symbol for 1000, called the "Cifrão."[2] It looks somewhat like our modern dollar mark $. But its function in writing numbers was identical with that of the calderón. Moreover, we have seen forms of this Spanish "thousand" which need only to be turned through a right angle to appear like the Portuguese symbol for 1000. Changes of that sort are not unknown. For instance, the Arabic numeral 5 appears upside down in some Spanish books and manuscripts as late as the eighteenth and nineteenth centuries.

A GENERAL CONSTRUCTION FOR CIRCULAR CUBICS.

By R. M. MATHEWS, Wesleyan University.

1. It is well known that any cubic curve can be generated as the locus of the intersections of a pencil of conics on four points with a projective pencil of lines. In practical work it is difficult to draw the conics and to effect the correlation involved in this construction. As a circular cubic passes through the circular points at infinity, the conics for such a curve may be specialized to a pencil of circles on two finite points. It remains, then, to find a simple and general method for effecting the correlation with the pencil of lines.

Schroeter and Durège in simultaneous papers[1] have shown that if each line pass through the center of the corresponding circle, the locus contains its singular

[1] Adriano Cappelli, *op. cit.*, p. 436, first column, Nos. 5 and 6.

[2] See the word "Cifrão" in Antonio de Moraes Silva, *Dicc. de Lingua Portuguesa*, 1877; in Vieira, *Grande Dicc. Portuguez*, 1873; in *Dicc. Comtemp. da Lingua Portuguesa*, 1881.

[1] H. Schröter, "Ueber eine besondere Curve 3^{ter} Ordnung und eine einfache Erzeugungsart der allgemeinen Curve 3^{ter} Ordnung," *Mathematische Annalen*, vol. 5, 1872, pp. 50–82.

H. Durège, "Ueber die Curve 3^{ter} Ordnung, welche den geometrischen Ort der Brennpunkte einer Kegelschnittschaar bildet," *ibid.*, 83–94.

focus, which is the intersection of the tangents at the circular points. Moreover, the general construction has been considered for special base points[1]. This paper explains a general and simple compass and ruler construction for the case of an arbitrary pair of base points.

2. Let Q and R be two fixed points on a variable circle which is cut again in S and T by two arbitrary fixed lines s and t through Q and R, respectively. The variable line ST cuts an arbitrary fixed line g in U. The variable line PU, drawn from a fixed point P, cuts the circle in A and B. The locus of A and B for the pencil of circles through Q and R is a cubic curve. We find the following properties of the figure.

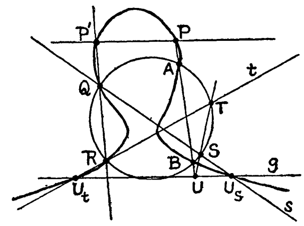

FIG. 1.

1. The lines ST are parallel, for $\angle\, TSQ = \angle\, QRT$ = constant.

2. There is a one-to-one correspondence between the circles and the pencil at P. For each circle cuts s and t in points S and T; the line ST determines U on g and so PU. Conversely, each line from P cuts g in a point U which determines one line in the fixed direction ST and so two points S and T concyclic with Q and R.

3. Q and R are on the locus. For PQ cuts g at U_Q and so the corresponding circle is determined; a similar consideration applies to R.

4. The circles of the pencil pass through the circular points at infinity, and thus the isotropic lines through P place them on the locus, which is, then, a circular cubic.

5. The circle through P puts this point on the curve.

6. When U is at infinity on g, then S and T are at infinity, and the circle on Q and R is replaced by the line QR and the line at infinity. Let the line through P parallel to g cut QR at P' and the line at infinity at F_∞, which is also on g.

7. The line g cuts the locus at F_∞ and at the points U_s and U_t where it meets s and t, respectively.

3. The given construction can now be shown applicable to any circular cubic. A circle through three points Q, R and A of the curve will cut it again in a finite point B. The line QR cuts the curve again in P' while AB does so in P. Then it is well known that PP' is parallel to the real asymptote. Let g, an arbitrary parallel to PP', cut the cubic in U_s and U_t. Thus lines s and t are determined as QU_s and RU_t, respectively. Our construction applied now on this basis will give a circular cubic which has in common with the given cubic the nine points, Q, R, P', A, B, P, F_∞, U_s and U_t besides the circular points at infinity. Accordingly, the cubics are identical.

4. The construction which Teixeira[2] derived analytically may be shown to be a special case of the foregoing. Take g through R and let it cut the cubic

[1] G. Loria, *Spezielle algebraische und transzendente ebene Kurven*. Leipzig, 1910, vol. 1, pp. 34–35.

[2] F. Gomes Teixeira, "Sur une manière de construire les cubiques circulaires," *Nouvelles Annales de Mathématiques*, fourth series, vol. 16, 1916, pp. 449–454.

again at U_s. Let the variable circle cut g again at H. Then as triangles QRU_s and SHU_s are similar:

$$\frac{U_sH}{U_sS} = \frac{U_sQ}{U_sR} = k, \text{ constant.}$$

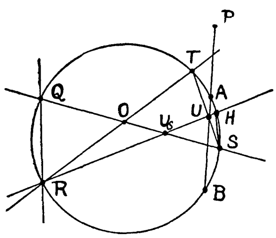

FIG. 2.

As the variable line ST moves parallel to itself, the segments it cuts on the fixed lines from U_s are proportional, so

$$\frac{U_sU}{U_sS} = k', \text{ constant.}$$

Thence

$$\frac{U_sH}{U_sU} = \frac{k}{k'} = c, \text{ constant.}$$

The procedure given is to draw PU arbitrarily from P to cut g at U; to determine H on g from the constant ratio just proved; and then to construct the circle through QRH to cut PU in points of the curve.

A GENERALIZATION OF THE STROPHOID.

By J. H. WEAVER, Ohio State University.

W. W. Johnson has given the following generalization for the strophoid.[1] Let A and B be two fixed points, and let two variable lines PA and PB make with AB angles ϕ and ψ, respectively. Let α be a constant angle and let P move so that

$$n\phi \pm m\psi = \alpha. \tag{1}$$

Then the locus of P is a strophoid. Equation (1) shows that there is associated with this set of curves a circle having a segment with AB as base in which the angle α may be inscribed.

In the following discussion some curves are developed which have associated with them the three conic sections.

Elliptic Case. Let there be an ellipse E with major axis AB, and from A and B let variable lines AP and BP be drawn making angles θ_1 and θ_2 respectively with AB. Let AQ and BQ be so drawn as to make angles $\pm m\theta_1$ and $\pm n\theta_2$ with AB. When the locus of Q is the ellipse E, the locus of P is a curve whose equation may be developed as follows. (In this development we will consider m and n as positive integers and relatively prime to each other.)

The slope of AQ is $\tan(m\theta_1)$ and of BQ is $\tan(n\theta_2)$. Then since Q is on E we have

$$\tan(m\theta_1)\cdot\tan(n\theta_2) = -b^2/a^2, \tag{2}$$

[1] "The Strophoids," *American Journal of Mathematics*, vol. 3, 1880, pp. 320–325. See also G. Loria, *Spezielle algebraische und transcendente ebene Kurven*, Berlin, vol. 1, 1910, p. 73. This lcass of curves includes the sextrix curves as a subclass. See Loria, l.c., p. 390.

where the equation of E is $x^2/a^2 + y^2/b^2 = 1$. Then using the same axes, the equations of AP and BP are

$$(AP) \qquad y = (x + a) \tan \theta_1, \tag{3}$$

$$(BP) \qquad y = (x - a) \tan \theta_2. \tag{4}$$

Eliminating θ_1 and θ_2 from (2), (3) and (4) we have

$$\begin{aligned} & a^2\left[\binom{m}{1} y(x+a)^{m-1} - \binom{m}{3} y^3(x+a)^{m-3} \cdots\right] \\ & \cdot\left[\binom{n}{1} y(x-a)^{n-1} - \binom{n}{3} y^3(x-a)^{n-3} \cdots\right] \\ & + b^2\left[(x+a)^m - \binom{m}{2} y^2(x+a)^{m-2} \cdots\right]\left[(x-a)^n - \binom{n}{2} y^2(x-a)^{n-2} \cdots\right] \\ & = 0, \end{aligned} \tag{5}$$

an equation of degree $m + n$.

We will designate the curve (5) as $C_{m,n}$. The curve $C_{m,n}$ has $m - n$ asymptotes. These pass through the same point[1] if $a = b$. If $m - n$ is odd, there will be one asymptote parallel to the y axis. Its equation will be

$$x = \frac{nb^2 + ma^2}{nb^2 - ma^2} a, \qquad \text{or} \qquad x = \frac{na^2 + mb^2}{na^2 - mb^2} a,$$

the former when m is even and n odd, the latter when m is odd and n even.

THEOREM I. *The curve $C_{m,n}$ has an m-tuple point at A and an n-tuple point at B, and the tangents to the curve at each of these points make equal angles with each other.*

This theorem may be readily proved in the usual way.

THEOREM II. *The rth polar of B with respect to $C_{m,n}$ is $C_{m-r,n}$ $(r \leqq m)$, and of A is $C_{m,n-r}$ $(r \leqq n)$.*

By substituting $-b^2$ for b^2 we obtain the hyperbolic case.

Parabolic Case. Let there be given a parabola and let there be drawn through the vertex a line making an angle $n\phi$ with the axis and cutting the parabola in the point Q. Through Q draw a line parallel to the axis, and let this line be cut in the point P by a line through the vertex of the parabola and making an angle ϕ with the axis of the parabola. Then the locus of P is a parabolic strophoid. Its equation may be determined as follows: Let O be the origin, and let the vertex of the parabola be at the origin, and its axis the x-axis. Then the equations of the parabola and of the lines OQ and OP are

$$y^2 = 4ax, \tag{6}$$

$$(OQ) \qquad y = x \tan (n\phi), \tag{7}$$

$$(OP) \qquad y = x \tan \phi. \tag{8}$$

[1] See Loria, l.c., p. 392.

From (6) and (7) we may find the equation of the line through Q parallel to the x-axis, and eliminating $\tan\phi$ from this equation and (8) we get for the equation of the curve

$$y = \frac{4a\left[x^n - \binom{n}{2}x^{n-2}y^2 \cdots\right]}{\binom{n}{1}x^{n-1}y - \binom{n}{3}x^{n-3}y^3 \cdots}. \tag{9}$$

The curve (9) has an n-tuple point at the origin, and the tangents at this point make equal angles with each other.

Since (9) is of degree $n + 1$ we will designate it as C_{n+1}.

THEOREM III. *The $(r + 1)$th polar of the point at infinity on the axis of the parabola $y^2 = 4ax$ with respect to C_{n+1} is C_{n-r}.*

This may be proved by finding the polars in the ordinary way.

A Special Elliptic Case. In equation (5) let $m = 1$ and $n = 2$ and we obtain the cubic

$$b^2y^2(x + a) - 2a^2y^2(x - a) - b^2(x + a)(x - a)^2 = 0. \tag{10}$$

THEOREM IV. *If in the curve* (10) *a line is drawn through A cutting the curve in the points P and P', and the ellipse in C, then the tangents to* (10) *at P and P' and to E at C are concurrent.*

Proof: Draw through A another line cutting (10) in the points P_1 and P_1' and E in C_1. From the construction of the curve (10) it follows that (AC, PP') and (AC_1, P_1P_1') are harmonic ranges. Since these ranges have a point A in common they are perspective. Therefore PP_1, CC_1 and $P'P_1'$ are concurrent. If now we consider the lines AP and AP_1 to become coincident, the lines PP_1, etc., become tangents.[1]

The two following theorems may also be proved without difficulty:

THEOREM V. *The first polar of A with respect to* (10) *is the ellipse E.*

THEOREM VI. *The locus of the point of concurrence of the tangents in theorem IV is a cuspidal cubic, having B as cusp and the asymptote of* (10) *as asymptote.*

If in (10) $a = b$, the curve is the right strophoid, and if $b^2 = a^2/2$ it is the folium of Descartes.

A Special Parabolic Case. Let $n = 2$ in (9) and it becomes

$$2a(x^2 - y^2) = xy^2. \tag{11}$$

THEOREM VII. *Let a line be drawn parallel to the axis of the parabola $y^2 = 4ax$ cutting the parabola in the point C and the cubic* (11) *in the points A and B, and let tangents be drawn to* (11) *at the points A and B and to the parabola at C. These tangents are concurrent.*

The proof is similar to that of theorem IV.

THEOREM VIII. *The locus of the point of concurrence of the tangents in theorem*

[1] See Bassett, *An Elementary Treatise on Cubic and Quartic Curves*, Cambridge, 1901, p. 67.

VII is a cissoid which has its cusp at the vertex of the parabola and whose asymptote is the asymptote of (11).

THEOREM IX. *The parabola $y^2 = 4ax$ is the curvilinear diameter of the cubic* (11).

The proofs of theorems VIII and IX offer no difficulty and are therefore omitted.

HONOR TO PROFESSOR E. H. MOORE.

By H. E. SLAUGHT, University of Chicago.

At the two hundred twenty-second regular meeting of the American Mathematical Society, held at the University of Chicago April 14, 15, 1922, there was celebrated the twenty-fifth anniversary of the founding of the Chicago Section of this Society. The history of this Chicago Section is almost coincident with that of the University of Chicago where the great majority of its forty-nine meetings have been held. These meetings, which began in 1896 with the reading of fourteen scientific papers at the first informal session, gradually increased in attendance and importance until they were recognized by the Society as of co-ordinate standing with those of the parent organization in New York and were officially designated as regular Western meetings of the Society. The one just held was the seventeenth and largest of this kind, over one hundred members being in attendance.

The young and vigorous department of mathematics of the University of Chicago in 1896, with its remarkable trio of leaders, Professors E. H. Moore, Oskar Bolza, and Heinrich Maschke, naturally assumed the important rôle of leadership in fostering mathematical research in the Middle West, as reflected in the phenomenal growth of the Chicago Section. To those who know Professor Moore scientifically and personally, it is no surprise that he at once became, and still remains, the leader of leaders in this great work. It is universally recognized that he stands quite alone as regards the scope and strength of his influence on the development of mathematics in America, not only through his own researches but also through his impress upon the hundred and more men and women who have gone out with the Chicago doctorate in mathematics, upon the hundreds of Chicago masters and other graduate students, and upon all others who have directly or indirectly come within his dynamic presence and captivating friendship.

It was a foregone conclusion that any celebration of the last quarter of a century of mathematical activity in this country would center about Professor Moore, and hence a committee of his former students began more than a year ago to consider what kind of a testimonial would be most appropriate to present to him on this occasion. It was at once decided that none of the ordinary forms of gold or silver gifts, nor even a painted portrait, would adequately express the sentiments of his grateful admirers. They sought rather some token which

Society, and upon the recommendation of a committee appointed from time to time for this purpose, in furtherance of such mathematical interests as

(*a*) The publication of important mathematical books and memoirs,

(*b*) The award of prizes for important contributions to mathematics;

It being further recommended that during the next ten years preference be given to the former and that publication of Professor Moore's researches in General Analysis or in other fields shall have precedence over all other claims.

The fund is to be kept intact by the American Mathematical Society except in so far as it is used to aid in the publication of Professor Moore's researches. For this special purpose a part of the principal, not exceeding one third, may be used, provided the interest on the remainder be allowed to accumulate until the fund has been restored to its original value.

(Here follow the names of 174 contributors.)

CHICAGO, April 25, 1922.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

24. SIR WILLIAM ROWAN HAMILTON AND THE EARLY DAYS OF QUATERNIONS.

It is well known that Sir William Rowan Hamilton (1805–1865) was one of the world's infant prodigies. Professor Alexander Macfarlane tells us of his considerable advancement in arithmetic at the age of three, of his ability to read Latin, Greek, and Hebrew at the age of five, and of his intimate familiarity with Italian, Sanskrit, and Arabic at the age of ten. At twelve he was contesting with Zerah Colburn, the American "calculating boy," and at sixteen he was reading Laplace's *Mécanique Céleste* and pointing out an error in the work of the great French master. He was thirty-eight, however, before he discovered the underlying principles of quaternions and it was not until 1853, ten years later, that he published his lectures on the subject. These lectures, in their initial form, were delivered at Trinity College, Dublin, in 1848.

The letter which follows was written in 1847 and has considerable interest in that it shows Hamilton's scientific activities at this time and refers to the monographs on quaternions which he had already published.

The letter, now in my collection, was written to Dr. James Booth and is as follows:

OBS^{Y} DUBLIN
March 5th, 1847

My dear Sir

Your note, dated the 12th of January, with its interesting accompaniments, did not reach me till a day or two ago,—which is the cause of its not having been sooner acknowledged. Allow me, as some return, to request your acceptance of my last contribution to the Proceedings of the Royal Irish Academy. There are, I think, some other papers of mine, of later date than any which you are likely to have seen—at least than any which I can have given you in 1845, as I printed several little things last year, availing myself so of the increased leisure gained by my retirement from the chair of the Academy. If, at *your* leisure, you will tell me the date of the last paper of mine on quaternions in your possession, I will try whether I cannot make your set

more complete:—since you have done me the honour of addressing the Society of which you are President, on the subject of my researches.

It could not, of course, be otherwise than honourable & gratifying to me, if the Literary and Philosophical Society of Liverpool should think fit, on your recommendation, to elect me a Corresponding Member. I do not know, however, that with the present claims upon my time, I have anything to promise in return.

Should you have mentioned the subject to any of the resident members, I hope that you will tell them that, through some accident, I did not hear of your intention till the day before yesterday.

I remain
my dear Sir
very truly yours
WILLIAM ROWAN HAMILTON

Rev. JAMES BOOTH, LL.D.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

NEW QUESTION.

The following question should call forth some interesting answers, whether complete or not, from readers having a taste for constructive geometry.

46. A geometrical construction will often become impracticable in special cases where some of the construction points are imaginary, although the final result is real. Is there any system according to which a construction failing in this way may be replaced by one that will work?

For example, let a line l_1 cut a circle c_1 in P_1 and Q_1, and let a line l_2 cut a circle c_2 in P_2 and Q_2. If the four points P_1, Q_1, P_2, and Q_2 are real, the intersections of P_1P_2 with Q_1Q_2 and P_1Q_2 with P_2Q_1 may be found directly. If the four points are imaginary, the intersections named will still be real, and there ought to be a simple way of getting them as the points common to a line and a circle.

DISCUSSIONS.

In PROBLEM-NOTES (*1921*, 278) a problem in Skeleton Division, proposed by Professor F. Schuh, was quoted from *Nieuw Tijdschrift voor Wiskunde*, volume 8.

Professor Curtiss shows that this problem has the unique solution 7752341/667334. In Professor Bennett's "Remarks" it is shown that the same conclusion is reached without the assumption of irreducibility. Solutions of a number of puzzles of this kind by W. E. H. Berwick may be found in *The Mathematical Gazette*, January, 1922, pages 8 and 9. In all of these certain digits were given beforehand, and in several cases there is more than one solution. For other references see Note 2, in Problems and Solutions (*1921*, 37).

Mr. Ballantine develops some ideas on linear systems of functions with a view to generalizing the notion of derivative. Considering a Taylor expansion as a linear combination of the successive powers of the variable, he discusses some ways in which a linear combination of a more general sequence of functions may be regarded as a generalized Taylor expansion.

I. Solution of a Problem in Skeleton Division.

By D. R. Curtiss, Northwestern University.

In the June–July number of this Monthly (*1921*, 278) mention was made of a problem in skeleton division:

"The following calculation represents the development of an irreducible fraction to a repeating decimal:

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Crosses indicate digits—all of which are possible. The repeating digits are shown by a line above the crosses. What irreducible fraction has been developed?"

The divisor is 667334, the dividend 7752341, the quotient $11.6\overline{168830001}\cdots$, the repeating digits being the nine under the dash.

First, as to the divisor:—The quotient, as an unending decimal, can be expressed as an infinite series

$$Q = \frac{m}{10} + \frac{n}{10^{10}} + \frac{n}{10^{19}} + \cdots,$$

where m and n are whole numbers. By the formula for the sum of a geometric series we have

$$Q = \frac{m}{10} + \frac{n}{10^{10}} \cdot \frac{1}{1 - 10^{-9}} = \frac{M}{10(10^9 - 1)},$$

where M is a whole number. Then, since dividend and divisor are relative primes by hypothesis, the divisor is a whole number which is a factor of $10(10^9 - 1)$. The prime factors of this number are 2, 3^4, 5, 37 and 333667. Since our divisor has six digits, it must be either 333667 or 667334. The former of these, however, is not the divisor, because the first three digits of line 7, and therefore of line 19,

would have to be 100 if the divisor were 333667, and the last indicated digit of the quotient would be 1 or 2. Multiply either into 333667 and subtract the result from a number of six digits whose last four are zeros, and the remainder cannot be a number beginning with 100. Thus 667334 is the only possible divisor.

We now show that the number in line 17 is 780000. The last indicated digit of the quotient must be 1, or there would be seven digits in line 18, and 1 is also the fourth digit of the quotient. This means that line 7 and line 19 begin with 10 or 11 or 12 or 13. The number in line 17, which is the sum of those in lines 18 and 19 (the former is the divisor itself), must then begin with 77 or 78 or 79 or 80. But the numbers in lines 15 and 16 are both even (the former terminating in a 0, and the divisor being even), hence their difference is even, and we have only 78 or 80 as possible first two digits in line 17. We dispose of 80 by noting that the numbers in lines 13 and 14 are even, so line 15 has an even number and a zero as its last two digits; to begin line 17 with 80 line 16 must end with a zero so that the eighth digit of the quotient would be 5, and line 16 ends with 70. Subtract from line 15, which ends with an even digit followed by a zero and the remainder cannot be 80.

We now know lines 17 and 18, and in consequence 19 and 7, the latter being 1126660. As line 5 ends with a zero, line 6 ends with a 4, and the third digit of the quotient is 6, since line 6 has seven digits. Line 5 is therefore 4116670, the sum of 112666 and 4004004. The first and second digits of the quotient are each 1, since lines 2 and 4 have but six digits apiece. These lines are, then, 667334, and a couple of additions brings us back to the dividend 7752341.

Note that the two decisive points are the number of factors in the repetend and the small remainder at line 17. If one wishes to amuse himself by constructing similar problems he would best pick this small remainder first, put it near the end of his operations, and build up the problem from it.

Remarks by A. A. Bennett, University of Texas.

This puzzle in skeleton division appears at first glance to be unadapted to any direct attack and to be probably capable of many solutions. A very brief examination is sufficient, however, to identify the division as that indicated by 7752341/667334. It may first be noticed that the continued decimal, which may be generated by $\times\times\times\times\times 000 \times/999{,}999{,}999$, and therefore by $\times\times\times\times\times\times/999{,}999{,}999$, is also generated by a fraction of the form $\times\times/\times\times\times\times\times\times$, as seen by the next to the last remainder. In this last fraction, if the common factors of numerator and denominator be removed, one has in the denominator a factor of 999,999,999, and this factor is at least a four-digit number. The factorization of 999,999,999 gives $3^4 \times 37 \times 333{,}667$, the last factor being a prime. A brief examination of 333,667 shows that this cannot itself be the divisor. It remains to test $2 \times 333{,}667$, any higher multiples being necessarily of seven figures. Now every six-figure multiple of 667,334 must be this number itself. Using this fact the steps may be rapidly retraced and the gaps filled.

The only alternative would be for the fraction $\times\times/\times\times\times\times\times\times$ to be of the

form $ab/(2997a) = \times\times\times\times\times\times/999{,}999{,}999$. Then $b \times 333{,}667 = \times\times\times\times\times\times$. Hence $b = 1$, or 2 and $\times\times\times\times\times\times = 333{,}667$ or 667,334. Both suggested possibilities are readily tested and excluded, by virtue of the fact that the last remainder has six digits prior to carrying down the next zero. Thus for $b = 1$, we have a in the form $\times\times$ and $3a \times 2997$ or $8991a$ as the last quantity to subtract. In other words $(10000 - 8991)a$ is the remainder, or $1009a$, where a is a two-digit number. Even 99 is too small for a, so that $b = 1$ leads to a contradiction as before. The solution then is unique.

II. What is a Calculus?

By J. P. Ballantine, University of Michigan.

There is much room for speculation as to the possible ways in which the calculus can be generalized. With a hope of stimulating other answers, rather than any presumption of closing the question, I suggest the following answer.

Consider the functions (1), which we may call the elementary functions of the Newtonian calculus, or briefly the Newtonian elementary functions:

$$0, \quad 1, \quad \frac{x}{1!}, \quad \frac{x^2}{2!}, \quad \frac{x^3}{3!} \cdots. \tag{1}$$

Notice that the derivative of each of the set is its predecessor. In fact, the process of differentiation may be regarded superficially as the process of replacing any of the above functions as they appear in a linear combination by its predecessor.

Thus, if $F(x)$ is given by the equation:

$$F(x) = a + \frac{bx}{1!} + \frac{cx^2}{2!} + \frac{dx^3}{3!} + \cdots, \tag{2}$$

we may infer (under certain conditions of convergence) that the derivative $F'(x)$ is given by the equation

$$F'(x) = b + \frac{cx}{1!} + \frac{dx^2}{2!} + \cdots. \tag{2'}$$

The object of this interpretation of the process of forming a derivative is that it admits readily of a generalization. One can substitute in place of the functions (1) another set of elementary functions, and if as a set they satisfy certain conditions, one can build up a new calculus based on them. The process of differentiation is not altered, but the geometrical interpretation is. Take for instance the set of functions,

$$0, \quad 1, \quad \frac{x}{1}, \quad \frac{x(x-1)}{1\cdot 2}, \quad \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}. \tag{1'}$$

The calculus based on these is the well-known calculus of finite differences.

Perhaps the most interesting questions raised by the discussion are those concerning the relations between the different schemes, and the extent to which these overlap or supplement one another. In particular, are all cumulative operations obtained under the second plan also obtainable under the first?

RECENT PUBLICATIONS.

REVIEWS.

Plane Trigonometry with Practical Applications. By L. E. Dickson. Chicago, Benj. H. Sanborn and Co., 1922. 4to. 11 + 176 + 35 pages. Price $1.52.

Extracts from the Preface: "This book introduces at an early stage concrete applications of trigonometry to the elementary parts of navigation and surveying, which are the two simplest exact sciences, as well as to the two elementary topics of physics which are known as composition of forces and refraction of light. There is, too, a full explanation of the theory and construction of a Mercator map, a subject of great importance also in geography. Three separate chapters are devoted to these subjects. The necessary terms and ideas are explained at length and illustrated concretely. We thereby obtain an abundance of simple problems whose importance is so convincing that they cannot fail to arouse real interest. Actual experience with classes has firmly convinced the author that these practical applications offer the best means to drive home the principles of trigonometry and to make the subject truly vital. . . .

"The development of the subject is leisurely and the student is given ample time in which to digest each idea. There are given full and lucid explanations of all new terms and ideas. Lack of the precise knowledge of the mathematical meaning of terms is one of the chief sources of difficulty in the study of mathematics. Various terms which should be already familiar to students are re-defined. On the basis of careful readings both of the manuscript and proof sheets by various experienced teachers in high schools and colleges, it is believed that the presentation is throughout both simple and clear.

"The tables are as simple as possible, and accurate for computation to four significant figures, which are ample for all ordinary practical purposes. It is true that some delicate astronomical measurements justify computations with 5, 6, or 7 place tables; but no new theory is involved. The traverse table, which is necessary for navigation and surveying, is really a systematic list of the sides and angles of all right triangles of moderate size. Its additional headings aid in making the present exposition of navigation much simpler than was possible heretofore. The traverse table is extremely useful in all parts of trigonometry and its applications, partly by relieving the monotony of logarithmic computation, but chiefly for the instantaneous checking of computations.

"The chapters on navigation and surveying are each divided into two parts, this making possible either a brief, wholly untechnical, introduction to those applications, or a fuller treatment."

Contents—Chapter I: Trigonometric functions of acute angles, 1–18; Chapter II: Solution of right triangles by means of tables of the natural functions, 19–25; Chapter III: Traverse table; solution of right triangles by inspection; problems on forces and refraction of light, 26–33; Chapter IV: Logarithms, slide rules, 34–47; Chapter V: Solution of right triangles by logarithms, 48–58; Chapter VI: Navigation: dead reckoning. Part I, the sailings (true course assumed). Part II, finding the true course; compass corrections, 59–81; Chapter VII: Land surveying. Part I, balancing a survey, area (true bearings assumed). Part II, surveying instruments; finding true bearings, 82–102; Chapter VIII: Trigonometric functions of any angle, 103–114; Chapter IX: Solution of oblique triangles, 115–132; Chapter X: Relations between functions of several angles, 133–149; Chapter XI: Graphs of the trigonometric functions and their inverses, radians, 150–167; List and index of formulas, 169–170; Index, including index to definitions, 171–172; Answers to certain of the first five exercises of each set, 173–176; Tables, 1–35.

The numerous textbooks that have been prepared by Professor Dickson, to say nothing of his more technical work, have throughout been characterized by a

logical directness and independence of treatment that are as refreshing as they are rare. First year college students, while seldom consciously critical, reflect by their loss of interest and decreased comprehension every carelessness on the part of the textbook writer and of the teacher. The merits of such a treatment as that here given are so obvious that any remark that might seem to convey a reflection upon the minor details of style may be out of place. For the prospective user of this text the following points may be mentioned. The treatment differs markedly from that customary for engineering texts. Perhaps the difference may be balanced by the method of presentation, but as a text, two features are fairly obvious. Most of the numerous relations that the good student prefers to work out as occasion may demand, but which the poor student, unfortunately, prefers to memorize, are not tabulated conspicuously in this text. This has the disadvantages of failing to impress the important features at a casual glance despite the warnings in the text, and also of making future reference inconvenient. That this remark is not an unfair one will doubtless be granted by any one who glances at the text. For example, the definitions of the cotangent, secant and cosecant are not given directly in terms of the adjacent side, opposite side and hypotenuse, but these functions are defined as the reciprocals of the tangent, cosine, and sine respectively. The values of the functions of 30°, 45°, and 60° are left to exercises as are also the signs of the functions in the several quadrants. Even the formulas which are labeled are not displayed in heavier type to catch the eye. The second conspicuous feature of this text is the number of technical suggestions made but left undeveloped. These undoubtedly add interest but lengthen the amount of time spent. For example, there is a description of the slide rule but no exercises in its use. It is to be presumed that the author does not expect the class to make actual use of this most convenient instrument. From the description given, it is at least dubious if a student could pick up a facility in the handling of a slide rule, and in fact it is hinted that the booklet which usually accompanies the sale of the instrument is to be relied upon. One may question the real utility of descriptions of instruments which the student does not see and handle, while if the instruments are at hand for examination the brief description in the text is superfluous. Similar remarks apply to the surveying instruments. It would have been of interest to have included also the sextant on account of its interesting geometrical theory, if so much less interesting an instrument as the surveyor's transit is regarded as deserving three pages of discussion and a full page cut.

The author says in his preface, "The great majority of students of trigonometry, whether in the high school or the college, take it as their final course in mathematics. Hence the course should justify itself at the time and not be merely a stepping stone to further mathematical subjects. Without overlooking the needs of the few who will go further in mathematics, we may justify trigonometry to the others by demonstrating its great utility by means of simple applications to various subjects that are vital in the world today." For the student of considerable imagination if not for all students this purpose has been well achieved

here. One might be inclined to question the value of this point of view for certain classes of students. Might not one regard the few "practical" problems always found in current texts as a sufficient hint of the practical side of the subject? Might not a good student obtain an even clearer insight into the utility of the subject, granted the time to cover such a book as this, if spherical trigonometry and its applications were included at the expense of problems in plane navigation and the refraction of light? Such queries must be answered by the individual teacher, as well as the more fundamental question as to whether the subject of trigonometry merits the time that a book of this sort will require to cover satisfactorily. In any case this text is urged upon the consideration of prospective teachers of the subject for its original and beautifully logical treatment in simple, clear style of a hackneyed but important course in elementary mathematics.

ALBERT A. BENNETT.

Loose-Leaf Outlines in Mathematics. By ROBERT R. GOFF. Boston, The Palmer Company. *Algebra,* 1920, 31 pages. Price $ .50. *Observational Geometry and Numerical Trigonometry,* 1922, 35 pages. Price $ .50.

The ideal method of conducting a mathematical class may depend upon the personality of the teacher even more than upon the content of the course or the preparation of the pupil. While lecture courses in which the students take notes as a basis for subsequent private study may prove entirely satisfactory for graduate courses in a university, there may be a question as to whether this is usually an ideal procedure in a subject like mathematics. Lectures are well adapted to advanced students if the subject matter is readily absorbed and merely an intelligent memory is required. The standard technique in undergraduate instruction in mathematics is undoubtedly to assign portions of a text for home study, designate certain problems to be completed outside of class, discuss and explain difficult points, and spend the remainder of the class period in testing the mastery that the students may have acquired of topics already assigned.

Other modes of attack are possible. In some places home work is reduced to a minimum. Laboratory methods are occasionally very successful in the hands of an enthusiastic teacher. A feature may be made of open discussion, or the subject may be taken up in a "reading course." The work may be handled in the character of a puzzle. A significant question to be asked is, how far does the student in fact utilize the explanations in the text. It is the reviewer's experience that the mind of the college freshman, at least, reacts far more vividly to a single illustrative example than to any discussion, long or short, of the principles involved. The college student is at first, if not eventually, inductive and general statements while readily memorized are applied only with difficulty until numerous examples have emphasized their significance. A question which the teacher must keep constantly in mind is as to just how far the student should be allowed to acquire mere dexterity accompanied by memorized rules (something that the otherwise poor student enjoys), and how far the

processes of deduction in their more abstract aspects should be forced upon the youthful and not too receptive mind.

One can judge somewhat the popular taste by noting the wide sale of texts which are often little better than collections of well graded exercises and carefully detailed illustrative examples. Many books which are frankly exercises and nothing else fail in a wider appeal merely because they are not accompanied by sufficient material to be memorized. They are unpopular not because they have strained the material too thin, but because they give too little restful variety to the laboring student.

In high school work, independent thought on the part of the pupil must be even less stressed than in the first year at college. Rules and exercises must constitute the bulk of a course in mathematics. Must that be all? The loose-leaf system under present discussion appears to expect an answer in the affirmative. I will quote from the compiler's "Explanatory Note."

"Some of the aims of these Outlines are to help the student:

1. Organize the material.
2. Separate essentials from non-essentials.
3. Emphasize fundamentals and methods.
4. Summarize related facts and processes.

These Outlines can be used with any text book. Make the Outlines the basis of the recitation and use the text book for a reference and supply book.

A topic is begun by a discussion of its nature and methods. Let the pupils do the work guided by the teacher. In the simpler operations, several particular examples from arithmetic should be carried through; then the generalized conclusion made in algebraic form and put on the pages of the Outlines or on an inserted page. "Chalk and talk" is good here. Then there should follow the application to exercises.

Models should be made of every definition and process.

Noticeable results of the use of these Outlines are:

1. They increase interest as work done with them is the pupil's own creation.
2. They make a reference book which the pupil knows how to use.
3. Their use develops habits of systematic effort.
4. They give a clear view of each topic as a whole."

Except for the idea of a loose-leaf method the material is handled in a standard conservative fashion and with a minimum of words. There is little beyond a set of definitional abstractions and a tabulation of type formulas and cases, with numerous blanks to be filled out by the pupil. There are few details which can be regarded as exercises in the narrow sense. Rather we have here an outline of the sort that when completed the professional tutor would call a syllabus. This makes the use of a separate source of problems essential but entirely replaces for the average pupil any text book discussion.

The sort of movement which this and numerous similar outlines evidence is of importance in the educational life of this country,—vastly more important than any single outline could be.

ALBERT A. BENNETT.

Plane Geometry: Experiment, Classification, Discovery, Application. By C. ADDISON WILLIS. Philadelphia, Pa., Blakiston's Son & Co., 1922. 16 mo. 301 pages. Price $1.32.

Preface: "This text represents the experience and developing ideals of a quarter of a century

of geometry teaching. The philosophy and the methods set forth in it are those of the author's own class-room. He believes that a pupil learns more of the subject when it is presented as in these pages, and he is certain that every pupil who is taught by this method under the guidance of a teacher who is himself interested in his subject becomes inspired to learn and is willing to work to learn. An awakening of a love of the subject is a service that may be rendered to our pupils of every intellectual grade. The text is so arranged that it may be variously used. . . . A short course . . . A laboratory course . . . A complete course . . . Exercises . . . For exceptional pupils . . . Dedication: The conception of this work runs far back in the years. It is the author's contribution in the service of his profession. It is dedicated to all those many students of the grand old science, . . . , and to all those teachers who have keenly felt the need of a more sympathetic interpretation of its immortal truths.

"In the hope that it will influence better teaching and learning, and will help to upbuild and uphold a generation which will reverence our beloved geometry for its own sake, as well as for its uses inseparable from a great constructive civilization—the author sends this little volume forth upon its mission."

Introduction to Ballistics.[1] [By ALBERT A. BENNETT.] Washington, Ordnance Department, U. S. Army, 1921. 8vo. 259 + 23 pages. Price $1.35.

Contents—Preface, 1–2; § 1: General remarks, 3–5; § 2: Some elementary concepts in mechanics, 6–25; § 3: Some elements of interior ballistics, 26–41; § 4: An introductory discussion of exterior ballistics, 41–46; § 5: The equations of the trajectory in vacuo, 46–53; § 6: Action of air resistance, 53–68; § 7: Measurements of air resistance, 69–72; § 8: Tabulation of air resistance, 73–77; § 9: The ballistic coefficient, 78–82; § 10: The trajectory in air, 83–85; § 11: Elements of the trajectory, 85–98; § 12: Siacci's method, 99–110; § 13: The use of Ingalls' Tables II, 111–117; § 14: Ballistic tables based on numerical integration, 118–131; § 15: The computation of trajectories by numerical integration, 131–139; § 16: Deviation of the trajectory from the plane of fire, 140–147; § 17: Errors and the application of the theory of probability, 148–168; § 18: Range table conditions, 169–183; § 19: Variations from range table conditions, 184–188; § 20: Differences in altitude of gun and target, 189–198; 21: Wind effects, 198–210; § 22: Meteorological changes, 211–217; § 23: Rotation of the earth, 218–227; § 24: The meteorological message, 228–238; § 25: Proving Ground data, 239–244; § 26: Range tables, 245–258; Table of contents, 259. Supplement: Comparative sketch of the normal ballistic coefficient, 1–23.

A Course in Exterior Ballistics. Ordnance Textbook. [By R. S. HOAR.] Washington, Government Printing Office, 1921. 8vo. 127 pages. Price $1.00.

Extracts from the Introduction—"The work of the ballistic computer is divided into three parts: (1) the computation of the elements of standard trajectories; (2) the computation of differential corrections, whereby the elements of a standard trajectory may be corrected for nonstandard conditions; and (3) the utilization of the foregoing to construct range tables from firing records. . . . The first course of instruction in these new ballistic methods ever given in this country was given at the Ordnance School of Application in the winter of 1919–20 by Capt. Roger Sherman Hoar, Coast Artillery, then in charge of the Ballistic Section of the Proof Department at Aberdeen. This present book is based upon the papers used in that course, and uses the standard symbology and nomenclature established as above. It is assumed that the student is thoroughly grounded in algebra and plane trigonometry, and knows enough calculus to appreciate the meaning of a derivative, a differential, and a definite integral. On that basis, this book gives, in Chapters I to IV, the irreducible minimum of higher mathematics necessary to understand all points involved in the later chapters."

Contents—Introduction, 7–9; Chapter I: Partial differentiation, 10–13; II: Successive approximations, 14–17; III: Effect of differential variations, 18–27; IV: Finite differences, 28–33; V: Elements of the trajectory, 34–37; VI: History of exterior ballistics, 38–42; VII: The motion of a projectile, 43–46; VIII: Computation of trajectories, 47–52; IX: Derivation of auxiliary variables, 53–58; X: Range correction formulas, 59–69; XI: Angle of departure correc-

[1] For notice of *Physical Bases of Ballistic Table Computation* by Professor Bennett see this MONTHLY, *1920*, 372.

tion formulas, 70–72; XII: Deflection formulas, 73–75; XIII: Rotation of the earth, 76–82; XIV: Computation of differential corrections, 83–90; XV: Weighting factors, 91–95; XVI: Construction of a range table, 96–102; Supplement A: Trajectory computation by the tangent reciprocal method, 103–104; B: Explanation of the signs in the computation of differential corrections, 105–106; C: Dimensions of ballistic symbols, 107–108; D: Antiaircraft fire, 109–110; E: Derivation of two equations of Chapter VII, 111–113; F: A derivation of Theorem I, 114–115; G: New methods of trajectory computation, 116–119; H: Note on advancing difference formulas, 120–121; Index, 123–127.

NOTES.

The third volume of *Fundamenta Mathematicae* (see *1921*, 318) appeared in April. It contains 323 pages, consisting of 29 papers + problems proposed and lists of exchanges. The only American authors are R. L. Moore, "Concerning connectedness im kleinen and a related property," 232–237, and J. R. Kline, "A theorem concerning connected point sets," 238–239.

Volumes 85 and 86 of *Mathematische Annalen* (see *1921*, 135) have been published. The following articles are by American writers—In volume 85: A. J. Kempner, "Über die Separation komplexer Wurzeln algebraischer Gleichungen," 49–59; E. R. Hedrick and W. D. A. Westfall, "The existence domain of implicit functions," 74–77; E. J. Wilczynski, "Charakteristische Eigenschaften der isothermkonjugierten Kurvennetze," 208–212; E. Kasner, "The solar gravitational field completely determined by its light rays," 227–236. In volume 86: O. D. Kellogg, "On the existence and closure of sets of characteristic functions," 14–17; E. H. Moore, "On power series in general analysis," 30–39.

A. Hermann, 6 rue de la Sorbonne, Paris, is offering for sale the volumes in the mathematical libraries of Monsieur Lebon and Professor Camille Jordan, and Professor Pierre Boutroux.

The mathematical library of the late Paul Mansion (d. 1919), long professor of mathematics at the University of Ghent, is to be presented to the University of Louvain as soon as a library building has been erected. One of Professor Mansion's sons is a professor at the University of Louvain, and another is professor at the University of Liége.

In accordance with the terms of the Treaty, volumes from the various libraries of Germany are now being assembled at Leipzig for the University of Louvain. The ruins of the old L-shaped library still stand much as after the fire. The corner stone of the building for the great new library was laid July 28, 1921, and work on the foundations is now being carried on. The inscription on the corner stone is: "Lapis primarius Bibliothecae Lovaniensis nobiliter reficiendæ."

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 43, October, 1921: "On some properties of general manifolds relating to Einstein's theory of gravitation" by J. A. Schouten and D. J. Struik, 213–216; "Geometrical theorems on Einstein's cosmological equations" by E. Kasner, 217–221; "On the Fermat and Hessian points for the non-euclidean triangle and their analogues for the tetrahedron" by C. M. Sparrow, 222–225; "The Cauchy-Lipschitz method for infinite systems of differential equations" by W. L. Hart, 226–231; "Boundary value and expansion problems; formulation of various transcendental problems" by R. D. Carmichael, 232–270; "Reciprocity in a problem of relative maxima and minima" by J. K. Whittemore, 271–290.

Bulletin des Sciences Mathématiques, volume 46, February, 1922: "A propos de quelques livres sur la théorie de la relativité" by J. Villey, 61–96—March: "Vue d'ensemble sur les machines à calculer" by M. D'Ocagne, 102–144—April: "Sur les fonctions qui admettent un théorème d'addition algébrique" by H. Mineur, 156–176.

Il Bollettino di Matematica, new series, volume 1, nos. 1–3, 1922: "Su alcuni problemi relativi all'equivalenza dei poligoni" by C. Ciamberlini, 1–4; "Sui criteri di eguaglianza dei triangoli" by R. La Marca, 5–9; "Le soluzioni intere della equazione $1/x + 1/y = 1/z$" by C. Ruggeri, 10–15; "Aggiunte ad una nota su 'Particolari progressioni aritmetiche e geometriche'" by L. Tenca, 16–21. "Sezione storico-bibliografica": review by G. Vivanti of Hadamard, *Four Lectures on Mathematics delivered at Columbia University in 1911* (New York, 1915), xi–xiii; review by G. L(oria) of C. J. Keyser, *The Humor Worth of Rigorous Thinking. Essays and Addresses* (New York, 1916), xix–xxi.

Esercitazioni Matematiche, volume 1, November–December, 1921: "La dimensionalità dello spazio" by B. Levi, 203–208; "Riduzione di una sostituzione lineare a forma canonica" by M. Cipolla, 209–220—Volume 2, January–February, 1922: Review by G. Mignosi of J. W. Young, *Fundamental Concepts of Algebra and Geometry,* Italian translation by D. Mercogliano, with preface by G. Loria, 50–53—March–April: "Le sostituzioni lineare che non alterano una quadrica" by G. Mignosi, 80–88; "La commemorazione di Ulisse Dini," 104–108; Letter from D. Mercogliano in regard to the review by G. Mignosi of Young's *Fundamental Concepts of Algebra and Geometry,* 121–123.

Nature, volume 109, May 6, 1922: "Text-books of elementary mathematics" by H. B. H., 574–576 [Brief notice of several text-books including Osgood, *Elementary Calculus* (Macmillan, New York, 1921)]; "Pythagoras's theorem as a repeating pattern" by J. R. Cotter, 579—May 20: "The general theory of relativity" by A. S. E., 634–636 [Review of H. Weyl, *Space—Time—Matter* (London, 1922)]; "On immediate solutions of some dynamical problems" by A. Gray, 645–647—June 3: "G.B.M." by A. Gray, 712 [Reminiscences of G. B. Mathews]; "The elliptic logarithmic spiral—a new curve" by H. S. Rowell, 716.

La Nature, volume 50_1, April 29, 1922: "Le télescope géant de M. Mac Afee et les canaux de Mars" by A. T., 270—May 13: "Einstein au Collège de France, Conférence de M. Einstein" by M. Morand, 298–303 (to be continued)—May 20: "Einstein au Collège de France, Discussion sur les théories" by M. Morand, 315–320.

Periodico di Matematiche, series 4, volume 2, March, 1922: "Sulla teoria della relatività" by G. Todesco, 125–135 ["Dalle conferenze di Einstein a Bologna. Prima conferenza: Il principio speciale di relatività"]; "L' invenzione dei logaritmi" by A. Agostini, 135–150; "Assiomi e ipotesi dei *Fondamenti di Geometria* del Veronese" by F. Palatini, 150–163; "Sopra una proprietà del cerchio dei nove punti" by R. Zappa, 164–173; "Prof. Umberto Scarpis," 219–220 [Tribute of the Council of the Italian Society, Mathesis]—May: "Sulla teoria della relatività" by G. Todesco, 221–236 ["Seconda conferenza: La relatività generale"]; "Gli incommensurabili e il procedimento euclideo del massimo comun divisore" by P. Cattaneo, 236–245; "Sulle funzioni iperboliche" by V. Notari, 246–260.

Philosophical Magazine, sixth series, volume 42, November, 1921: "On the transformation of the equation of motion of the dynamics of continuous media in the restricted principle of relativity" by H. T. Flint, 794–799; "The relativity of field and matter" by A. S. Eddington, 800–806; "On the postulates and conclusions of the theory of relativity" by L. T. More, 841–852—Volume 43, January, 1922: "The derivation of symmetrical gravitational fields" by F. D. Murnaghan, 19–31 ["In the usual treatment of this problem a field is said to be symmetrical about a point if the form for $(ds)^2$ is invariant under linear orthogonal transformations of the 'Cartesian coördinates' (x_1, x_2, x_3); then a transformation is made to 'polar coördinates' r, θ, ϕ, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, etc., and an appeal is made, in choosing a form for $(ds)^2$, to the corresponding form in Euclidean space. Now the essential assumption in the relativity theory of a permanent gravitational field is that the physical space-time continuum of four dimensions is *non*-Euclidean; the term 'Cartesian coördinates' for a non-Euclidean space requires definition, and the equation for r given above assumes an underlying Pythagorean theory that is not tenable for non-Euclidean spaces in general. Since the only existing experimental verifications of the relativity *gravitational theory* are based on the expression for $(ds)^2$ in a permanent symmetrical gravitational field, it is desirable that the assumptions in the mathematical treatment should be clearly stated."]; "On the significance of Einstein's gravitational equations in terms of the curvature of the world" by A. S. Eddington, 174–177 [Revision of a proof in volume 42, p. 800];

"On a graphical solution of a class of differential equations occurring in wireless telegraphy" by A. A. Robb, 206–215 [Given an equation of the type $d^2p/d\omega^2 + f(p)\cdot dp/d\omega + p = 0$, the expression $d^2p/d\omega^2 + p$ may be interpreted as the radius of curvature of a curve from which by mechanical means a second curve is drawn whose polar equation is a solution of the given equation. (Compare F. V. Morley's derivation of the differential equation arising from the curve of pursuit problem, *1921*, 55)]—February: "Method of tracing caustic curves" by A. S. Percival, 258–277—March: "The relation between the projective and the metrical scales, and its bearing on the theory of parallels" by L. Silberstein, 420–429; "The deflexion of a ray of light in the solar gravitational field" by F. D. Murnaghan, 580–588; "The identical relations in Einstein's theory" by G. B. Jeffery, 600–603—April: "On products of Legendre functions" by J. W. Nicholson, 768–784—May: "Circular plates of variable thickness" by G. D. Birkhoff, 953–962.

Popular Astronomy, volume 30, June–July, 1922: "Photograph of a small tornado," 325–326; "The present position of the island universe theory of the spiral nebulæ" (concluded) by D. B. McLaughlin, 327–339; "Aberration and relativity" by W. H. Pickering, 340–343; "A universal scale for the solution of astronomical problems" by J. E. G. Yalden, 352–356 [Describes a device for the graphical solution of astronomical problems].

Proceedings of the London Mathematical Society, series 2, volume 20, part 6, February, 1922: "On double surfaces" by B. M. Sen, 417–434—Part 7, April, Prefatory Matter: "Lord Rayleigh" by H. L., xliii–xlvii; "Adolf Hurwitz" by W. H. Y., xlviii–liv.

Proceedings of the Royal Society, series A, volume 100, December 1, 1921: "The principles of internal ballistics" by J. Proudman, 289–305—Volume 101, April 1, 1922: "The design of repeating patterns.—Part I" by P. A. MacMahon and W. P. D. MacMahon, 80–94 [Abstract in *Nature*, November 24, 1921, p. 421 (see *1922*, 75)]; "Correlation between arrays in a table of correlation" by C. Spearman, 94–100.

Proceedings of the Royal Society of Edinburgh, volume 41, part 2, 1920–21: "Note on a continuant of Cayley's of the year 1874" by T. Muir, 111–116.

Quarterly Journal of Pure and Applied Mathematics, volume 49, December, 1921: "On recurrences for sums of divisors" by E. T. Bell, 186–192.

Rendiconti del Circolo Matematico di Palermo, volume 45, 1921: "On the relation between the Hilbert space and the calcul fonctionnel of Fréchet" by E. W. Chittenden, 265–270; "Orbits asymptotic to the straight line equilibrium points in the problem of three finite bodies" by D. Buchanan, 332–356—Volume 46, fasc. 1, 1922: "The Cauchy problem on the equation of telegraphy" by T. Hayashi, 117–121.

Revue Générale des Sciences, volume 33, April 15, 1922: "A propos de la théorie d'Einstein" by J. Richard, 193–194; "Sur la possibilité de se représenter l'espace fini et sans bornes de la théorie de la relativité" by R. Thiry, 205–209—April 30: "Sur la notion de courbure de l'espace" by R. Thiry, 225; "Coup d'œil sur les principes fondamentaux de la nomographie en quels cas et comment ils permettent de réduire à une représentation plane des dimensions en nombre supérieur à trois" by M. d'Ocagne, 230–239; "La démonstration scientifique" by R. d'Adhémar, 239–244 (to be continued)—May 15: "La démonstration scientifique" by R. d'Adhémar, 268–276.

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2972. Proposed by J. H. M. Wedderburn, Princeton University.

In *Mathematical Education*, p. 361, B. Branford gives the following approximate construction

for the trisection of an angle $A\hat{B}C$. Take any point D on AB produced and make $BC = BD$; through C draw a parallel to BA and on it lay off a length $CP = CD$; then $A\hat{B}P$ is approximately $\frac{1}{3}A\hat{B}C$. Show that this construction is rendered very much more accurate by laying off on CP a distance $CQ = 2BD$ and taking in place of P, the point R which divides PQ in the ratio 5 : 4. Also show that the error does not exceed 10″ for angles less than 45°.

2973. Proposed by N. K. CHAFFEE, Rutland, Vt.

A straight brass bar 800 feet long expands 8 inches. The ends are fixed, so that it is distorted. If the new shape is that of an arc of a circle of which the original bar is the chord, how far above the center of the bar in its original position will the center be in the new position?

It is instructive to guess at this distance before attempting to solve the problem.

2974. Proposed by the late L. G. WELD.

A man standing on a straight railway track watches a train starting from a station half a mile distant and notices that it approaches at such speed that each puff of the exhaust is heard at the same instant that the next succeeding puff is seen. How long will it be before the train reaches him, the drive wheels of the locomotive being sixteen feet in circumference?

2975. Proposed by the late ARTEMAS MARTIN.

What must be the form of the mold-board of a plow, considered as a warped surface, so that it will run most easily through the ground?

2976. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

The base of a variable triangle is fixed, the opposite vertex describing a straight line. Find the locus of the symmedian point, the locus of the center of the nine-point circle, and the envelope of the Euler line.

SOLUTIONS.

2825 [1920, 186]. Proposed by the late L. G. WELD.

A ball, having a coefficient of resilience α, strikes a rigid plane surface, inclined at an angle θ from the horizontal, after falling through a height h. What is the distance from the first to the second point of impact with the plane?

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio, and H. P. MANNING, Providence, R. I.

In this problem, the forces are impulsive. Let $u = \sqrt{2gh}$ be the velocity of the center of the ball when it impinges upon the plane for the first time at O; let the straight line through O directly down the plane be the axis of x, and the normal to the plane at O the axis of y; let v_x, v_y be the x and y components of the space velocity immediately after the first impact; and let X, Y be the corresponding reactions between the ball and plane, the mutual friction being such that the ball does not slip. Let m be the mass, and a the radius of the ball, and let k be its radius of gyration about a diameter.

First suppose $\alpha = 0$. Then, resolving parallel to the x and y axes, we have

$$m(v_x - u \sin \theta) = - X, \qquad m(v_y + u \cos \theta) = Y;$$

and, taking moments about the center of the ball, and putting $\omega =$ the angular velocity, we have also

$$mk^2\omega = Xa.$$

For the condition of no slipping, we have $v_x = a\omega$, and for $\alpha = 0$, $v_y = 0$. These equations give

$$X = \frac{mk^2u \sin \theta}{k^2 + a^2}, \qquad Y = mu \cos \theta. \tag{1}$$

Let v_x', v_y' be the velocities corresponding to v_x, v_y in the second case, when resilience is taken into account; then for X, Y we write $(1 + \alpha)X$, $(1 + \alpha)Y$, and the equations for linear motion become

$$m(v_x' - u \sin \theta) = - (1 + \alpha)X, \qquad m(v_y' + u \cos \theta) = (1 + \alpha)Y.$$

These combined with (1) give

$$v_x' = u \sin\theta \frac{a^2 - \alpha k^2}{a^2 + k^2}, \qquad v_y' = + u\alpha \cos\theta. \tag{2}$$

The equations of motion are $\ddot{x} = g \sin\theta$, $\ddot{y} = -g\cos\theta$, or $\dot{x} = gt\sin\theta + c_1$, $\dot{y} = -gt\cos\theta + c_2$. When $t = 0$, $\dot{x} = v_x'$ and $\dot{y} = v_y'$. Hence determining c_1 and c_2 from (2), we get

$$\dot{x} = \sin\theta\left(gt + u\frac{a^2 - \alpha k^2}{a^2 + k^2}\right), \qquad \dot{y} = \cos\theta\,(u\alpha - gt).$$

Integrating again, with the conditions $x = 0$ and $y = a$ when $t = 0$,

$$x = \sin\theta\left(\tfrac{1}{2}gt^2 + tu\frac{a^2 - \alpha k^2}{a^2 + k^2}\right), \qquad y = a + \cos\theta\,(u\alpha t - \tfrac{1}{2}gt^2).$$

The ball hits the plane a second time when $y = a$, i.e. when $t = 2u\alpha/g$. The corresponding value of x becomes, after simplification,

$$\frac{2a^2\alpha(1+\alpha)u^2}{g(a^2+k^2)}\sin\theta.$$

Also solved by H. L. OLSON, ARTHUR PELLETIER, and J. B. REYNOLDS.

2860 [1920, 428]. Proposed by E. O. BROWN, Chicago, Ill.

A frustum of a right circular cone has a volume v. The lateral area added to the lesser base is a sum which is a minimum. Determine the dimensions of the frustum in terms of v.

SOLUTION BY H. S. UHLER, Yale University.

Let h, r, R, and s denote respectively the altitude, the radius of the smaller base, the radius of the larger base, and the slant height of the frustum. Also let σ symbolize the sum in question. Then

$$\sigma = \pi(Rs + rs + r^2), \tag{1}$$

$$v = \tfrac{1}{3}\pi h(R^2 + Rr + r^2), \tag{2}$$

$$s = +[h^2 + (R-r)^2]^{1/2}. \tag{3}$$

By virtue of equations (2) and (3), σ may be considered as a function of the two independent variables R and r, hence the necessary conditions for a minimum of σ are

$$\frac{\partial\sigma}{\partial R} = 0, \qquad \frac{\partial\sigma}{\partial r} = 0.$$

Differentiating equations (1), (2), and (3) with respect to R and combining the results we obtain

$$\frac{\partial\sigma}{\partial R} = \frac{\pi}{s}\left\{s^2 + (R+r)\left[R - r - \frac{9v^2(2R+r)}{\pi^2(R^2+Rr+r^2)^3}\right]\right\} = 0,$$

whence

$$\frac{9v^2}{\pi^2(R^2+Rr+r^2)^3} = \frac{2(R-r)}{R+2r}. \tag{4}$$

In like manner we find

$$\frac{\partial\sigma}{\partial r} = \frac{\pi r}{s}\left[2s - 2(R-r) - \frac{9v^2(2R+r)}{\pi^2(R^2+Rr+r^2)^3}\right] = 0. \tag{5}$$

The ratio $r/R \equiv \rho$ may be obtained in the following manner. First substitute the equal of the left member of equation (4) in the third or fractional term of equation (5) to get

$$s = \frac{3(R^2 - r^2)}{R + 2r}. \tag{6}$$

Then square the last equation, eliminate v^2 by again using relation (4), and suppress the com-

mon factor $R - r$ to obtain $3\rho^3 + 5\rho^2 - 2 = 0$, the roots of this equation being

$$-1, \qquad -(\sqrt{7}+1)/3, \qquad (\sqrt{7}-1)/3.$$

Substituting the positive root in the following modified form of equation (4)

$$R^6 = \frac{9(1+2\rho)v^2}{2(1-\rho^3)(1+\rho+\rho^2)^2\pi^2},$$

we get

$$R = \frac{(88+13\sqrt{7})^{1/6}v^{1/3}}{2^{1/6}\cdot 7^{1/4}\pi^{1/3}} \doteq 0.833293v^{1/3},$$

and $r = \frac{1}{3}(\sqrt{7}-1)R \doteq 0.457131v^{1/3}$.

The values of h and s may now be obtained from formulas (2) and (6) respectively. They are $h = [2(\sqrt{7}-2)v/\pi]^{1/3} \doteq 0.743559v^{1/3}$, $s = R$.

The last relation is the most significant geometrically. Let θ denote the acute angle between the axis and the slant height of the frustum. Then

$$\theta = \sin^{-1}[(R-r)/s] = \sin^{-1}[(4-\sqrt{7})/3] \doteq 26° \, 50' \, 4.5''.$$

From the practical point of view, the problem amounts to asking for the smallest amount of sheet metal out of which a cup or deep cake pan of given capacity can be made.

2870 [1921, 36]. Proposed by WARREN WEAVER, University of Wisconsin.

A pendulum bob of mass m is attached to one end of a weightless and inextensible string of length l and swings as a conical pendulum with an angular velocity ω_1 about a vertical line through a fixed point to which the other end of the string is attached. If the angular velocity is increased to ω_2, the height through which the bob rises is independent of the length l. Consider, then, a very long and a very short pendulum. Suppose they are each swung first with an angular velocity ω_1 and then with a larger angular velocity, ω_2, the difference between these two values being great enough so that the longer pendulum rises through a height greater than the length of the shorter pendulum. According to the above result, the shorter one should rise through this same height, which is obviously impossible. Explain this apparent paradox.

SOLUTION BY I. MAIZLISH, University of Minnesota.

Let l be the length of the string, α_1 the angle which the equilibrium position of the pendulum makes with the vertical when the angular velocity is ω_1, and α_2 the corresponding angle when the angular velocity is ω_2. Assume $\omega_2 > \omega_1$, that the pendulum is free to take a vertical position when not revolving, and that ω_1 and ω_2 are finite. It is readily seen that for equilibrium the following equation must hold:

$$\tan\alpha_1 = (\omega_1^2 l \sin\alpha_1)/g, \tag{1}$$

and if $\alpha_1 > 0$ we must have

$$\omega_1 > \sqrt{(g/l)}. \tag{2}$$

The height of the bob from the horizontal plane passing through it when the pendulum is in its vertical position is $h_1 = l(1-\cos\alpha_1) = l - (g/\omega_1^2)$, and if the angular velocity is increased from ω_1 to ω_2, α being increased to α_2, the height through which the bob rises will be

$$h_2 - h_1 = (g/\omega_1^2) - (g/\omega_2^2), \tag{3}$$

as long as $\omega_1 \geqq \sqrt{(g/l)}$.

Let l' be the length of the short pendulum. If ω_1 and ω_2 are chosen so that $(g/\omega_1^2) - (g/\omega_2^2) > l'$ we shall have $\omega_1 < \sqrt{(g/l')}$ and we cannot apply (3) to this pendulum. If $\omega_2 > \sqrt{(g/l')}$ and the bob of the short pendulum rises at all, the height to which it rises will be $l' - g/\omega_2^2$, but if $\omega_2 \leqq \sqrt{(g/l')}$, it will not rise at all.

Also solved by H. L. OLSON, F. L. WILMER and the PROPOSER.

2872 [1921, 36]. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

The rectangular coördinates of a point P at the time t are given by the equations

$$x = k\cos\gamma\cos(nt-\alpha), \qquad y = k\sin\gamma\cos(nt-\beta),$$

where k, γ, n, α, and β are constants and γ is taken in the first quadrant. An auxiliary angle δ, in the first quadrant, is defined by the equation

$$\sin \delta = \sin 2\gamma |\sin (\alpha - \beta)|.$$

(1) Show that P describes an ellipse the lengths of whose semi-axes are $k \cos \frac{1}{2}\delta$ and $k \sin \frac{1}{2}\delta$ and the inclinations of whose axes to the x-axis are given by

$$\tan 2\theta = \tan 2\gamma \cos (\alpha - \beta).$$

(2) Show that the time T when P is at the end of an axis is given by

$$\tan (2nT - \alpha - \beta) = \cos 2\gamma \tan (\alpha - \beta).$$

(3) Obtain criteria for distinguishing between the major and minor axes and for the direction of rotation of P and show that the quantities γ, $\alpha - \beta$, δ, θ and $nT - \alpha - \beta$ are simply related as the parts of a right spherical triangle, thus providing a partial check on the computation.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

1. Eliminating t from the given parametric equations, we have

$$\alpha + \cos^{-1} \frac{x \sec \gamma}{k} = \beta + \cos^{-1} \frac{y \csc \gamma}{k},$$

or

$$\cos (\alpha - \beta) = \frac{xy \sec \gamma \csc \gamma}{k^2} + \frac{1}{k^2} \sqrt{k^2 - x^2 \sec^2 \gamma}\,\sqrt{k^2 - y^2 \csc^2 \gamma};$$

whence

$$x^2 \tan^2 \gamma + y^2 - 2xy \cos (\alpha - \beta) \tan \gamma - k^2 \sin^2 (\alpha - \beta) \sin^2 \gamma = 0 \qquad (1)$$

is the rectangular equation of the curve, and is that of an ellipse with center at the origin. To ascertain the length s of the axes then, we need find only the maximum and minimum values of $r = \sqrt{x^2 + y^2}$. Now,

$$r^2 = x^2 + y^2 = k^2[\cos^2 \gamma \cos^2 (nt - \alpha) + \sin^2 \gamma \cos^2 (nt - \beta)]. \qquad (2)$$

Setting the $d(r^2)/dt = 0$, we get

$$\tan 2nt = \frac{\cos^2 \gamma \sin 2\alpha + \sin^2 \gamma \sin 2\beta}{\cos^2 \gamma \cos 2\alpha + \sin^2 \gamma \cos 2\beta} = \frac{S}{C}, \quad \text{say}; \qquad (3)$$

and we find

$$\sqrt{S^2 + C^2} = \sqrt{1 - \sin^2 2\gamma \sin^2 (\alpha - \beta)} = \cos \delta \text{ by the data given};$$

whence

$$\sin 2nt = \pm \frac{S}{\cos \delta} \qquad \text{and} \qquad \cos 2nt = \pm \frac{C}{\cos \delta}.$$

Putting these values in (2), we get for maximum and minimum values of r^2,

$$\frac{k^2}{2}\left[1 \pm \left(\frac{S^2 + C^2}{\cos \delta}\right)\right] = \frac{k^2}{2}(1 \pm \cos \delta),$$

so that the major and minor semi-axes are $k \cos \delta/2$ and $k \sin \delta/2$. The angle made by these axes with the x-axis comes immediately from (1) by writing the condition for elimination of the term in xy or[1]

$$\tan 2\theta = \frac{-2 \cos (\alpha - \beta) \tan \gamma}{\tan^2 \gamma - 1} = \tan 2\gamma \cos (\alpha - \beta).$$

2. When the ends of the axes are reached, r is a maximum or minimum. In getting $d(r^2)/dt = 0$, we have

$$\cos^2 \gamma \sin (2nt - 2\alpha) + \sin^2 \gamma \sin (2nt - 2\beta) = 0.$$

[1] Similarly the expression for the semi-axes could be obtained by use of the invariants of rotation, and then the equation for T could be derived by algebraic processes. On the other hand, the equation for θ ($\tan \theta$ being y/x) could be deduced from the equation $d(r^2)/dt = 0$ without any reference to the theory of rotation of axes.—EDITORS.

Or, putting $\frac{1}{2}(1 + \cos 2\gamma)$ and $\frac{1}{2}(1 - \cos 2\gamma)$ for $\cos^2 \gamma$ and $\sin^2 \gamma$,

$$\sin (2nt - \alpha - \beta) \cos (\alpha - \beta) - \cos 2\gamma \cos (2nt - \alpha - \beta) \sin (\alpha - \beta) = 0;$$

and then replacing t by the particular value T at this time,

$$\tan (2nT - \alpha - \beta) = \cos 2\gamma \tan (\alpha - \beta).$$

3. The major semi-axis is $k \cos \delta/2$, δ being by hypothesis an angle in the first quadrant. Differentiating

$$\frac{y}{x} = \tan \gamma \frac{\cos (nt - \beta)}{\cos (nt - \alpha)},$$

we have for the angular velocity of P

$$\frac{n \tan \gamma \sin (\beta - \alpha)}{\cos^2 (nt - \alpha)},$$

positive or negative with $\sin (\beta - \alpha)$.

We can place 2γ on the hypotenuse, 2θ and δ on the legs of a right spherical triangle, $90° - (2nT - \alpha - \beta)$ opposite 2θ and $\alpha - \beta$ opposite δ on the angles; for the three equations

$$\sin \delta = \sin 2\gamma \sin (\alpha - \beta), \qquad \cos (\alpha - \beta) = \tan 2\theta \cot 2\gamma,$$
$$\cos 2\gamma = \cot (\alpha - \beta) \tan (2nT - \alpha - \beta)$$

will then conform to Napier's rules.[1]

2878 [1921, 89]. Proposed by R. S. HOAR, Fort Banks, Mass.

Consider the integers $0, 1, 2, 3 \cdots n - 1, n$. Consider all possible permutations of combinations of these taken r at a time, allowing any integer to occur more than once. Select from these permutations all groups the sum of whose integers is n. Form the reciprocal of the product of the factorials of the r integers of each of these selected groups. Then the sum of all of these reciprocals will equal $r^n/n!$. Prove that this must be so.

Example. $n = 2, r = 3$.

$$\frac{1}{2!0!0!} + \frac{1}{0!0!2!} + \frac{1}{0!2!0!} + \frac{1}{1!0!1!} + \frac{1}{1!1!0!} + \frac{1}{0!1!1!} = \frac{3^2}{2!}, \qquad \text{if} \qquad 0! = 1.$$

SOLUTION BY H. L. OLSON, University of Michigan, and PHILIP FRANKLIN, Princeton University.

By the multinomial theorem, we have

$$(x_1 + x_2 + \cdots + x_r)^n = \Sigma \frac{n!}{i_1!i_2!\cdots i_r!} x_1^{i_1}x_2^{i_2}\cdots x_r^{i_r},$$

where $i_1 + i_2 + \cdots + i_r = n$ and $i_1, i_2, \cdots, i_r \geqq 0$. If we set each x_i equal to unity and divide the two members of the equation by $n!$, we have the theorem stated.

2880 [1921, 89]. Proposed by SIDNEY DORB, Detroit, Mich.

Solve the simultaneous equations $xy = 2$, and

$$\left(3 - \frac{6y}{x-y}\right)^2 + \left(3 - \frac{6y}{x+y}\right)^2 = 82.$$

SOLUTION BY A. A. BENNETT, University of Texas.

It is assumed for various reasons that the second equation was intended in the form

$$\left(3 + \frac{6y}{x-y}\right)^2 + \left(3 - \frac{6y}{x+y}\right)^2 = 82,$$

[1] This supposes $\sin (\alpha - \beta)$ positive, and it would not essentially restrict the problem to assume that α and β are angles in the first quadrant, $\alpha > \beta$.

which makes the problem of interest. With this supposition, the pair of equations may be written as

$$xy = a,$$
$$\left(\frac{x+y}{x-y}\right)^2 + \left(\frac{x-y}{x+y}\right)^2 = 2b,$$

where $a = 2$, $b = 41/9$. It is then to be noted that if (r, s) be one solution, the following are the eight distinct solutions:

(r, s), (s, r), $(-r, -s)$, $(-s, -r)$, $(ir, -is)$, $(is, -ir)$, $(-ir, is)$, $(-is, ir)$.

Thus there is a group of eight operations carrying any solution into a solution. This remark applies for all finite non-vanishing values of a, b. Solutions can coincide in pairs only for a, or b, zero or infinite, or for $b = -1$. Since (1, 2) is by inspection a solution, the eight solutions may be written out at once.

By direct methods also the solution is readily obtained. Put $u = (x + y)/(x - y)$; whence

$$x^2 = a(u + 1)/(u - 1),$$
$$y^2 = a(u - 1)/(u + 1),$$

where the signs of x and y are to be so taken that $xy = a$. Then

$$u^2 + 1/u^2 = 2b, \qquad u = \frac{\pm\sqrt{b+1} \pm \sqrt{b-1}}{\sqrt{2}}.$$

In the given example $u = 3, -3, \frac{1}{3}$, or $-\frac{1}{3}$, $x^2 = 4, 1, -4, -1$, $y^2 = 1, 4, -1, -4$, and the solutions are

$(2, 1)$, $(-2, -1)$, $(1, 2)$, $(-1, -2)$, $(2i, -i)$, $(-2i, i)$, $(i, -2i)$, $(-i, 2i)$.

NOTE.—Miss Berta M. King remarks that if the sign in the second equation is changed as stated in Professor Bennett's solution, the problem is in Hawkes's *Advanced Algebra*. She gets the same results as Professor Bennett.

H. N. CARLETON, A. R. NAUER, J. W. SHUMAN and F. L. WILMER solved the problem as it is printed above.

2898 [1921, 228]. Proposed by J. W. CLAWSON, Ursinus College.

Four straight lines determine four triangles. It is well known that the circumcenters of these triangles lie on a circle and that the circumcircles intersect this circle in a point, called the Wallace point. It is also well known that the orthocenters of the four triangles lie in a straight line, which is perpendicular to the line on which lie the middle points of the three diagonals of the quadrilateral determined by the four given straight lines.

Prove that the centroids of the four triangles lie on a parabola whose axis is parallel to the mid-diagonal line; and that the distance from the Wallace point to the mid-diagonal line is two thirds of the distance from the Wallace point to the axis of the parabola.

SOLUTION BY THE PROPOSER.

In the statement of the problem "two thirds" should be "three halves."

Given the four lines, let the origin be at the Wallace point. The Wallace or pedal lines of this point with respect to the four triangles are coincident. Let the y-axis be parallel to this line, its equation being, say, $x = p$. Then the perpendicular from the origin on the four given lines will meet them in four points on this line, which we may call (p, a), (p, b), (p, c), (p, d), and the equations of the four given lines will be

$$px + ay = p^2 + a^2, \text{ etc.}$$

The intersections of these lines are the six points

$$p - \frac{ab}{p}, \; a + b, \text{ etc.,}$$

and the mid-points of the three diagonals are

$$p - \frac{ab + cd}{2p}, \; \tfrac{1}{2}(a + b + c + d), \text{ etc.,}$$

so that the equation of the mid-diagonal line is $y = \frac{1}{2}(a + b + c + d)$. Also the centroids of the four triangles are

$$p - \frac{ab + ac + bc}{3p}, \ \tfrac{2}{3}(a + b + c), \text{ etc.}$$

The equation of a parabola whose axis is praallel to the x-axis is of the form[1]

$$y^2 + Dx + Ey + F = 0,$$

and if we substitute the coördinates of any three of the centroids, evaluate D, E, F, and simplify, we obtain the equation

$$\left(y - \frac{a + b + c + d}{3}\right)^2 = -\frac{4p}{3}\left(x - p - \frac{a^2 + b^2 + c^2 + d^2}{12p} + \frac{ab + ac + ad + bc + bd + cd}{6p}\right),$$

symmetrical with respect to a, b, c, d.

The axis of this parabola is $y = \frac{1}{3}(a + b + c + d)$; therefore, the distance of the mid-diagonal line from O is three halves of the distance of this axis from O.

We may note that the four orthocenters are

$$2p, \ a + b + c + \frac{abc}{p^2}, \text{ etc.},$$

all lying on the line $x = 2p$, which is twice as far from o as the pedal line.

2791 [1919, 414; 1921, 143–145].

A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom, the wine drips into the water, and the mixture drips out at the same rate. When the wine cup is empty, what part of the contents of the lower cup is water? [Proposed by Charles Gilpin, Jr., Philadelphia, as Problem 287 in *The Mathematical Visitor*, January, 1881, volume 1, page 193. No solution was published in the *Visitor*.]

Remarks by R. E. Moritz, University of Washington.

Each of the solutions of the problem published in this Monthly (*1921*) proceeds on the assumption that the liquids flow from the cups, in other words, that the process by which the wine cup is emptied is a continuous process. This assumption seems to me unwarranted, being a flat contradiction of one of the conditions of the problem. The problem states explicitly that "the wine drips into the water, and the mixture drips out at the same rate." Dripping is essentially a discontinuous process, the solutions referred to must therefore be considered approximations rather than exact solutions.

Curiously enough, the exact solution is very much simpler than the approximations in question. Suppose that each cup contains at the outset n drops of equal size. After the first drop has fallen into the lower cup, the mixture consists of n drops of water and one drop of wine,[2] $n/(n + 1)$ of the contents of the lower cup is therefore water; after the second drop of wine has been added, the water content is $[n/(n + 1)]^2$; after the third drop has been added, $[n/(n + 1)]^3$; after the nth drop has been added, that is when the cup has been emptied, $[n/(n + 1)]^n$. It is clear then that the exact answer to the problem is a function of n, the number of drops in the wine cup at the outset.

The limit of $[n/(n + 1)]^n$ as n approaches ∞ is e^{-1}, which is the result obtained in the previous solutions. But n cannot be infinite, for if it were, the wine cup could not be emptied by a dripping process.

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[1] That the coördinates of the four centroids satisfy an equation of this type may be seen from the fact that if we substitute them all and eliminate D, E and F, we get a polynomial of degree 5 in a, b, c, d, divisible by the six differences, $a - b$, etc., and, therefore, zero—Editors.

[2] This assumes that the cup can hold $n + 1$ drops till they are mixed—Editors.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

VICTOR DANTSCHER VON KOLLESBERG, born at Innsbruck, Austria, October 29, 1847, died July 26, 1921. He studied at Innsbruck, Berlin, and Vienna where he received his doctorate in 1870. His connection with the University of Graz dates from 1879; he became ordinary professor of mathematics there in 1894. The number of his mathematical papers is not numerous; apart from three in the *Mathematische Annalen*, most of them appeared in Austrian publications. His eighty-page pamphlet, *Vorlesungen über die Weierstrasssche Theorie der irrationalen Zahlen*, appeared at Leipzig, in 1908.

JACOB ROSANES, born at Brody, Austria, August 16, 1842, died January 7, 1922. Receiving his doctorate at Breslau in 1865, he became extraordinary professor of mathematics there in 1873, and ordinary professor in 1870. Since 1911 he had been excused from delivery of lectures. Most of his mathematical papers were published in *Journal für reine und angewandte Mathematik* and *Mathematische Annalen*, and they contain some notable results in algebraic geometry; compare *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 24, 1915, page 78.

GUSTAV KOHN, born in Reichenan, Bohemia, April 22, 1859, died December 15, 1921. He studied at Brelin, Strasbourg, and Vienna where he received his doctor's degree in 1881. In 1884 he became Privatdozent at the University of Vienna, and in 1894 extraordinary professor of mathematics. Most of his mathematical papers were published in *Sitzungsberichte der mathematisch-naturwissenschaftlichen Klasse der kaiserl. Akademie der Wissenschaften*, Vienna, and *Monatshefte für Mathematik und Physik*. He is perhaps best known, however, for the part of the *Encyklopädie der mathematischen Wissenschaften*, III–2–4, 5 (pages 457–634, 1908–1914), on "Spezielle ebene algebraische Kurven," prepared in collaboration with Gino Loria.

HUGO FERDINAND BUCHHOLZ, born in Lübeck, Germany in 1866, died November 24, 1921. He was educated in Jena, Stockholm, Berlin, and Munich, where he received his doctorate in 1894. He was assistant at the observatory in the University of Göttingen 1897–1900 and was later appointed "Titular-professor," while dozent in astronomy and applied mathematics, at the University of Halle. His first book (Leipzig, 1908) was *Das mechanische Potential. Nach Vorlesungen von L. Boltzmann bearbeitet und die Theorie der Figur der Erde. Zur Einführung in die höhere Geodäsie* (*angewandte Mathematik*), Theil 1; no other part appeared. A new edition (1916), nearly doubled in size (38 + 820 pages), had the title: *Angewandte Mathematik. Das mechanische Potential und seine Anwendung zur Bestimmung der Figur der Erde* (*höhere Geodäsie*). It will be recalled also that he was the editor of the third and enlarged edition (Leipzig, 1912) of the monumental *Theoretische Astronomie* by W. Klinkerfues.

KARL HERMANN AMANDUS SCHWARZ died in Berlin November 29th, 1921. He was born at Hermsdorf on January 25th, 1843. He took his doctorate at

Berlin in 1864; taught at Halle 1867–1869; at Zürich 1869–1875; and became professor at Göttingen in 1875. In 1892 he succeeded Weierstrass at Berlin. In 1902 he was awarded an honorary degree of Doctor of Mathematics by the University of Christiania. His most notable work was in geometry in connection with the theory of minimal surfaces; but his name will be familiar to students of mathematics through the Schwarz inequality and the Schwarz-Christoffel mapping of a linear polygon on the axis of reals in the theory of functions of a complex variable. The fiftieth anniversary of his doctorate was celebrated at Berlin on August 6, 1914, and was commemorated by a notable volume edited by Carathéodory, Hessenberg, Landau and Lichtenstein. His collected papers were published in two volumes in 1890 (Berlin).

MAX NÖTHER died at Erlangen on December 13, 1921. He was born at Mannheim on September 24, 1844. He attended the Universities of Heidelberg, Giessen and Göttingen; took his doctorate at Heidelberg in 1868; taught at Heidelberg 1874–1875 and went to Erlangen in 1875 where he was made professor in 1888. He made notable contributions to the geometry of curves and surfaces and the connected theory of algebraic functions; his name has been given to the fundamental theorem that every plane algebraic curve can be transformed birationally into one whose only singularities are double points with distinct tangents. In 1882 he shared with Halphen the Steiner prize of the Berlin Academy. He read a paper at the Chicago Mathematical Congress (1893). A detailed biography can be found in *Mathematische Annalen* (of which he was an editor), volume 85.

LEO KÖNIGSBERGER died at Heidelberg on December 15, 1921. He was born at Posen on October 15, 1837. He took his doctorate at Berlin in 1860; taught at Greifswald 1864–1866; was professor there 1866–1869; then at Heidelberg 1869–1875, Dresden Polytechnic 1875–1877, Vienna 1877–1884. Since 1884 he had been professor at Heidelberg. He published texts on ordinary differential equations (in which theory his best work was done) and the principles of mechanics. A paper by him appeared in *American Journal of Mathematics,* volume 11 (1889). A detailed biography and a photograph may be found in the index to volumes 1–35 of *Acta Mathematica.*

PAUL RUDOLPH EUGEN JAHNKE died at Berlin October 18, 1921. He was born at Berlin on November 30, 1863. He studied at Berlin 1881–1885 and took his doctorate at Halle 1889. He became Kneser's successor, in 1905, as professor of higher mathematics at the Bergakademie, Berlin. When this was taken over by the Technische Hochschule, Charlottenburg in 1916, he became a professor in the latter institution. His work was mainly in elliptic function theory; a short note by him appeared in the *American Journal of Mathematics,* volume 21 (1899). His name is familiar to workers in mathematical physics through the excellent book, Jahnke und Emde, *Funktionentafeln mit Formeln und Kurven* (Leipzig, 1909).

GEORGE BALLARD MATHEWS died March 19, 1922. He was born at London on February 23, 1861. He studied for a year under Henrici at Uni-

versity College, London, and entering St. John's College, Cambridge, took first place in the Tripos of 1883. In 1884 he was made professor in the newly created University of North Wales at Bangor. In 1897 he was elected a Fellow of the Royal Society and appointed University Lecturer in Mathematics at Cambridge. He returned to Bangor in 1906 where he taught till 1919, the remaining years of his life being clouded by ill health. He received the honorary degree of LL.D. from the University of Glasgow in 1915. His principal contributions to mathematics were in the theory of numbers and projective geometry. He published a *Theory of Numbers* (Cambridge, 1892); a *Treatise on Bessel Functions* (in collaboration with Professor Andrew Gray, London, 1895); *Algebraic Equations* (Cambridge Tracts, 1907); *Projective Geometry* (London, 1914), and also re-edited Scott's *Determinants* (Cambridge, 1904). He contributed frequent reviews and articles to *Nature* (London) and more detailed biographies may be found in that journal, volume 109, page 450, and in *Proceedings of the Royal Society of London*, September, 1922. In the latter the biographer comments on his powers of controversy in Latin elegiacs, his notable translations of Arabic poetry, and his extensive knowledge of music.

James Henry Cotterill died on January 8, 1922. He was born at Norfolk, England, on November 2, 1836. In 1866 he was appointed lecturer and in 1870 vice-principal of the Royal School of Naval Architecture and Marine Engineering at South Kensington. In 1873 this school was moved to Greenwich and became part of the Royal Naval College, in which Professor Cotterill was professor of Applied Mathematics until his retirement in 1897. He was elected honorary vice-president of the Institution of Naval Architects in 1905. In 1878 he published a treatise on *The Steam Engine* (London) and in 1884 his *Applied Mechanics* (London). These have passed through several editions and have been widely used in the engineering schools of this country.

Published November 7, 1922.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

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BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December 28, 29, 1922

The following are dates of Section meetings of the Association in 1922 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29
IOWA, Des Moines, November 3; Cornell College, Mount Vernon, April 27-28, 1923
KANSAS, Topeka, January 21
KENTUCKY, Georgetown College, April 8; University of Kentucky, April, 1923
MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Annapolis, May 13; Washington, December
MINNESOTA, St. Paul, June 4, 1921; St. Paul, May 27
MISSOURI, St. Louis, November 25–26, 1921; Kansas City Junior College, November 18
OHIO, Columbus, March 25–26, 1921; Columbus, April 14–15; March, 1923
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SOUTHEASTERN, Atlanta, Ga., April 29; February, 1923
TEXAS, Dallas, November 25, 1921; Houston, December 1–2

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GENEVA. Durfee.
GOVERNOR'S ISLAND. Gronwall.
HAMILTON. DoBell, Fredericksen, A. W. Smith.
ITHACA. Boothroyd, A. D. Campbell, Carver, Gillespie, Hurwitz, D. S. Morse, H. M, Morse, Osborne, F. W. Owens, H. B. Owens, Poritsky, Ranum, Reed, Robison, V. Snyder, Tanner, Vandiver.
MOUNT VERNON. Breckenridge.
NEW YORK. J. Allen, Auerbach, Autenreith, Blair, Brewster, G. A. Campbell, Coffin, C. H. Douglas, Eckersley, Edmonson, Fiske, Fite, R. M. Foster, Frankel, Fry, Hawkes, Henderson, Himwich, Hirsch, Hodgdon, Hoppe, Joffe, Kasner, Langellotti, Langman, Latham, Lehmann, Linehan, Merriman, H. B. Mitchell, Molina, Mullins, Paaswell, Pedersen, Penn, Pooler, Reddick, Requa, F. G. Reynolds, Saurel, Schmall, Schorling, Schub, Siceloff, Simons, D. E. Smith, R. F. Smith, R. R. Smith, H. Thompson, Thorne, Tienzo, Upton, Waldo, E. Walker, L. M. Webster, Wechsler, E. E. Whitford, W. O. Wiley.
OSSINING. Mirick.
PLEASANTVILLE. B. G. Westfall.
POUGHKEEPSIE. Cowley, Cummings, M. E. Wells.
ROCHESTER. Betz, Gale, Hall, Harding, Long, Silberstein, Watkeys, D. E. Whitford.
SCHENECTADY. Newkirk, A. D. Snyder, Vedder.
SYRACUSE. W. G. Bullard, Decker, Lindsey, Secy. Pi Mu Epsilon Frat., Roe, M. J. Sperry, W. E. Taylor.
TROY. Lundin.
TUCKAHOE. Mac Neish.
WEBSTER. Becker.
WEST POINT. C. P. Echols.
WHITE PLAINS. T. H. Brown.
WOODMERE. C. B. Walsh.
YONKERS. Hubert.

NORTH CAROLINA. (18)

CHAPEL HILL. Browne, W. Cain, A. Henderson, A. W. Hobbs, Lasley.
DAVIDSON. J. L. Douglas.
DURHAM. M. R. Richardson.
ELON COLLEGE. Amick.
GREENSBORO. G. W. Mendenhall, F. S. Mitchell, Pegram.
GREENVILLE. M. D. Graham.
GUILFORD COLLEGE. Newlin, Pancoast.
HICKORY. Fritz.
JAMESTOWN. Ragsdale.
MURFREESBORO. Caldwell.
WILMINGTON. H. B. Smith.

NORTH DAKOTA. (5)

FARGO. Householder, I. W. Smith.
JAMESTOWN. T. W. Jackson.
UNIVERSITY. Hitchcock.
VALLEY CITY. Meyer.

OHIO. (78)

ADA. W. S. Beckwith, Fairchild.

ALLIANCE. Trott.
ASHLAND. Haun.
ATHENS. Borger.
BEREA. Dustheimer.
BOWLING GREEN. Overman.
BLUFFTON. Hirschler.
CINCINNATI. Brand, Hancock, Kindle, C. N. Moore, Salkover, E. S. Smith, Yowell.
CLEVELAND. Focke, Freas, W. W. Johnson, M. Morris, Palmié Pitcher, Simon, C. F. Thomas, D. T. Wilson.
COLUMBUS. C. L. Arnold, Bareis, Bohannan, V. B. Caris, Kuhn, Manson, C. C. Morris, A. F. Preston, J. B. Preston, Rasor, Rickard, Singer, Wildermuth.
DAYTON. Hoffmann.
DEFIANCE. A. G. Caris, Loomis.
DELAWARE. Armstrong, Austin.
GAMBIER. R. B. Allen.
GRANVILLE. Lemon, Peckham, Sheets, F. B. Wiley.
HILLIARD. J. H. Weaver.
HIRAM. E. H. Clarke.
MARIETTA. Coar, M. C. Horn, P. L. Rea.
MONROE. McCoy.
NEW CONCORD. C. E. White.
OBERLIN. Cairns, Carr, Sinclair, Yeaton.
OXFORD. W. E. Anderson, Baudin, M. B. Carter, Lange, Pepper, Schoonmaker, Spenceley.
PAINESVILLE. A. D. Lewis.
PAYNE. Romig.
PUT-IN-BAY. Cottingham.
ROSS. Haldeman.
SPRINGFIELD. Tripp.
TIFFIN. C. N. Mills.
TOLEDO. Brandeberry.
WESTERVILLE. Glover.
WILBERFORCE. B. Sanders.
WILMINGTON. Spinks.
WOOSTER. Knight, Williamson.
YELLOW SPRINGS. Nordgaard.

OKLAHOMA. (11)

CHICKASHA. J. J. Miller.
NORMAN. Altshiller-Court, K. C. Barbour, Hassler, McFarland, Reaves, F. L. Smith.
OKLAHOMA CITY. Cornell.
SHAWNEE. W. T. Short.
STILLWATER. Gundersen.
WEATHERFORD. McCormick.

OREGON. (9)

ALBANY. Moore.
ASTORIA. H. M. Manning.
CORVALLIS. Beaty, C. L. Johnson.
EUGENE. DeCou, Milne.
PORTLAND. Griffin, Merriss, Ruby.

PENNSYLVANIA. (70)

ALLENTOWN. Bauman.
BEAVER FALLS. Colwell.
BETHLEHEM. J. B. Reynolds.
BRYN MAWR. Pell, C. Scott.
CAMP HILL. Foberg.
CARLISLE. Landis.
CHESTER. Church.
COLLEGEVILLE. Clawson.
CYNWYD. Sensenig.
DALLAS. Templin.
DEVON. J. A. Clarke.
EASTON. W. S. Hall, W. M. Smith.
GERMANTOWN. Mullikin.
GETTYSBURG. Granville.
GLEN MILLS. Gummere.
GROVE CITY. Ramsey.
HARRISBURG. Whited.
HAVERFORD. Reid, A. H. Wilson.
LANCASTER. Charles.
LANSDOWNE. Chambers, Glenn.
LARIMER. A. A. Jones.
LEWISBURG. Bartol, Everett, Gold, Lowry.
LINCOLN UNIVERSITY. W. L. Wright.
LOCK HAVEN. High.
MEADVILLE. Akers.
MYERSTOWN. Kiess.
NEW WILMINGTON. McCain.
PHILADELPHIA. Burley, Crawley, Eshleman, H. B. Evans, Gehman, Kline, Linton, Partridge, Rittenhouse, Safford.
PITTSBURGH. Baird, Barrett, Bishop, Bland, Foraker, R. P. Johnson, J. H. Mathews, Riggs, Rosenbach, Swartzel, Taber, Waltz, Webber.
POTTSTOWN. Arms.
SOUTH BETHLEHEM. Lambert, MacNutt.
STATE COLLEGE. Gravatt, L. S. Johnston, Shibli.
SWARTHMORE. Marriott, J. A. Miller.
UNIONTOWN. M. Buchanan.
WASHINGTON. Atchison, Bert, R. W. Thomas.
WEST PHILADELPHIA. Latshaw.

PHILIPPINE ISLANDS. (5)

MANILA. V. Mills, Ott, Rafferty, Schick, H. L. Smith.

PORTO RICO. (1)

MAYAGUEZ. C. E. Horne.

RHODE ISLAND. (11)

EDEN PARK. C. R. Adams.
PROVIDENCE. Archibald, R. W. Burgess, Burwell, Carlen, Chace, Currier, Gilman, Hicks, H. P. Manning, R. G. D. Richardson.

South Carolina. (7)

Charleston. O. J. Bond.
Columbia. Coleman, J. B. Jackson.
Greenville. Earle, R. B. Wood.
Hartsville. C. M. Reaves.
Saluda. Ramage.

South Dakota. (4)

Brookings. I. L. Miller.
Rapid City. McLaury.
Vermilion. McKinney.
Yankton. Faught.

Tennessee. (8)

Chattanooga. Hooper.
Clarksville. Higgins.
Knoxville. J. D. Bond, Brezler.
Maryville. Knapp.
Nashville. S. I. Jones, Schuerman.
Sewanee. S. M. Barton.

Texas. (52)

Abilene. Burnam.
Arlington. Baten.
Austin. Batchelder, H. Y. Benedict, A. A. Bennett, Calhoun, Cleveland, Cooper, Decherd, Ettlinger, Hammer, Holmes, Horton, Jacobs, Michie, R. L. Moore, I. I. Nelson, M. B. Porter, Rockwell, R. L. Wilder.
Boerne. Hathaway.
Brownsville. de la Garza.
Canyon. L. G. Allen.
College Station. Cox, P. K. Smith.
Commerce. Cowling.
Dallas. Dice, E. H. Jones, Mahoney.
Fort Worth. Hargett, E. R. Tucker.
Galveston. Burrell, P. H. Underwood.
Georgetown. Wunder.
Houston. Daniell, Dean, G. C. Evans, L. R. Ford, Lovett, Michal.
Palestine. Eason.
Port Arthur. G. S. Smith.
San Angelo. Hagelstein.
San Antonio. B. R. Allen, Roach.
San Marcos. J. S. Brown, Sewell.
Sherman. P. F. Henry.
Stephenville. J. L. Riley.
Waco. W. A. Nelson, P. C. Porter.
Wichita Falls. H. Porter.

Utah. (5)

Logan. Saxer.
Salt Lake City. J. L. Gibson, Horsfall, Pehrson, Unseld.

Vermont. (5)

Burlington. Swift, E. Thomas.
Essex Junction. Donahue.
Middlebury. Bonney, Perkins.

Virginia. (27)

Abingdon. V. L. Wright.
Ashland. T. McN. Simpson.
Blacksburg. Brodie, Gudheim, O'Shaughnessy, J. E. Williams.
Bridgewater. Shull.
Bristol. Holtwick
Charlottesville. F. A. Wells.
Clifton Station. O. Stone.
Emory. J. S. Miller.
Hollins. Dickinson, Plapp.
Lexington. L. W. Smith, C. W. Watts.
Lynchburg. Larew, Shackelford, E. V. Watts.
Monterey. Colaw.
Richmond College. Gaines.
Salem. Carpenter.
South Hill. Walton.
Sweet Briar. Morenus.
University. W. H. Echols, Luck.
Williamsburg. Rowe.
Woodberry Forest. W. L. Lord.

Washington. (13)

Aberdeen. V. Young.
Everett. Robb.
Pullman. Freeman, Isaacs.
Seattle. E. T. Bell, Moritz, Neikirk, Stager.
Spokane. Hess.
St. John. Hays.
Tacoma. Hanawalt.
Walla Walla. Bratton, Eells.

West Virginia. (7)

Huntington. Hackney.
Morgantown. H. A. Davis, Eiesland, Harkins, Hodgson, C. N. Reynolds, Torrey.

Wisconsin. (33)

Beloit. H. H. Conwell, W. A. Hamilton, Haynes.
Madison. F. E. Allen, Chapman, H. T. Davis, Dowling, Dresden, Fowlkes, W. W. Hart, Lane, E. B. Miller, Rechard, Skinner, Slichter, Van Vleck, W. Weaver.
Mercer. Kendrigan.
Milton. A. E. Whitford.
Milwaukee. Atwater, F. A. Bixby, Ericson, P. H. Evans, Frumveller, L. E. McCarty, C. G. Simpson, Thayer.
Platteville. Warner.

RIPON. Woodmansee.
RIVER FALLS. McMillan.
SINSINAWA. Sr. Dobbin.
SOUTH MILWAUKEE. Hoar.
SUPERIOR. C. W. Smith.

WYOMING. (6)

CASPER. Counts.
LARAMIE. Bellamy, Feddersen, Fehn, Fitterer, Stromquist.

FOREIGN MEMBERS. (Other than Canada.)

ARGENTINE. (2)

BUENOS AIRES. Baidaff, Broggi.

BELGIUM. (1)

LIEGE. Van Hee.

CHINA. (8)

CHANGSHA. Leavens.
FOOCHOW. Yen.
PEKING. Chang Shen-Fu, Heinz, Hsia, Konantz.
SHANGHAI. Ely.
TANGSHAN. Patten.

ENGLAND. (8)

CAMBRIDGE. Ball, Richmond, P. W. Wood.
HOVE. Chepmell.
LONDON. Forsyth, Greenhill.
OXFORD. Hardy.
PETERSFIELD. Zilliacus.

FRANCE. (5)

BESANÇON. Lebeuf.
NANCY. Gérardin.
PARIS. Borel, Hadamard.
STRASBOURG. Fréchet.

INDIA. (4)

ALLAHABAD CITY. Mitra.
CALCUTTA. Bose.
DHARWAR. Saldanha.
VEDARANIAM. Ramana-Sastrin.

ITALY. (6)

BOLOGNA. Bortolotti, Enriques, Pincherle.
CATANIA. Cipolla.
PISA. Bianchi.
TURIN. Fubini.

JAPAN. (5)

PYENGYANG. Parker.
KOREA. Andrew.
SAITAMA-KEN. Sekiyama.
TOKYO. Mikami, Ono.

NEW ZEALAND. (1)

DUNEDIN. Martyn.

POLAND. (1)

WARSAW. Dickstein.

PORTUGAL. (1)

LISBON. da Cunha.

SCOTLAND. (1)

EDINBURGH. Horsburgh.

SOUTH AFRICA. (3)

BLOEMFONTEIN. Arndt.
JOHANNESBURG. Dalton.
RONDEBOSCH. Muir.

SPAIN. (2)

MADRID. de Toledo.
ZARAGOZA. de Galdeano.

SWITZERLAND. (3)

FRIBOURG. Bays.
GENEVA. Fehr.
NEUCHATEL. DuPasquier.

SYRIA. (1)

BEIRUT. Jurdak.

TURKEY. (2)

CONSTANTINOPLE. K. S. Arnold, Mourad.

RECAPITULATION OF MEMBERSHIP.

Individual members November 1, 1922	1,382	
Institutional members November 1, 1922	94	
Total membership November 1, 1922		1,476
Total membership October 1, 1919		1,187

CHARTER MEMBERSHIP.

Individual charter members	1,045	
Institutional charter members	52	
Total charter membership		1,097
Net gain in individual members	337	
Net gain in institutional members	42	
Total net gain over charter membership		379
Total net gain since October 1, 1919		289

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED).

(As amended and adopted by unanimous vote at a special meeting of members called for the purpose at Rochester, N. Y., September 7, 1922, a quorum being present.)

Article I—Name, Purpose and Corporate Seal.

1. This organization shall be known as

The Mathematical Association of America (Incorporated).

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

Article II—Membership.

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Any institution in which the Calculus is regularly taught shall be eligible for election to institutional membership in the Association. Such an institution shall have the privilege of sending a voting delegate to the meetings of the Association.

3. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission, endorsed in the case of individuals by two members of the Association.

4. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

Article III—Board of Trustees and Officers.

1. The Officers of the Association shall be a President, two (2) Vice-Presidents, a Secretary-Treasurer, a Librarian and three (3) members of a Committee on Official Journal.

2. The control and management of the affairs and funds of the Association shall be vested in a Board of twenty (20) Trustees (hereinafter called the "Board"), who shall be members of the Association. This Board shall consist of the officers of the Association and twelve (12) additional members.

3. The President and Vice-Presidents shall be elected by the Association's members annually for a term of one year, and four members of the Board shall be elected by the Association's members annually for a term of three years. They shall be eligible for reëlection, but not for more than two (2) consecutive terms. The Secretary-Treasurer, the Librarian, and the Committee on Official Journal, consisting of the Editor-in-Chief, the Manager and one other member, shall be appointed by the Board. All Officers and other Trustees shall hold over until their respective successors are elected or appointed and qualify.

4. The Board shall transact the official business of the Association and shall report its actions at the annual business meeting of the Association and in the official journal. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board and in the Committee on Official Journal, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Trustees a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be

adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. Approximately two months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Board shall announce two candidates for each office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

8. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Trustees and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Trustees.

9. The Vice-Presidents shall, in the absence of the President, have and exercise the powers of the President, their order being determined alphabetically. The Board of Trustees may assign to the Vice-Presidents such duties as may from time to time be determined.

10. The Secretary-Treasurer shall have the usual duties pertaining to the office of Secretary and of Treasurer, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Trustees and of the annual meeting and special meetings of members, and giving of due notice of all regular and special meetings of the Association and of the Board of Trustees and the supervision and safe-keeping of the funds of the Association. The Secretary-Treasurer shall also have the duty of seeing that whenever Trustees are elected, including the election of Trustees to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificate shall be signed by the Secretary-Treasurer and verified by oath of the President.

11. The Committee on Official Journal shall have supervision of the official journal subject to the control of the Board of Trustees.

12. The Librarian shall have general charge of the library of the Association and shall direct its affairs, including the exchange of the publications of the Association, subject to the control of the Board.

Article IV—Meetings.

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The outgoing Board shall hold a meeting immediately preceding the annual meeting of the Association next succeeding their election, and the members of the new Board shall hold a meeting and organize, by completing the Board, immediately succeeding the annual meeting of the Association at which the new members thereof were elected. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary-Treasurer at least thirty (30) days prior to the date set for such meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections.

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings.

The Board shall have power to specify the conditions under which such authority shall be granted.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections.

ARTICLE VI—OFFICIAL PUBLICATIONS.

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. The official journal shall be under the general management of the Committee on Official Journal. There shall also be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal and under the direction of the Committee on Official Journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

ARTICLE VII—DUES.

1. Individual members of the Association shall pay an initiation fee of Two Dollars ($2) at the time of election.

2. The annual dues of each individual member shall be Four Dollars ($4), including a subscription to the official journal.

3. The annual dues of each institutional member shall be Seven Dollars ($7), including two (2) subscriptions to the official journal.

4. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list, after due notice.

5. New members entering the Association after April 1 of any year shall have their dues pro rated for the balance of the year, except when they desire to receive the full current volume of the official journal.

6. The life membership fee shall be the present value, according to McClintock's Male Annuitant Table based upon four (4) per cent. interest, of an immediate annuity of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the McClintock Male Four (4) Per Cent. Table; and the reserve thus computed shall be held as a liability.

ARTICLE VIII—AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BY-LAWS.

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (⅔) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary-Treasurer, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

AMENDMENT TO THE ARTICLES OF ASSOCIATION.

(This amendment was adopted by unanimous vote at a special meeting
of members called for the purpose at Rochester, N. Y.,
September 7, 1922, a quorum being present.)

Section 3 of the Articles of Association of The Mathematical Association of America (Incorporated) as amended on September 10, 1920, shall be further amended to read as follows: "The management of the aforesaid Association shall be vested in a Board of twenty (20) Trustees."

PERIODS OF SERVICE OF THE OFFICERS OF THE ASSOCIATION.

Presidents.

E. R. Hedrick	1916
Florian Cajori	1917
E. V. Huntington	1918
H. E. Slaught	1919
D. E. Smith	1920
G. A. Miller	1921
R. C. Archibald	1922

Vice-Presidents.

E. V. Huntington	1916
G. A. Miller	1916
D. N. Lehmer	1917, 1918
Oswald Veblen	1917
J. W. Young	1918
R. G. D. Richardson	1919
H. L. Rietz	1919
Helen A. Merrill	1920
E. J. Wilczynski	1920
R. C. Archibald	1921
R. D. Carmichael	1921, 1922
B. F. Finkel	1922

Secretary-Treasurer.

(Appointed by the Council or Board after 1918.)

W. D. Cairns	1916–

Committee on Publications.

(Appointed by the Council or Board.)

H. E. Slaught	1916–
R. D. Carmichael	1916–1918
W. H. Bussey	1916–1918
R. C. Archibald	1919–1921
W. A. Hurwitz	1919–1921
A. A. Bennett	1922–
H. P. Manning	1922–

Elected Members of the Council or Board.

D. N. Lehmer	1916–1918, 1922–
R. E. Moritz	1916–1918
K. D. Swartzel	1916
Oswald Veblen	1916, 1920–
R. C. Archibald	1916–1917
Florian Cajori	1916, 1918–
M. B. Porter	1916–1917
J. W. Young	1916–1917, 1920–
B. F. Finkel	1916–1921
E. H. Moore	1916–1921
J. N. Van der Vries	1916–1918
Alexander Ziwet	1916–1918
E. R. Hedrick	1917–1922
Helen A. Merrill	1917–1919
D. E. Smith	1917–1919, 1921–
Elizabeth B. Cowley	1918–1920
G. A. Miller	1918–1920, 1922–
E. J. Wilczynski	1918–1919, 1922–
L. P. Eisenhart	1919–
E. V. Huntington	1917, 1919–
E. L. Dodd	1920
R. D. Carmichael	1920
A. A. Bennett	1921
H. L. Rietz	1921–
C. F. Gummer	1921–

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the..

....Fund, and to be used[2]

for { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: Signature..

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

// THE AMERICAN MATHEMATICAL MONTHLY

OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

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THE APRIL MEETING OF THE ILLINOIS SECTION.

The third annual meeting of the Illinois Section of the Mathematical Association of America was held at Rockford College, Rockford, Illinois, on April 28 and 29, 1922, in conjunction with the Illinois State Academy of Science. There were three sessions; on Friday afternoon Chairman Comstock presided, on Friday night at a joint meeting of the Academy and the Illinois Section President Knipp of the Academy presided, and at the Saturday morning session Professor Miller presided.

There were forty-two in attendance, including the following nineteen members of the Association:

C. E. Comstock, M. W. Coultrap, A. Emch, R. M. Ginnings, W. A. Hamilton, E. S. Haynes, J. M. Kinney, E. B. Lytle, W. D. MacMillan, L. E. Mensenkamp, Bessie I. Miller, E. J. Moulton, Mary W. Newson, C. I. Palmer, S. F. Parson, I. Roman, L. S. Shively, H. E. Slaught, and G. E. Wahlin.

The following officers were elected: G. T. SELLEW, chairman; C. E. COMSTOCK, vice-chairman; G. H. SCOTT, secretary-treasurer. It was suggested by unanimous vote that the next meeting of the Illinois Section be held again with the annual meeting of the Illinois State Academy of Science but the power to fix the time and place of the next meeting was given to the executive committee.

A committee consisting of E. J. MOULTON, chairman, C. E. COMSTOCK and E. B. LYTLE was appointed to report at the next meeting on the best courses in mathematics for college freshmen consistent with present standards of high school preparation.

The Illinois Section and the State Academy were entertained at a cafeteria dinner Thursday evening at the Rockford High School and at a banquet on Friday evening at Rockford College.

The following papers were presented:

(1) "Constructive methods in geometry" by Professor ARNOLD EMCH;

(2) "Some aspects of correlation theory" by Mr. L. E. MENSENKAMP;

(3) "Romance in science. An experimental course offered by a mathematics department" by Professor BESSIE I. MILLER;

(4) "Consistency in grading mathematics papers" by Professor E. J. MOULTON;

(5)[1] An illustrated address on "Cosmogony" by Professor W. D. MACMILLAN;

Discussion of the National Committee's Report on College Entrance Requirements.

(6) Professor G. E. WAHLIN;

(7) Professor S. F. PARSON;

(8) Dr. J. M. KINNEY;

Discussion of how many and what mathematics courses should be offered to college freshmen.

(9) Professor E. B. LYTLE;

[1] This paper was presented at the joint session on Friday evening.

(10) Professor R. M. GINNINGS;

(11) Professor M. W. COULTRAP.

Professor MacMillan's paper is to be published in *Scientia*, Milan, Italy. Abstracts follow of the first four papers and of numbers 8 and 10:

1. In his paper Professor Emch emphasized the importance of the constructive side of geometrical instruction in elementary as well as advanced courses. Most of the geometrical teaching, with a few commendable exceptions, is conspicuously deficient in this respect. This neglect of the constructive phase of geometry is, in most cases, not intended by the teacher; it is due to the fact that the teacher of the teacher was deficient himself on constructive methods. Professor Emch showed by some typical cases how the teaching of geometry can be made more effective and interesting. In solid analytic geometry the study of the hyperboloid of revolution of one sheet, for example, usually consists in the discussions of the plane sections parallel to the coördinate planes (including these). Very little attention is paid to the geometric organism of the surface. In the first place this surface may be generated kinematically, *i.e.*, by a generating line rotating about a fixed non-intersecting axis, relatively unchanged in its relation to the axis. From this generating principle the equation of the surface is easily obtained. Then the existence of another set of generators follows immediately. Finally a very effective graphical representation of the surface is obtained by means of an isometric projection. In this projection the principle of affinity is incidentally introduced. In this manner an exhaustive elementary study of a simple surface at once reveals a number of geometrical principles of fundamental importance. As another example, Professor Emch discussed the effective construction of an important class of plane sextics, a construction so simple that ordinary college mathematics is all that is required for its understanding.[1]

2. After briefly developing the theory of the measurement of correlation by means of the correlation coefficient and the correlation ratio, Mr. Mensenkamp proceeded to a discussion of the formulæ for the probable errors of these quantities. He dwelt at some length on the assumptions underlying their derivation and pointed out the consequent limitations to which they are subject. His paper will be published in full in a forthcoming issue of *The Mathematics Teacher*.

3. In Professor Miller's paper a new course was described as to its origin, organization, methods, and results. It is a two-hour semester course open to freshmen and aims to introduce them to scientific literature such as Poincaré's "Science and Hypothesis," Keyser's essays, etc., to such modern theories as the Einstein theory, and to some of the most recent mathematical developments and applications.

4. Professor Moulton reported on the consistency in grading mathematics papers shown in an experiment at Northwestern University two years ago. Fifty papers on a five-question test in the theory of exponents were graded independently and in their usual manner by eight members of the mathematics

[1] A great number of other examples of this sort might be produced which would add further strength to the contention that constructive methods in geometry are of great heuristic value, and that their importance should not be underestimated.

department. The average grade for the fifty papers by the eight men was 61.4 per cent.; six of the eight instructors had average grades within 2 per cent. of this, the mean deviation for all eight being 2.1 per cent. A detailed study of the grades indicates, insofar as these grades are typical of mathematics grades in general, that if a number of instructors grade a number of papers: (*a*) The mean deviation on grades below 60 per cent. is about twice as great as on grades above 80 per cent. (*b*) The mean deviation of grades on definitions is nearly twice as great as of grades on formal algebraic work. (*c*) For good or excellent papers the mean deviation of grades on a paper containing answers to five questions is about 3 per cent., and the probability is 5 to 1 that for such a paper a random grade lies within 6 per cent. of the average. If the paper contains answers to twenty-five instead of five questions (perhaps a semester grade), this 6 per cent. is reduced to 4 per cent. (*d*) Instructors have a tendency to grade habitually high (or low), and a personal equation can be found which reduces their grades to normalcy. While the experiment may not be conclusive concerning the consistency of a group of instructors, it is pretty conclusive in testing the consistency of each with himself.

8. Dr. Kinney developed the following points: Instruction in secondary mathematics in the United States is characterized by an extreme formalism. This condition probably arises from the fact that the college mathematics of earlier times has been shoved down into the high school. There has been some improvement, as viewed from the pedagogical standpoint, during the past hundred years in the presentation of the subject matter. In geometry originals have been introduced and in algebra the so-called verbal problems. However, many of the originals have little value and many of the verbal problems do not raise legitimate mathematical questions. The writer believes that a great improvement can be made in secondary mathematical instruction by following the recommendations of the National Committee in regard to college entrance requirements.

10. Professor Ginnings discussed the effects of the present crusade against high school mathematics and the lowering of college entrance requirements in this subject. Under present conditions in Illinois, freshmen students offering two semesters of high school algebra should take a five-hour course, including algebra, the first semester and a two- or three-hour course in trigonometry the second semester. If three semesters of high school algebra is offered, then algebra and trigonometry should be offered the first semester and analytic geometry the second. Mathematics organizations should endeavor (*a*) to settle and get into high schools the best mathematics as a preparation both for college and for life (the two aims are not inconsistent); (*b*) to get a high school certificating law making it necessary for teachers to have special and ample preparation to teach the subjects they are actually allowed to teach. Future mathematical history will probably refer to the period immediately ahead of us as the "Renaissance period of the basal forms of mathematics in America."

E. B. LYTLE, *Secretary-Treasurer.*

THE MAY MEETING OF THE MINNESOTA SECTION.

The regular meeting of the Minnesota Section was held at Macalester College, St. Paul, on Saturday, May 27th. There were thirty in attendance, including the following fifteen members of the association:

W. O. Beal, W. E. Brooke, W. H. Bussey, H. H. Dalaker, Gladys E. C. Gibbens, W. L. Hart, C. A. Herrick, Dunham Jackson, R. A. Johnson, W. H. Kirchner, A. Reuterdahl, F. J. Taylor, Ella A. M. Thorp, A. L. Underhill, H. B. Wilcox.

Dinner was served at 6:30 P.M. at the Women's Dormitory of Macalester College. At the business meeting, which came after the dinner, the following officers were elected: Chairman, Professor V. H. WELLS, Carleton College; Secretary-Treasurer, Professor R. W. BRINK, University of Minnesota; Members of the executive committee, Professor W. H. KIRCHNER, University of Minnesota; L. E. LUNN, Superintendent of Schools, Heron Lake, Minnesota. At the afternoon session the following papers were read and discussed:

(1) "Single parameter mixed groups of polar fields" by Professor V. H. WELLS;

(2) "A short method of division" by Mr. SOLOMON GUTTMAN, Minneapolis (by invitation);

(3) "Logarithmic tables of the 17th and 18th centuries" by Mr. W. D. MORGAN, St. Paul (by invitation);

(4) "An analytic geometry treatment of the nature of conics generated by projective ranges and pencils" by Miss ELIZABETH CARLSON, University of Minnesota (by invitation);

(5) "Nomograms" by Professor R. W. BRINK;

(6) "The Finite and Infinite, Space and Time" by Professor ARVID REUTERDAHL.

Abstracts of the papers are given below, the numbers corresponding to the numbers in the list of titles:

1. Professor Wells gave a development of three single parameter systems of polar fields which lie between the pencil and the range of polar fields. The polars of a point envelop a conic or a fourth class curve and the poles of a line lie on a fourth order curve or on a conic.

2. Mr. Guttman presented a new arrangement of arithmetical long division in which the usual subtractions are turned into additions. This is done by making use of the arithmetical complement, defined as $(10^n - D)$, where D is the divisor and n the number of digits in the divisor. The chief advantage of the method consists in a check for accuracy which depends on the identity $A + Q(10^n - D) = Q \cdot 10^n + R$, A being the dividend, Q the quotient and R the remainder. *Example:* The compact form of the division of 48375 by 57, by this method, would be as follows:

$$\begin{array}{r|l} 57 & 48375 \\ 43 & 344 \\ & \;\;172 \\ & \;\;\;\;344 \\ \cline{2-2} & 84839 \end{array}$$

The quotient is 848 and the remainder is 39.

3. Mr. Morgan's paper concerned the history and construction of five of the earlier logarithmic tables, namely, Henry Briggs, *Arithmetica Logarithmica,* London, 1624; Adrian Vlacq, *Arithmetica Logarithmica,* Goudae, 1628; Briggs-Gellibrand, *Trigonometria Britannica,* Goudae, 1633;. Dodson, *Anti-Logarithmic Canon,* London, 1742, and Vega, *Thesaurus Logarithmorum,* Leipzig, 1794. Copies of each of these works which are now all entirely out of print and very scarce were exhibited at the meeting. The *Arithmetica Logarithmica* of Briggs and the *Trigonometria Britannica* are the only fourteen place tables ever published. The *Anti-Logarithmic Canon* is an eleven place table of anti-logarithms, the most extensive ever constructed. The *Arithmetica Logarithmica* of Vlacq and the *Thesaurus* of Vega are both ten place tables. The latter gives also the Wolframmi table of Naperian logarithms to forty-eight decimal places.

4. The nature of the conic generated by projective ranges and pencils, as determined by the projective relation involved in its generation, has been obtained by Steiner by the use of synthetic methods. In this paper Miss Carlson obtained the same and other results by the use of analytic geometry methods.

5. In his paper Professor Brink explained the purpose of nomograms and the method of using them. He exhibited nomograms for the solution of right triangles and other problems, and indicated their usefulness in teaching college mathematics.

6. Professor Reuterdahl discussed the significance of Space and Time for the Finite and Infinite of mathematics. Space and Time may be regarded as the cosmic background of all knowledge. The finite and infinite are inseparable by-products of Space-Time. As a dual cosmic principle, Space-Time is a dynamic totality. Time is the space-separator and space is the time-binder. The onward urge of time conveys the impression of incompleteness and thus time constitutes the dynamic element of the dual principle Space-Time. Time is like a shearing blade which dissects space. Time can shear space only in three distinct ways and therefore physical space is three dimensional. Spatial manifolds of higher dimensionality are merely conceptual spatial extensions. Space-Time constitutes the only true continuum in the cosmos. The study of the finite and the infinite, consequently, becomes an investigation of the nature of this dynamic continuum. The paradox of a process-infinite composed of continuously associated finite phases finds its solution in the dynamic continuum of Space-Time.

GLADYS GIBBENS, *Acting Secretary.*

A SIMPLE FORM OF DUHAMEL'S THEOREM AND SOME NEW APPLICATIONS.[1]

By H. J. ETTLINGER, University of Texas.

1. Introduction. It is an open question whether Duhamel's[2] theorem should be given to a beginning class in the calculus. That one can avoid the use of this

[1] Read before the Mathematical Association of America, September, 1921.

[2] Duhamel, *Eléments de calcul infinitésimal,* Paris, 1856, p. 35.

theorem in dealing with *some* problems involving definite integrals such as length, volume, pressure, etc., must be recognized.[1] On the other hand, Duhamels' theorem is such a powerful tool for use in evaluating the limit of a sum by means of a definite integral and provides such a simple test as to when one infinitesimal may be replaced by another of the same order in this type of problem, that it is highly desirable to have it among the elementary theorems of the worker in mathematics. The theorem, furthermore, has important applications to the transformation of a double integral and the solution of integral equations.

W. F. Osgood[2] was the first to call attention to a rigorous formulation of the theorem. His proof makes use of uniform convergence to a limit and does not readily adapt itself to elementary treatment. The proof itself, based on Abel's lemma, is rather direct and simple,[3] though it is generally accepted that a proof involving uniform convergence should be taboo in a first course in calculus. However direct the proof may be, the *application* of the theorem is too difficult for a beginner and is not attempted in this form by Osgood in his text-book.[4]

R. L. Moore[5] has given a very general form of Duhamel's theorem whose importance, because of its generality, has been entirely overlooked. Some of the later writers simply mention Moore's form in passing. G. A. Bliss[6] presented a theorem equivalent to Duhamel's but involving the concept of uniform continuity. Recently E. V. Huntington[7] and H. B. Fine[8] presented simplified forms of Bliss's substitute. Finally G. James[9] has added a substitute different in form from the others.

It is the object of this paper to present Duhamel's theorem by means of a geometric lemma, which has the following advantages: (a) it is so simple in form that the beginning student in calculus will, from intuition, accept it as correct, subject to the statement that it will be proved in a later course; (b) it avoids the question of double limits and its concomitant concept, uniform convergence;[10]

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[1] Cf. a suggestion by one who has "suffered" from the effects of Duhamel's theorem, Benjamin Graham, "Some calculus suggestions by a student," in this MONTHLY, *1917*, pp. 265–271.

[2] "The integral as the limit of a sum, and a theorem of Duhamel's," *Annals of Mathematics*, second series, vol. 4, 1903, pp. 161–178.

[3] Cf. W. F. Osgood, *A First Course in the Differential and Integral Calculus*, New York, 1907, pp. 164–165. The uniformity condition in the statement of the theorem is here omitted.

[4] *Ibid.*, pp. 166, 167, 168, 171, 173, 178, 182. In the applications no test of uniform convergence is made.

[5] "On Duhamel's theorem," *Annals of Mathematics*, second series, vol. 13, 1912, pp. 161–166. This paper, though mentioned by Bliss, Huntington and James, does not seem to have attracted the attention it deserves.

[6] "A substitute for Duhamel's theorem," *Annals of Mathematics*, second series, vol. 16, 1914, pp. 45–49.

[7] "On setting up a definite integral without the use of Duhamel's theorem," this MONTHLY, *1917*, pp. 271–275.

[8] "Note on a substitute for Duhamel's theorem," *Annals of Mathematics*, second series, vol. 19, 1918, pp. 172–173.

[9] "A substitute for Duhamel's theorem," *Tôhoku Mathematical Journal*, vol. 17, 1920, pp. 7–9.

[10] The secretary's report of the Wellesley meeting records an observation having been made indicating doubt as to the truth of this statement (see *1921*, 359–360). Responsibility for the origin of this observation could not be fixed. That there is no basis for this doubt is evident

(c) it is more general than the other forms;[1] (d) it is easily applied. In 2 the various forms of Duhamel's theorem are set forth for comparison. In 3 the geometric lemma is stated and Duhamel's theorem derived from it. In 4 an application is made to prove the existence of a definite integral for a continuous function and an example worked out for the usual class of problems in definite integrals. In 5 indications are given of applications of a more general form of the theorem to the transformation of a double integral and the solution of integral equations.

2. Various Forms of the Theorem. Osgood's[2] form of Duhamel's theorem is the following:

"*Let*

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n \tag{A}$$

be a sum of infinitesimals and let α_i *differ uniformly by an infinitesimal of higher order than* Δx_i *from the summand* $f(x_i)\Delta x_i$ *of the definite integral*

$$\int_a^b f(x)dx \tag{B}$$

of the function $f(x)$, *this function being continuous throughout the interval* $a \leqq x \leqq b$. *Then the sum* (*A*) *approaches a limit when* $n = \infty$, *and the value of this limit is the definite integral* (*B*):

$$\lim_{n=\infty} \sum_{i=1}^{n} \alpha_i = \int_a^b f(x)dx."$$

The application of this theorem to any problem would call for a proof that

$$\lim_{n=\infty} \left[\frac{\alpha_k}{\Delta x} - f(x_k) \right] = 0$$

uniformly with respect to k. This is not to be done with ease in particular examples.

The following is Moore's[3] form:

"HYPOTHESIS: (*a*) *E is a limited point-set in a space of n-dimensions.* $E_{1n}, E_{2n}, \cdots, E_{nn}$ *are* (*for each value of the positive integer n*) *non-overlapping sub-sets of E of interior measures* $e_{1n}, e_{2n}, \cdots, e_{nn}$, *respectively.* r_{in}, r_{in}' $(i = 1, \cdots, n)$ *are numbers such that the set* $\{|r_{in}' - r_{in}|\}$ *is a bounded set, i.e., there exists a number c such that for all values of n and i* $(i \leqq n)$, $|r_{in}' - r_{in}| \leqq c$. (*b*) $\operatorname{Lim}_{n=\infty} \sum_{i=1}^{n} r_{in}e_{in}$ *exists.* (*c*) E_0 *is a subset of E of measure* 0. (*d*) *If P is a point of E not belonging to* E_0, *then*

$$\operatorname{Lim}_{n=\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0.$$

upon reading the proofs of the lemma. Also compare Osgood's theorem on non-uniformly convergent series, p. 244, footnote, and Moore's theorem, p. 241, referred to in this paper.

[1] Moore's form, of which the lemma is a corollary, is, of course, to be excepted.

[2] See *Annals* paper referred to above, p. 173.

[3] *L.c.*, pp. 162–163.

CONCLUSION: $\lim_{n=\infty} \sum_{i=1}^{n} r_{in}' e_{in}$ *exists and equals*

$$\lim_{n=\infty} \sum_{i=1}^{n} r_{in} e_{in}."$$

That this theorem is more general than Osgood's form has been shown by an independence example by Moore.[1] Unfortunately the extreme generality and nomenclature of Moore's theorem has caused its importance to be overlooked.

The substitute theorem of Bliss[2] is the following:

"*Let us consider a function of the form $f(p, p', p'')$ where p is a symbol for a set of values (x, y, z) and p' and p'' for analogous sets. The points p, p', p'' are to range over a closed measurable region V in xyz-space in which f is continuous and hence uniformly continuous.*

"*If the region V is divided into measurable sub-regions with maximum diameters less than δ and with volumes denoted by ΔV_k $(k = 1, 2, \cdots, n)$, and if in each region three points p_k, p_k', p_k'' are chosen, then*

$$\lim_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k', p_k'')\Delta V_k = \lim_{\delta=0} \sum_{k=1}^{n} f(p_k, p_k, p_k)\Delta V_k = \int_V f(p, p, p)dV."$$

The proof depends on the fact that f is uniformly continuous, a concept which Bliss frankly admits has little meaning to the average sophomore, but which he hopes will "leave a framework of proof in the mind of the student."

Huntington[3] simplifies Bliss's theorem in the following form:

"*Suppose that a required quantity P is associated with a real interval, $x = a$ to $x = b$, in such a way that we are led to divide the interval into n small parts or 'elements,' Δx, and to regard P as the sum of n separate contributions, one from each element. Suppose also that a set of one or more functions, $F(x), f(x), \cdots$, can be found, such that, no matter what value of x is considered, and no matter how small Δx may be, the contribution from a typical element, $x = x$ to $x = x + \Delta x$, can be expressed 'approximately' (see note 1) in the form*

$$[F(x)f(x) \cdots]\Delta x.$$

Then the required quantity P will be correctly given by the value of the definite integral

$$P = \int_a^b [F(x)f(x) \cdots]dx,$$

whenever the functions $F(x), f(x), \cdots$ are continuous from $x = a$ to $x = b$.

"*Note 1.* The word 'approximately' is here used in a technical sense, meaning that the exact value of the contribution in question lies between $[\underline{F} \cdot \underline{f} \cdots]\Delta x$ and $[\overline{F} \cdot \overline{f} \cdots]\Delta x$, where $\underline{F}, \underline{f}, \cdots$ are the smallest and $\overline{F}, \overline{f}, \cdots$ the largest values of $F(x), f(x), \cdots$ in the element."

Fine[4] also has given a simplification of Bliss's substitute. His statement is as follows:

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[1] *L.c.*, p. 166.

[2] *L.c.*, p. 46.

[3] *L.c.*, pp. 273 and 274. The proof as given in the footnote 1 on p. 273 is different from Bliss's.

[4] *L.c.*, p. 172.

"*Let $f_1(x)$, $f_2(x)$, $\cdots$, $f_p(x)$ denote any set of functions of x, finite in number, which are continuous in the interval* (ab) *and let*

$$F(x) = f_1(x)f_2(x) \cdots f_p(x).$$

"*Suppose the interval* (ab) *to be divided and redivided into parts in any manner such that as the process is indefinitely continued the greatest of the parts will approach* 0 *as a limit, and at any stage in the process let* h_1, h_2, $\cdots$, h_n *represent the parts in length and position; also let* ξ_i', ξ_i'', $\cdots$, $\xi_i^{(p)}$ *and* ξ_i *denote any numbers in the part* h_i $(i = 1, 2, \cdots, n)$. *Then*

$$\lim_{n \doteq \infty} \sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'') \cdots f_p(\xi_i^{(p)})h_i = \lim_{n \doteq \infty} \sum_{i=1}^{n} F(\xi_i)h_i = \int_a^b F(x)dx."$$

The proof[1] of this theorem, like the proof of Bliss's, is based on uniform continuity.

James[2] gives the following substitute:

"*Divide the interval, a to b, into n positive subintervals,* Δx_i, $i = 1, 2, 3, \cdots, n$, *each of which converges to zero as n increases. If*

$$\lim_{n \to \infty} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x)dx,$$

then

$$\lim_{n \to \infty} \sum_1^n [f(x_i)\Delta x_i + \phi_i(n)\Delta x_i]$$

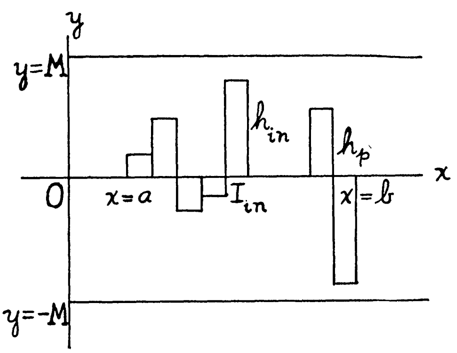
exists and is equal to

$$\int_a^b f(x)dx,$$

provided there exists a sequence of constants c_1, c_2, c_3, $\cdots$ (*independent of* n),[3] *with limit zero, such that* $|\phi_i(n)| \leqq c_i$, $i = 1, 2, \cdots n$."

This form though more general than Osgood's is less general than Moore's, and the simplicity of the proof and ease of application are debatable.

3. A Geometric Lemma and a Simple Form of Duhamel's Theorem. The following lemma is stated in geometric form because it is believed that it will appeal to the student's intuition and will be one which he will readily accept, subject to the remark that the proof must be deferred until a later time.



LEMMA: — *Let the interval* $I : a \leqq x \leqq b$ *be divided into n equal closed subdivisions,* I_{in}, *of length,* $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. *Upon* I_{in} *as base*

[1] *L.c.*, p. 173.
[2] *L.c.*, pp. 7–8.
[3] Parenthesis inserted by the present writer.

draw a rectangle of area, R_{in}, and height, h_{in}, such that

$$|h_{in}| \leqq M \quad (i = 1, 2, \cdots n)$$

for all values of n, where M is a constant independent of i and n. If $\overline{P}$ is any fixed point of I, there is for each value of n at least one rectangle whose base contains $\overline{P}$. If for each fixed $\overline{P}$ the altitude of this rectangle, $h_{\overline{P}}$, approaches 0 as a limit as $n \rightarrow \infty$, $\lim_{n\rightarrow\infty} h_{\overline{P}} = 0$, then $\lim_{n\rightarrow\infty} \sum_{i=1}^{n} R_{in} = 0$.

The proof[1] of this lemma is omitted, because it has no immediate relation to the purpose of this paper, but references to proofs are given in the footnote. The acceptance of the above lemma as true may be compared to the acceptance of Rolle's theorem as true from geometric intuition, or the acceptance of the following theorem: *If $f(x)$ is continuous in $a \leqq x \leqq b$ and $f(a) \neq f(b)$, then the equation $f(x) = N$, where N is between $f(a)$ and $f(b)$, has at least one root in $a < x < b$.* In fact Huntington[2] explicitly assumes this latter theorem without proof, and Graham,[3] who also uses it, refers to Goursat's *Cours d'Analyse Mathêmatique* for a proof.

On the basis of this geometric lemma we can readily establish

DUHAMEL'S THEOREM:—*Let the interval $I : a \leqq x \leqq b$ be divided into n equal subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base draw two rectangles, R_{in}' and R_{in}'', of height h_{in}' and h_{in}'', respectively, such that*

$$|h_{in}' - h_{in}''| \leqq M \quad (i = 1, 2, \cdots n)$$

for all values of n, where M is a constant, independent of i and n. If $\overline{P}$ is any fixed point of I, there is for each value of n at least one rectangle, R_{in}', whose base contains $\overline{P}$ and a corresponding rectangle, R_{in}'', with the same base. If, for each fixed $\overline{P}$, the difference of the altitudes of these rectangles, $h_{\overline{P}}'$ and $h_{\overline{P}}''$, approaches zero as a limit,

$$\lim_{n\rightarrow\infty} (h_{\overline{P}}' - h_{\overline{P}}'') = 0,$$

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[1] A proof may be given similar to Moore's (*l.c.*, p. 163), based on the application of a theorem due to W. H. Young (*Proceedings of the London Mathematical Society*, series 2, vol. 2, 1905, p. 25). Still another proof is that of E. Landau (*Mathematische Zeitschrift*, vol. 2, 1918, pp. 350–351). Professor Osgood, in a recent letter, pointed out that the lemma is virtually contained in a theorem which he proved in the *Amer. Jour. of Math.*, vol. 19, 1897, p. 188. He showed that, *if $s_n(x)$ be continuous in the interval $a \leqq x \leqq b$ for all values of n, and if $s_n(x)$ approach a continuous limit; if, furthermore, $s_n(x)$, regarded as a function of the two independent variables x and n, remain finite, then*

$$\lim_{n=\infty} \int_a^b s_n(x)dx = \int_a^b \lim_{n=\infty} s_n(x)dx.$$

From this theorem the lemma follows at once. Let $s_n(x)$ be defined as follows: On I_{in} as base erect an isosceles triangle whose altitude is $2h_{in}$. The equal legs of these triangles shall form the graph of $s_n(x)$. And now the area under $s_n(x)$ is precisely $\sum_{i=1}^{n} R_{in}$. But $\lim_{n=\infty} s_n(x) = 0$. Hence $\lim_{n=\infty} \sum_{i=1}^{n} R_{in} = 0$.

Since this paper was written the author has given an elementary proof based on the usual fundamental limit theorems of the beginning of the calculus (presented to the American Mathematical Society, September 7, 1922).

[2] *L.c.*, p. 273, footnote 1. [3] *L.c.*, p. 268, Theorem 3, Lemma.

and if $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} h_{in}'\Delta x_n$ *exists, then* $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} h_{in}''\Delta x_n$ *exists and*

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n = \lim_{n\to\infty}\sum_{i=1}^{n} h_{in}''\Delta x_n.$$

Let

$$h_{in} = h_{in}' - h_{in}''.$$

Then $|h_{in}| \leqq M$ for all values of n, and $\lim\limits_{n\to\infty} h_{\bar{P}} = 0$ for every fixed P. Hence by the lemma,

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}\Delta x_n = 0 \qquad \text{or} \qquad \lim_{n\to\infty}\sum_{i=1}^{n} (h_{in}' - h_{in}'')\Delta x_n = 0.$$

Hence

$$\lim_{n\to\infty}\sum_{i=1}^{n} h_{in}'\Delta x_n = \lim_{n\to\infty}\sum_{i-1}^{n} h_{in}''\Delta x_n.$$

4. **Applications to Definite Integrals.** We proceed to use the theorem of 3 to show that if $f(x)$ is continuous in the interval $I: a \leqq x \leqq b$, then $f(x)$ is integrable in I. Let I be subdivided into n equal subdivisions, I_{in} $(i = 1, 2, \cdots n)$, of length $\Delta x_n = (b - a)/n$. Let H be the maximum value of $f(x)$ in I; and h the minimum value of $f(x)$ in I. Let $\bar{P}$ be any fixed point of I and let $I_{\bar{P}n}$, for each value of n, be a closed subinterval I_{in} which contains $\bar{P}$, Let $H_{\bar{P}n}$ be the largest value of $f(x)$ in $I_{\bar{P}n}$ and let $h_{\bar{P}n}$ be the smallest value of the set $H_{\bar{P}1}$, $H_{\bar{P}2}, \cdots, H_{Pn}$. Let h_{in} be defined for each I_{in} as the number which for each $\bar{P}$ in I_{in} is equal to the value of $h_{\bar{P}n}$ which corresponds to $\bar{P}$. For each point $\bar{P}$, h_{in} never increases as n increases. Hence $A_n = \sum\limits_{i=1}^{n} h_{in}\Delta x_n$ never increases as n increases. But $A_n \geqq h(b - a)$. Therefore as $n \to \infty$, A_n approaches a limit or $\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n} h_{in}\Delta x_n$ exists.

Let ξ_{in} be any value of x in I_{in} and $\bar{\xi}_{in}$, the value of ξ_{in} which corresponds to $I_{\bar{P}n}$. Now $\lim\limits_{n\to\infty} h_{\bar{P}n} = f(\bar{x})$ where $\bar{x}$ is the abscissa of $\bar{P}$, and $\lim\limits_{n\to\infty} f(\bar{\xi}_{in}) = f(\bar{x})$ since $f(x)$ is continuous in I. Also $|h_{in} - f(\xi_{in})| \leqq H - h$ for every value of $i(\leqq n)$ and n. Hence by the theorem of 3,

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_{in})\Delta x_n$$

exists. This limit is independent of the choice of ξ_{in}, and of the method of subdivision.

Hence by the usual argument

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_{in})\Delta x_n = \int_a^b f(x)dx.$$

We may now restate Duhamel's theorem:

Let $f(x)$ be a continuous function of x in the interval $I: a \leqq x \leqq b$. Let I be subdivided into n equal closed subdivisions, I_{in}, of length, $\Delta x_n = (b-a)/n$, $(i = 1, 2, \cdots n)$. Upon I_{in} as base draw a rectangle of height β_{in}. Let ξ_{in} be any value of x in I_{in}. If $\bar{P}$ is any fixed point of I whose $x = \bar{x}$, there is for each value of n at least one rectangle with height, $\beta_{\bar{P}}$, whose base $\bar{I}_{in}$ contains $\bar{P}$. If $|\beta_{in}| \leqq M$ $(i = 1, 2, \cdots n)$ for all values of n, where M is a constant, independent of i and n, and if for each fixed point $\bar{P}$ $\lim_{n\to\infty} \beta_{\bar{P}} = f(\bar{x})$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \beta_{in}\Delta x_n = \int_a^b f(x)dx.$$

If we identify $f(\xi_{in}) = h_{in}'$ and $\beta_{in} = h_{in}''$ of the theorem in 3, the derivation of the preceding theorem is immediate.

For the application of this form of Duhamel's theorem to the usual problems of the first course of the calculus, it is only necessary to refer to the examples in Osgood's[1] text-book with some modifications. In each of the examples cited the theorem of 3 is tacitly assumed by Osgood and applied. For $x = x_k$ in these instances cannot be taken as designating the right hand end of the kth subdivision since $\lim_{n\to\infty} x_k = a$ and $\lim_{n\to\infty} (\beta_k/\alpha_k) = 1$ (where $\beta_k = \beta_{kn}\Delta x_n$, $\alpha_k = f(x_{kn})\Delta x_n$ in our notation) become meaningless as far as the application of Osgood's form of the theorem is concerned. Professor Osgood has indicated in another letter that he regards x_k as fixed. This is impossible unless k varies with n and n increases without limit in some special manner. What is evidently intended is that x_k shall be regarded as fixed entirely independent of any change in n and, furthermore, that it shall represent any value of x in I, *i.e.*, x_k is the $\bar{x}$ of our theorem. It is also desirable to replace $\lim_{n\to\infty} (\beta_k/\alpha_k) = 1$ by $\lim_{n\to\infty} (\beta_k - \alpha_k) = 0$, in our notation $\lim_{n\to\infty} (h_{\bar{P}}' - h_{\bar{P}}'') = 0$, since this avoids the difficulty of $\alpha_k = 0$. With these interpretations we reproduce the example on p. 166 in Osgood's text-book.

To find the length of the curve $y = F(x)$ from $x = a$ to $x = b$. For simplicity let $F(x)$ be continuous with a continuous derivative in $I: a \leqq x \leqq b$. Subdivide I into n equal parts; erect ordinates at the points of division; and inscribe a broken line in the arc to be measured. The length of this line is

$$\sum_{i=0}^{n-1} \sqrt{\Delta x_n^2 + \Delta y_{in}^2} = \sum_{i=0}^{n-1} \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}\,\Delta x_n.$$

Now, to make connections with the foregoing theorem, let

$$f(x) = \sqrt{1 + [F'(x)]^2}$$

and

$$\beta_{in} = \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}.$$

[1] *L.c.*, see examples on pp. 166, 167, 168, 171, 173, 178, 182.

Now $|\beta_{in}| = \sqrt{1 + \left(\frac{\Delta y_{in}}{\Delta x_n}\right)^2}$ remains finite and $\lim_{n\to\infty} \beta_{\bar{P}} = \sqrt{1 + [F'(\bar{x})]^2} = f(\bar{x})$. Hence the conditions of Duhamel's theorem are satisfied and

$$s = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\Delta x_n^2 + \Delta y_n^2} = \int_a^b \sqrt{1 + [F'(x)]^2}\, dx.$$

It seems to the writer that the form of Duhamel's theorem here given, on account of its simplicity and ease of application, should earn a place in our elementary calculus texts.[1]

5. Applications to the Transformation of a Double Integral and the Solution of an Integral Equation.[2] The geometric lemma and the statement of Duhamel's theorem of 3 may be generalized in several directions. It is quite obvious that it is not essential that the lengths of the subdivisions be equal, and instead of a linear interval we may have a closed space of any number of dimensions. These indications are merely special cases of Moore's general form which was reproduced in 2. We proceed to apply this form to the transformation of a double integral.[3]

HYPOTHESES: 1. *$f(x, y)$ is a function having an upper and lower bound defined for all points (x, y) of a closed, connected domain, G, contained in H, a bounded connected domain, the boundary of G having content zero;*

2. *x and y are expressed in terms of ξ and η by the relations*

$$x = f_1(\xi, \eta), \qquad y = f_2(\xi, \eta), \tag{T}$$

such that the transformation (T), as well as its inverse

$$\xi = \phi_1(x, y), \qquad \eta = \phi_2(x, y), \tag{T'}$$

is continuous and one-to-one. Let $F(\xi, \eta) \equiv f[f_1(\xi, \eta), f_2(\xi, \eta)]$ and let G' in H' represent the transform of G in H. L is a set of points of G of measure zero;

3. *$f(x, y)$ is continuous in G except at points of L;*

4. *the partial derivatives $\frac{\partial f_1}{\partial \xi}, \frac{\partial f_1}{\partial \eta}, \frac{\partial f_2}{\partial \xi}, \frac{\partial f_2}{\partial \eta}$ exist and are continuous at every point of H', except at points of the set L' corresponding to L;*

5. *the Jacobian of $f_1(\xi, \eta)$, $f_2(\xi, \eta)$ with respect to ξ and η, $J = \frac{\partial(f_1, f_2)}{\partial(\xi, \eta)}$, is different from zero in H', except at points of L'.*

[1] The ordinary "carefree" treatment of Duhamel's theorem may be exemplified in an otherwise most excellent text, G. A. Gibson, *Elementary Treatise on the Calculus*, London, 1919. In many respects this book is superior to the average calculus text-book. On p. 198 the author states that Duhamel's theorem "is not necessarily true if the infinitesimals are not all of the same sign." That this is incorrect is established by the present paper. He adds that "from its use in integration this theorem is often called the fundamental theorem of the integral calculus."

[2] Since this paper was written, the author has made other important applications (1) to the transformation of a simple integral, (2) to the differentiation or integration termwise of a non-uniformly convergent series, (3) (by a pupil) to the approximate solution of differential equations, (4) to Green's theorem in potential theory and related problems.

[3] See E. W. Hobson, *The Theory of Functions of a Real Variable*, Cambridge, 1907, pp. 445–452.

CONCLUSION:

$$\iint_G f(x, y)dxdy = \iint_{G'} F(\xi, \eta)\left|\frac{\partial(f_1, f_2)}{\partial(\xi, \eta)}\right| d\xi d\eta.$$

The outline of the proof is as follows:

$$\begin{aligned}\iint_G f(x, y)dxdy &= \lim_{n\to\infty}\sum_{i=1}^{n^2} f(x_{in}, y_{in})\Delta G_{in}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n^2} f[f_1(\xi_{in}, \eta_{in}), f_2(\xi_{in}, \eta_{in})]\Delta G_{in}\\ &= \lim_{n\to\infty}\sum_{i=1}^{n^2} F(\xi_{in}, \eta_{in})\frac{\Delta G_{in}}{\Delta G_{in}'}\cdot\Delta G_{in}'.\end{aligned}$$

If $\bar{P}(\bar{x}, \bar{y})$ is a fixed point of G not in L, then for each value of n there is (at least) one $\Delta \bar{G}_{in}$ containing $\bar{P}$. If $\Delta \bar{G}_{in}'$ is the transform of $\Delta \bar{G}_{in}$ by T, then it may be proved that

$$\lim_{n\to\infty}\left[\frac{\Delta\bar{G}_{in}}{\Delta\bar{G}_{in}'} - J_{|\xi=\bar{\xi}, \eta=\bar{\eta}|}\right] = 0^1 \qquad \text{and} \qquad \left|\frac{\Delta G_{in}}{\Delta G_{in}'} - J_{|\xi=\bar{\xi}, \eta=\bar{\eta}|}\right| \leqq M$$

for each value of n.

Hence by Moore's form of Duhamel's theorem the transformation is established.

Fredholm[2] solved the integral equation of the second kind,

$$u(x) = f(x) + \int_a^b K(x, \xi)u(\xi)d\xi, \tag{E}$$

by means of a system of algebraic equations in an infinite number of unknowns. We give Bôcher's[3] treatment:

Let the interval $I : a \leqq x \leqq b$ be subdivided into n equal subdivisions, I_{in} $(i = 1, 2, \cdots n)$, of length $\Delta x_n = (b - a)/n$ and let $x_{0n} = a$, x_{1n}, x_{2n}, $\cdots$, $x_{nn} = b$ designate the points of division. If we replace (E) by the equation

$$u(x) = f(x) + \sum_{j=1}^{n} K(x, x_{jn})u_n(x_{jn})\Delta x_n, \tag{E'}$$

(E') is to be satisfied for the values x_{in} $(i = 1, 2, \cdots n)$. This yields the system of n equations

$$u_n(x_{in}) = f(x_{in}) + \sum_{j=1}^{n} K(x_{in}, x_{jn})u_n(x_{jn})\Delta x_n \quad (i = 1, 2, \cdots n) \tag{S}$$

[1] Cf. Goursat, *Cours d'Analyse Mathématique*, Paris, 1902, vol. 1, pp. 300–302.

[2] "Sur une nouvelle méthode pour la résolution du problème de Dirichlet," *Öfversigt af Kungl. Vetenskaps-Akademiens Förhandlingar*, vol. 57, 1900, p. 39. Also *Acta Mathematica*, vol. 27, 1903, pp. 365–390.

[3] *Cambridge Tracts in Mathematics and Mathematical Physics*, no. 10, *An Introduction to the Study of Integral Equations*, Cambridge, at the University Press, 1909, pp. 25–27.

in n unknown quantities, $u_n(x_{in})$, $(i = 1, 2, \cdots n)$. (S) may be written as

$$-\sum_{j=1}^{n} K(x_{in}, x_{jn})u_n(x_{jn})\Delta x_n + u_n(x_{in}) = f(x_{in}) \quad (i = 1, 2, \cdots n). \qquad (S')$$

By Cramer's formula the value of $u_n(x_{\mu n})$ is

$$u_n(x_{\mu n}) = \frac{1}{D_n}\sum_{i=1}^{n} f(x_{in})D_n(x_{\mu n}, x_{in}) \quad (\mu = 1, 2, \cdots n),$$

where $D_n(\neq 0)$ is the determinant of the system [1] and $D_n(x_{\mu n}, x_{\nu n})$ is the cofactor of the element in the νth row and the μth column. The solution of (E) is now obtained by letting n become infinite and at the same time allowing μ to vary with n in such a manner that $x_{\mu n}$ approaches a fixed value x in I. If $D = \lim_{n \to \infty} D_n$ and $D(x, \xi) = \lim_{n \to \infty} \dfrac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$, then it may be proved that

$$u(x) = f(x) + \frac{1}{D}\int_a^b f(\xi)D(x, \xi)d\xi.$$

This is the solution of (E).

The application of Duhamel's theorem in this proof is that of determining the limiting form of $\dfrac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$. The hypothesis [2] concerning $K(x, \xi)$ is that it shall be bounded in $S : a \leqq x \leqq b,\ a \leqq \xi \leqq b$ and continuous except for a set of points, L, of measure zero, and $K(x, x)$ shall be integrable in $I : a \leqq x \leqq b$. The typical term [3] of $\dfrac{D_n(x_{\mu n}, x_{\nu n})}{\Delta x_n}$, except for the omission of the numerical coefficient, is

$$\underbrace{\sum_{i, j, \cdots = 1}^{n}}_{m-1} \begin{vmatrix} K(x_{\mu n}, x_{\nu n}) & K(x_{\mu n}, x_{in}) & K(x_{\mu n}, x_{jn}) & \cdots m \text{ columns} \\ K(x_{in}, x_{\nu n}) & K(x_{in}, x_{in}) & K(x_{in}, x_{jn}) & \cdot \\ K(x_{jn}, x_{\nu n}) & K(x_{jn}, x_{in}) & K(x_{jn}, x_{jn}) & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m \text{ rows} \ldots & \ldots & \ldots & \cdot \end{vmatrix} \overline{\Delta x}_n^{m-1}. \qquad (A)$$

If (x, ξ) is a fixed point in S not in L such that as n increases $x_{\mu n}$ approaches x and $x_{\nu n}$ approaches ξ, then the limit of (A) would appear to be

$$\underbrace{\int_a^b \int_a^b \cdots \int_a^b}_{m-1} \begin{vmatrix} K(x, \xi) & K(x, \xi_1) & K(x, \xi_2) & \cdots m \text{ columns} \\ K(\xi_1, \xi) & K(\xi_1, \xi_1) & K(\xi_1, \xi_2) & \cdot \\ K(\xi_2, \xi) & K(\xi_2, \xi_1) & K(\xi_2, \xi_2) & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ m \text{ rows} \ldots & \ldots & \ldots & \cdot \end{vmatrix} d\xi_1 d\xi_2 \cdots d\xi_{m-1}. \qquad (B)$$

[1] Bôcher, *l.c.*, p. 26.
[2] *L.c.*, p. 36, theorem 3.
[3] *L.c.*, p. 27.

That this is the limit can be proved by identifying (A) above with $\Sigma r_{in}'e_{in}$ and (B) with $\lim\limits_{n\to\infty} \Sigma r_{in}e_{in}$ of the theorem of p. 241.

Now $K(x, \xi)$ is bounded; hence

$$|r_{in}' - r_{in}| \leqq C$$

and $\lim\limits_{n\to\infty} (r'_{i_{P_n}n} - r_{i_{P_n}n}) = 0$, if P is not a point of L, since $K(x, \xi)$ is continuous except for points of L. Hence Duhamel's theorem establishes the limiting form of (A) to be (B).

NOTE ON APPLICATION OF DIOPHANTINE ANALYSIS TO GEOMETRY.

By HORACE L. OLSON, University of Michigan.

It frequently becomes desirable, in teaching analytic geometry of three dimensions, to know a set of three mutually perpendicular lines each of which has all its direction cosines rational. It is well known that if two lines have direction cosines l_1, m_1, n_1 and l_2, m_2, n_2, respectively, the direction cosines of their common perpendicular are proportional to $(m_1n_2 - m_2n_1)$, $(n_1l_2 - n_2l_1)$, and $(l_1m_2 - l_2m_1)$, and that if the two first-mentioned lines are perpendicular the factor of proportionality is unity. Hence, if two of our three mutually perpendicular lines have rational direction cosines, the third will also have this property. Furthermore, it will presently appear that one of these lines can be taken to be any line having rational direction cosines.

We therefore have first to find rational solutions of the equation

$$l_1^2 + m_1^2 + n_1^2 = 1. \tag{1}$$

It is well known, and can easily be verified, that all such solutions are given by the formulæ

$$l_1 = \frac{p^2 + q^2 - r^2}{p^2 + q^2 + r^2}, \qquad m_1 = \frac{2pr}{p^2 + q^2 + r^2}, \qquad n_1 = \frac{2qr}{p^2 + q^2 + r^2},$$

where p, q, and r are any three integers.

Having so selected l_1, m_1, and n_1, we have next to solve, in rational numbers, the simultaneous equations

$$\begin{cases} l_2^2 + m_2^2 + n_2^2 = 1, \\ l_1l_2 + m_1m_2 + n_1n_2 = 0. \end{cases} \tag{2}$$

If we eliminate l_2 from these equations, the resulting equation can be put into the form

$$\left\{m_2 + \frac{m_1n_1n_2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1n_2}{l_1^2 + m_1^2}\right\}^2 = \left\{\frac{l_1^2}{l_1^2 + m_1^2}\right\}^2 + \left\{\frac{l_1m_1}{l_1^2 + m_1^2}\right\}^2, \tag{3}$$

since the second member may be written as $l_1^2/(l_1^2 + m_1^2)$. Two solutions of

equation (3) are evidently given by

$$\begin{cases}(l_1^2 + m_1^2)m_2 + m_1n_1n_2 = l_1^2, \\ \qquad\qquad\qquad\quad l_1n_2 = \pm\, l_1m_1,\end{cases}$$

and

$$\begin{cases}(l_1^2 + m_1^2)m_2 + m_1n_1n_2 = \pm\, l_1m_1, \\ \qquad\qquad\qquad\quad l_1n_2 = l_1^2.\end{cases}$$

From the first of these two solutions and the second of equations (2)

$$l_2 = \frac{-\,l_1m_1(1 \pm n_1)}{l_1^2 + m_1^2}, \qquad m_2 = \frac{l_1^2 \mp m_1^2n_1}{l_1^2 + m_1^2}, \qquad n_2 = \pm\, m_1,$$

and from the relations

$$l_3 = m_1n_2 - m_2n_1, \qquad m_3 = n_1l_2 - n_2l_1, \qquad n_3 = l_1m_2 - l_2m_1, \tag{4}$$

mentioned above, we have

$$l_3 = \frac{-\,l_1^2n_1 \pm m_1^2}{l_1^2 + m_1^2}, \qquad m_3 = \frac{-\,l_1m_1(n_1 \pm 1)}{l_1^2 + m_1^2}, \qquad n_3 = l_1.$$

Similarly, from the second of our two solutions and the second of equations (2)

$$l_2 = \frac{-\,l_1^2n_1 \mp m_1^2}{l_1^2 + m_1^2}, \qquad m_2 = \frac{-\,l_1m_1(n_1 \mp 1)}{l_1^2 + m_1^2}, \qquad n_2 = l_1,$$

and with (4)

$$l_3 = \frac{l_1m_1(1 \mp n_1)}{l_1^2 + m_1^2}, \qquad m_3 = \frac{-\,l_1^2 \mp m_1^2n_1}{l_1^2 + m_1^2}, \qquad n_3 = \pm\, m_1.$$

A more general solution of equation (3) is given by

$$\begin{cases}(l_1^2 + m_1^2)m_2 + m_1n_1n_2 = sl_1^2 \mp tl_1m_1, \\ \qquad\qquad\qquad\quad l_1n_2 = tl_1^2 \pm sl_1m_1,\end{cases} \tag{5}$$

where s and t are any two rational numbers satisfying the equation

$$s^2 + t^2 = 1.$$

Then s and t can always be expressed in the form

$$s = \frac{1 - k^2}{1 + k^2}, \qquad t = \frac{2k}{1 + k^2},$$

k any rational number (or $k = \infty$, which gives $s = -1, t = 0$). From equations (5) and the second of equations (2) we have, finally, the one-parameter family,

$$\begin{cases} l_2 = \dfrac{-\,sl_1m_1 \pm tm_1^2 - tl_1^2n_1 \mp sl_1m_1n_1}{l_1^2 + m_1^2}, \\ m_2 = \dfrac{sl_1^2 \mp tl_1m_1 - tl_1m_1n_1 \mp sm_1^2n_1}{l_1^2 + m_1^2}, \\ n_2 = tl_1 \pm sm_1, \end{cases}$$

equations (4) giving the direction cosines of the remaining line, viz.,

$$\begin{cases} l_3 = \dfrac{tl_1m_1 \pm sm_1^2 - sl_1^2n_1 \pm tl_1m_1n_1}{l_1^2 + m_1^2}, \\ m_3 = \dfrac{-\,sl_1m_1n_1 \pm tm_1^2n_1 - tl_1^2 \mp sl_1m_1}{l_1^2 + m_1^2}, \\ n_3 = sl_1 \mp tm_1. \end{cases}$$

The solution last given is the most general rational solution possible; for evidently any other solution must correspond to a solution of equation (3) more general than (5). To show that (5) is the most general solution, let us simplify the notation and consider the most general rational solution, for x and y, of the equation

$$x^2 + y^2 = a^2 + b^2, \qquad \text{or} \qquad x^2 - a^2 = -\,y^2 + b^2.$$

Then

$$x + a = \frac{1}{k}(y - b), \qquad x - a = -\,k(y + b),$$

or

$$x = \frac{(1 - k^2)a - 2kb}{1 + k^2}, \qquad y = \frac{2ka + (1 - k^2)b}{1 + k^2}.$$

Hence we have determined the most general set of these mutually perpendicular lines each of which has rational direction cosines.

A NOTE ON THE PROBLEM OF THE EIGHT QUEENS.

By W. H. BUSSEY, University of Minnesota.

Finite geometries were defined by Veblen and Bussey in the *Transactions of the American Mathematical Society*, volume 7 (1906), pp. 241–259. References to existing literature of the subject were given in this MONTHLY, *1921*, 85–86. The simplest case of a finite plane geometry based upon an odd prime, the euclidean plane geometry, modulo 3, was presented in detail by Bennett, in this MONTHLY, *1920*, 357–361.

The Problem of the Eight Queens is the determination of the number of ways in which eight queens can be placed on a chess board—or, more generally, in which n queens can be placed on a square board of n^2 cells—so that no queen can take any other. It was proposed originally by Franz Nauck.[1]

The object of this note is to show that in the special case in which n is a prime number p there is a connection between the problem of the queens and the lines of the finite plane geometry of p points to the line.

The cells of the chess board are represented by their middle points which

[1] For the history of the problem see Ahrens, *Mathematische Unterhaltungen und Spiele*, Leipzig, 1901, chapter 9. A brief discussion of the problem and its solution is given by Ball, *Mathematical Recreations and Essays*, fifth, sixth or seventh edition, pp. 113–118.

constitute a finite euclidean plane geometry of p points to the line. In order that one queen may take another, the two queens must occupy cells whose representative points are on a vertical line, on a horizontal line, on a line of slope 1, or on a line of slope $p - 1$. (The slope $p - 1$ is the same as the slope -1 since $p - 1 \equiv -1$, modulo p.)

Therefore, if p queens be placed on the cells whose representatives are the points of a line whose slope is any one of the integers $2, 3, 4, \cdots, p - 2$, no queen can take any other queen. The total number of such lines, each of which furnishes a solution of the problem of the p queens, is $p^2 + p - 4p$ or $p^2 - 3p$. When $p = 5$, the number of solutions obtained by this method is 10, which happens to be all the solutions that exist. When $p = 7$, the number of solutions furnished by this finite geometry method is 28; but, as a matter of fact, there are 40 solutions[1] when $p = 7$. For higher values of p, the lines of the finite geometry furnish some, but not all, of the solutions of the problem.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

25. Montucla's Closing Years.

The little that we know about the great men of our varied spheres of interest and the careless surmises that we make as to their lives are evident whenever we look even slightly below the surface that lies open to the world. Anyone who reads an ordinary biographical sketch of Montucla, for example, gains an impression of a man who was born, who wrote the first great history of mathematics, and who died in the fullness of years.[2] In an earlier article in this series[3] I have called attention to a little side-light thrown upon his life by a note from his collaborator Lalande. It seems worth the effort, however, to call further attention to his closing years by publishing a portion of a letter, now in my collection and written to some unnamed friend, which gives a nearer view of these last years of one whose portraits show as a well-fed "gentleman of the old school," bland, placid, content with the world, and appreciative of the praise that this same world had bestowed upon him.

The letter is garrulous to the point of being wearisome, but a portion will suffice to give a picture of a poor, harassed, discouraged old man, suffering in mind and body, neglected by his friends and deserted by his family,—a subject for the pity of a world not then given to pitying anyone, and of a later world

[1] Cf. Ball, *l.c.*, p. 116.

[2] Jean Étienne Montucla, born September 5, 1725; died at Paris, December 18, 1799. He was married at Grenoble in 1763. The first edition of his history appeared in 1758. It is a work of great impartiality and erudition. By the Revolution he lost his position in the civil service and was left with no means of livelihood. In 1794 he secured a place under the revolutionary government and he finally, four months before his death, secured a small pension of 2,400 francs.

[3] This Monthly, *1921*, 207.

which has well-nigh forgotten Montucla, the man. The letter was written on October 12, 1798, not many months before his death, and is in part as follows:

Tonnerre 20. vend.re 6.[1]

I have delayed an answer, my dear old friend, to your latest, in the hope that I might reply by word of mouth. I have now decided to send it in writing, traveling not having been very agreeable during the summer. I shall therefore follow your letter by article.

Not being able to get a word out of M. Denizet [2] I asked Madame Denoust [3] directly, and then through the mediation of the divine Lalande,[4] to stir him up. Madame Denoust replied: "It would be useless for me to take the matter up with my brother-in-law[5] and it would be necessary for me to go to Paris to take charge of the matter directly." The "Divine One" [6] did more than that, for he offered me a bed and some money, and let M. Denizet know about it.

M. Denizet said to Madame Denoust that her family had become bankrupt and told the divine one that he would only be getting her into trouble if he encouraged her to come to Paris. And then he suddenly wrote me a long letter.

Since the divine one told me that M. Denizet did not even read my letters to him, I answered him briefly and a few days later I received four lines from him. I was waiting for the result of his promises when Demottier, "the Picard"—my old coachman, wrote to me that I had become a widower and that my wife[7] had died at the abbé's at Gournai, near Compiègne.

A few weeks afterwards M. Denizet came to Tonnerre and told me that an Abbé[8] Carron had taken charge of the business affairs of my three daughters and that he was going to file a claim against my pension,—a pension which had not been paid since January, 1794. I offered to write to my eldest daughter, and, although the initiative should have come from her, I took the first step.

On the 15th of August M. Denizet wrote to me that he had obtained a pension of 800 livres, of which the first installment of 400 livres was paid immediately, that the second one of 400 livres would be paid on January first, and that he would take the matter up again,—this being looked upon only as a provisional payment.

I replied that I owed some money for my board at Tonnerre during the three years preceding and that I hoped he would carry out his plans.

As to my eldest daughter, she sent me a cool letter[9] from which I could see that she had been influenced against me. Indeed, is it possible to believe that this letter which I was compelled to write to her was the first that had passed between us since she was born!

These 400 livres paid on August 15 have not yet reached me because Madame Denizet, intending to take the trouble to send them to me, had received them all right, but being away on the 19th Fructidor,[10] the banks were closed and she used my money to come to Tonnerre, giving me the assurance that her husband would pay it to me in October. M. de Denizet has not come and I have just written to him to let me have at least a mouthful,—the only assistance that has been given me since January first, 1794. He has replied that I shall receive it on the 26th.

Moreover, M. Denizet, during his brief sojourn at Tonnerre last summer, told me that I might go to Paris when I had the means and that he saw no objection to my so doing. In anticipation of this I took the matter up with several people, but the 18th Fructidor[11] has upset everything and I do not know what will become of it all.

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[1] October 12, 1798. Tonnerre is about 80 miles southeast of Paris.

[2] The names in the letter are mostly those who left no impression upon the world and are now forgotten. M. Denizet was apparently connected with the bureau of civil pensions.

[3] Evidently the sister-in-law of M. Denizet.

[4] Joseph Jérôme Le Francais de Lalande, born at Bourg-en-Bresse, July 11, 1732; died at Paris, April 4, 1807. Montucla speaks of him ironically as "the Divine One," apparently from his manner.

[5] That is, by correspondence with M. Denizet.

[6] Capital, *Divin.*

[7] A joke, he and his wife not being on good terms. His wife survived him.

[8] Aé in the original.

[9] Une lettre respectuesement sèche.

[10] September 6. The *coup d'état* of Barras, Larévellière, and Reubell had taken place the day before and everything was at a temporary standstill.

[11] September 5, the day of the *coup d'état* already mentioned.

During his sojourn at Tonnerre M. Denizet saw Madame Quingeri and Madame Leprince, two frequent visitors there, saying in a lofty way that he never forgot his old friends.

The day of his departure he finished up the business and asked me to dinner. He was very affable and, I must say in justice to him, he paid me special attention. I do not know whether, with patience, I shall ever see the end of this business. If I do, I shall need to have shown a lot of it.

I now come to the second part of your letter.

I have always blamed the misplaced pride of Bailly[1] and Condorcet.[2] As to Duséjour[3] I was under the impression that he had been executed, but you tell me that he died of fear of terrorism.[4]

. . . I was right about complaining about Madame de Marcheval and Madame de Lesseville.[5] These two females[6] have never ceased to stir up my wife against me. I made a short call at Auteuil in August 1779[7] and took supper at my aunt's, but I did not succeed in making any better impression. She had already got fixed in my wife's head the necessity for leaving me and managed to obtain the *lettre de cachet*,[8] on the 30th of the following November, through her friend Amelot. Now talk of gratitude among relations, especially after all that I had done for them!

The rest of the letter relates to suggested changes in Montucla's history, then being revised by Lalande, and need not concern us. Enough has been given to allow us to picture to ourselves a very human man, enduring very human ills, and approaching his end with very human complaints,—a man the exact antithesis of what his contemporary conventional portraits show to the world at large.

QUESTIONS AND DISCUSSIONS

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

REPLIES.

36, 1. [*1919*, 69].[9] For what values of n can $\cos(2\pi/n)$ be expressed in the form $(a+\sqrt{b})/c$ where a, b and c are integers?

I. REPLY BY R. S. UNDERWOOD, Alabama Polytechnic Institute.

Using the formula[10]

$$2\cos n\theta = (2\cos\theta)^n - \frac{n}{1}(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \frac{n(n-4)(n-5)}{3!}(2\cos\theta)^{n-6} + \cdots,$$

we get

$$2 = (2\cos 2m\pi/n)^n - \frac{n}{1}(2\cos 2m\pi/n)^{n-2} + \cdots;$$

[1] Jean Sylvain Bailly, born at Paris in 1736, executed at Paris in 1793. Arago's biography is well known. The misplaced pride consisted in standing for moderate principles against a bloodthirsty mob. Both he and Condorcet were prominent as mathematicians and as statesmen.

[2] Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet, born at Ribemont in 1743; died at Bourg-la-Reine in 1794, a suicide by poison, in prison, to prevent being taken to Paris for execution.

Tallentyre wrote of him: "Since he never gave himself blindly to any one faction, all factions have distrusted and condemned him. To the Royalist he is a Revolutionist; to the Revolutionist he is an aristocrat. The thinker cannot forgive him that his thought led him to deeds and words; the man of action cannot forget that he was a thinker and dreamer to the end.

[3] Achille Pierre Dionis du Séjour (1734–1794), a celebrated astronomer. He fled to the country during the Terror, concealed himself, and died there.

[4] The recipient of the letter (apparently) has added the words, "He died of fear of death."

[5] The Marquis de Lesseville, now living at Châlons sur Marne, tells me that this was probably his great grandmother. The family is an old one.

[6] Femelles, female beasts, a term of great contempt.

[7] This shows that for the last twenty years of his life Montucla's family life was unpleasant. The event was only sixteen years after his marriage.

[8] Order for arrest.

[9] The other parts of this question were answered by the proposer, Professor Harris Hancock, *1919*, 292–295.

[10] See Problems—Notes 3, *1921*, 38.

so that the values of 2 $\cos(2m\pi/n)$ for $m = 1, 2, \cdots, n$ are the roots of the equation

$$\left\{ x^n - nx^{n-2} + \frac{n(n-3)}{2} x^{n-4} - \cdots \right\} - 2 = 0.$$

Since the coefficients in this equation are integers, the coefficients k and k_1 of any rational quadratic factor $x^2 + kx + k_1$ must also be integers. We may reject values of k and k_1 which make the roots of $x^2 + kx + k_1 = 0$ imaginary or greater than 2 in absolute value. Furthermore all the n roots of the original equation will be needed for the values of 2 cos $(2m\pi/n)$ when $m = 1, 2, \cdots, n$; hence neither one of the roots of the quadratic can be greater than 2.

With this restriction, the possible irreducible quadratic factors are limited to[1] $x^2 - 2$, $x^2 - 3$, and $x^2 \pm x - 1$, while the only possible rational linear factors are x, $x \pm 1$, and $x \pm 2$. Hence the only possible rational and quadratic surd values for cos $(2m\pi/n)$ are 0, $\pm 1/2$, ± 1, $\pm \sqrt{2}/2$, $\pm \sqrt{3}/2$, and $(\pm 1 \pm \sqrt{5})/4$; and the only values of n for which cos $(2\pi/n)$ is of the form $(a + \sqrt{b})/c$, including the case $b = 0$, are 1, 2, 3, 4, 5, 6, 8, 10, and 12.

II. Remarks by the Editor.

It is evident that Professor Underwood's method may be used to find the positive integers n for which cos $(2\pi/n)$ satisfies an equation of given degree k with integral coefficients. It is necessary that 2 cos $(2\pi/n)$ satisfy an equation of the form

$$x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots = 0, \tag{1}$$

in which (a) $a_1, a_2, \cdots$ are integers, and (b) all roots are real and not greater than 2 in absolute value. On account of (b), $|a_1| \leqq 2k$, $|a_2| \leqq 4\binom{k}{2}$, etc. Hence the number of equations (1) satisfying (a) and (b) is finite, and the possible equations may be examined in succession.

It will be noticed that in the cases $k = 1$ and $k = 2$ all "possible" equations led to actual solutions, and it might be thought that in the higher cases some difficulty would occur in recognizing those equations which, while conforming to (a) and (b), failed to satisfy the trigonometric test. Such difficulty however would not occur in view of the following interesting theorem, due to Kronecker:[2]

If an equation (1) *satisfies* (a) *and* (b), *each of its roots is of the form* 2 $\cos(2m\pi/n)$, *where m and n are integers.*

For proof, let $x = 2 \cos \theta$, and let $y = 2 \cos 2\theta = x^2 - 2$. On eliminating x between this equation and (1) by any of the ordinary methods, we get an equation in y of type (1) which is seen to satisfy (b). From the nature of the transformation it also satisfies (a). Hence, by applying successively transformations $z = y^2 - 2$, $u = z^2 - 2$, etc., we obtain always equations of type (1) satisfying (a) and (b). But such equations are finite in number. Therefore eventually we obtain two that are identical, and have the same roots. Now it may be that the roots of these two equations do not correspond to themselves, but are permuted by the transformations. In such a case we may apply more transformations until the permutation has been made $k!$ times, when every root will return to its own position. It will therefore happen that after, let us say, p and q transformations any root 2 cos θ of (1) will be transformed into the same number. Hence $2^p\theta = 2r\pi \pm 2^q\theta$, where r is an integer and $p \neq q$; and therefore θ is commensurable with 2π.

The method used hitherto would prove very laborious for values of k exceeding 2 or 3; but the question may be settled in another way by means of the theory of the primitive nth roots of unity. If the prime divisors of n are $p, q, \cdots$, the number of primitive roots of $x^n = 1$ is $\varphi(n) = n(1 - 1/p)(1 - 1/q) \cdots$, being the number of positive integers prime to and not greater than n. They satisfy an equation $f(x) = 0$ of degree $\varphi(n)$, having integral coefficients. If n is greater than 2, $\varphi(n)$ is even, and the coefficients of $f(x)$ are unaltered by writing in reversed order; so that the transformation $y = x + x^{-1}$ leads to an equation in y of degree $\frac{1}{2}\varphi(n)$. But one of the roots in x is $e^{2i\pi/n}$; and therefore one of the roots in y is 2 $\cos(2\pi/n)$. It follows that cos $(2\pi/n)$ is a root of a rational equation of degree $\frac{1}{2}\varphi(n)$.

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[1] The question of rational linear factors (and therefore of rational cosines) was discussed by Professor Underwood in a recent issue of this Monthly (*1921*, 374).—Editor.

[2] "Zwei Sätze über Gleichungen mit ganzzahligen Coefficienten", *Journal für die reine und angewandte Mathematik*, vol. 53, p. 173; *Werke*, vol. 1, 1895, p. 107. Kronecker deduced the present theorem from another concerned with roots of unity; but his method is essentially similar to that of the direct proof given above.

The question now is whether $\cos(2\pi/n)$ can be a root of a rational equation of lower degree. Let it be a root of an equation of degree k. The substitution of the exponential form for the cosine then shows that $e^{2i\pi/n}$ is a root of a rational equation of degree $2k$. But the equation of primitive nth roots of unity is not rationally reducible;[1] that is, $e^{2i\pi/n}$ is not a root of any rational equation of degree less than $\varphi(n)$. Therefore k is not less than $\frac{1}{2}\varphi(n)$. We have then the theorem: *If n is an integer greater than* 2, $\cos(2\pi/n)$ *is a root of an irreducible equation of degree* $\frac{1}{2}\varphi(n)$.

The simpler cases are:

$\varphi(n) = 2$ when $n = 3, 4, 6$; $\cos(2\pi/n)$ rational, as also for $n = 1, 2$.
$\varphi(n) = 4$ when $n = 5, 8, 10, 12$; $\cos(2\pi/n)$ a quadratic surd.
$\varphi(n) = 6$ when $n = 7, 9, 14, 18$; $\cos(2\pi/n)$ a cubic surd.
$\varphi(n) = 8$ when $n = 15, 16, 20, 24, 30$; $\cos(2\pi/n)$ a quartic surd.

DISCUSSIONS.

Professor Hathaway obtains an integral reduction formula which includes as special cases the formulas usually given in works on the integral calculus.[2] In a note following the paper it is shown how the formula may be regarded as a transform of one of these special cases. This, of course, does not prevent it from being also a generalization.

A General Type of Reduction Formula.

By A. S. Hathaway, Rose Polytechnic Institute.

Let X, Y, Z be functions of a single variable, such that

$$AX^2 + BY^2 + CZ^2 = 0, \tag{1}$$

A, B, C being constants. By differentiation,

$$AXdX + BYdY + CZdZ = 0; \tag{2}$$

and from (1) and (2)

$$\frac{YdZ - ZdY}{AX} = \frac{ZdX - XdZ}{BY} = \frac{XdY - YdX}{CZ} = dT \text{ (for brevity).} \tag{3}$$

The general integral considered is

$$\beta(L, M, N) = \int X^L Y^M Z^N dT, \text{ where } L + M + N + 1 = 0. \tag{4}$$

The integral is homogeneous of order zero. It is not altered by substituting for X, Y, Z any variable common multiples of them, VX, VY, VZ. These also satisfy (1) and (3).

Further the integral is not altered by permuting X, Y, Z, concurrently with A, B, C and L, M, N; except that the sign is changed when the permutation is not cyclic (from the definition of dT). This means that any formula of expansion of the integral in powers of two bases is equivalent to six different formulas

[1] Dedekind's proof for the general case (n composite) is given in H. Weber, *Lehrbuch der Algebra*, volume 1 (Braunschweig, 1898), p. 596, or *Traité d'Algèbre Supérieure* (French translation by J. Griess, Paris, 1898), p. 636. For proof by Arndt, see P. Bachmann, *Die Lehre von der Kreistheilung* (Leipzig, 1872), p. 38.

[2] W. A. Granville, *Elements of the Differential and Integral Calculus*, Boston, 1911, pp. 350-360.

obtained by the six different permutations of concurrent parts. We propose to establish such a formula, and show the great variety of its applications.

We have $d(X^{L+1}Y^{M-1}Z^{N+1}) = X^LY^{M-2}Z^N[(L+1)YZdX + (M-1)ZXdY + (N+1)XYdZ] = X^LY^{M-2}Z^N[(M-1)CZ^2 - (N+1)BY^2]dT.$

Integrating and solving for $\beta(L, M, N)$,

$$\beta(L, M, N) = \frac{X^{L+1}Y^{M-1}}{B(-N-1)Z^{-N-1}} + \frac{-C(M-1)}{B(-N-1)}\beta(L, M-2, N+2). \quad (5)$$

The reduced integral is of the same form, since one exponent is increased as much as the other is decreased; and (5) applies to it, with proper change of exponents, M to $M-2$, N to $N+2$. This can be continued in general indefinitely. The resulting formula[1] is:

$$\begin{aligned}\int X^LY^MZ^NdT = {} & \frac{X^{L+1}Y^{M-1}}{B(-N-1)Z^{-N-1}} + \frac{-C(M-1)X^{L+1}Y^{M-3}}{B^2(-N-1)(-N-3)Z^{-N-3}} + \cdots \\ & + \frac{(-C)^{k-1}(M-1)(M-3)\cdots(M-2k+3)X^{L+1}Y^{M-2k+1}}{B^k(-N-1)(-N-3)\cdots(-N-2k+1)Z^{-N-2k+1}} \\ & + \frac{(-C)^k(M-1)\cdots(M-2k+1)}{B^k(-N-1)\cdots(-N-2k+1)}\int X^LY^{M-2k}Z^{N+2k}dT. \quad (6)\end{aligned}$$

If M were an odd positive integer, we would have an algebraic integral of k terms, where $k = (M+1)/2$, since a factor zero appears in the coefficient of the reduced integral. We could not carry the reduction so far, however, if at the same time N were an odd negative integer not numerically larger than M, as a zero factor would appear first in the denominator, and the series must be stopped before that occurs. The effect of a possible zero factor in stopping reduction may be thus stated: *an odd exponent cannot be reduced so that its sign changes.* If positive, it must stop at 1; if negative, at -1. We have: *the integral $\beta(L, M, N)$ is algebraic in X, Y, Z when (and only when) one exponent is an odd*

[1] Equation (6) may be written:

$$\beta(L, M, N) = X^{L+1}\left\{\frac{-C\cdot M-1\cdot Y}{B\cdot -N-1\cdot Z}\,k\right\} + \left\{\frac{-C\cdot M-1}{B\cdot -N-1}\,k\right\}\beta(L, M-2k, N+2k)$$

The bracketed terms are "indices" for corresponding sums and products, and k is the number of integrated terms.

By permuting, Y and Z become *any two of the variables*, M and N, their exponents, B and C, their coefficients in (1), *with the factor* -1 *before that coefficient which follows the other in the cyclic order ABCA.*

In this notation, the illustrative example given later is,

$$\beta(4, 6, -11) = X^5\left\{\frac{3\cdot5\cdot Y}{2\cdot10\cdot Z}\,3\right\} + \left\{\frac{3\cdot5}{2\cdot10}\,3\right\}\beta(4, 0, -5).$$
$$\beta(4, 0, -5) = Y\left\{\frac{-3\cdot3\cdot X}{-7\cdot4\cdot Z}\,2\right\} + \left\{\frac{-3\cdot3}{-7\cdot4}\,2\right\}\beta(0, 0, -1).$$

A complete reduction may therefore be written in this notation, given variables, exponents, and coefficients. Preferably reduce the two *numerically greatest* exponents, one of which is positive, the other negative, by (4).

positive integer, the other two being any numbers subject to (4), *rational or irrational, that are not both negative odd integers.*

When one exponent is a positive odd integer and the other two negative odd integers, the integration may be made with a logarithmic term. For the exponents reduce to 1, -1, -1; and

$$\int \frac{XdT}{YZ} = \frac{1}{A}\int\left(\frac{dZ}{Z} - \frac{dY}{Y}\right) = \frac{1}{A}\log\frac{Z}{Y}.$$

Hence $\beta(L, M, N)$ *can always be integrated when one exponent is a positive odd integer.*

The integration is possible if one exponent be an odd negative integer and the other exponents rational numbers. For integral exponents (excluding an odd positive one, already considered), by (4), two must be even and one odd negative, so that the exponents reduce to -1, 0, 0. In this case, $\int dT/X = \int (YdZ - ZdY)/(AX^2) = -\int (YdZ - ZdY)/(BY^2 + CZ^2) = \int dx/(Bx^2 + C)$, where $x = Y/Z$ (an anti-tangent or logarithm as the signs of B, C are alike or not).

With one odd negative and two fractional exponents, the exponents reduce to -1, m/n, $-m/n$, by (4), and we have the preceding form multiplied by $x^{m/n}$ ($x = Y/Z$), which integrates as a rational fraction by the substitution $x = z^n$.

It seems probable that $\beta(L, M, N)$ is not integrable in elementary functions when no exponent is an odd integer. By (4), there must be an odd exponent when all are integers.

In reducing by (6), it is first necessary to determine bases, coefficients, and exponents. Powers of X, Y, Z must be factors of the given differential to be integrated, their relation (1) being found by inspection. Then dT is computed by (3), and the quotient of the differential by dT must be, to a constant factor, $X^LY^MZ^N$, where $L + M + N = -1$. If there are only two functions X, Y, with a relation $AX^2 + BY^2 + C = 0$, then $Z = 1$, $dT = dX/BY$, and the relation (4) is no restriction, but only a determination of the exponent of $Z = 1$, required in (6).

Of the six possible permutations of formula (6), only two can be used to reduce exponents to their smallest values (between 1 and -1). Namely, there must be a positive exponent in place of M, and a negative one in place of N (only two, since by (4) two exponents are positive and one negative, or vice versa, or all are negative and reduced). Generally either pair of exponents may be used to reduce its numerically smallest exponent (leaving the third unchanged), and in the reduced integral there is only one pair for further reduction, and this pair leaves the exponent just reduced unchanged.

For example, find $\int x^4(3 + 4x^2)^{5/2}(2 + 5x^2)^{-6}dx$. The variables are $X = x$, $Y = (3 + 4x^2)^{1/2}$, $Z = (2 + 5x^2)^{1/2}$, with $7X^2 + 2Y^2 - 3Z^2 = 0$. $dT = dx/(YZ)$, and the given integral is therefore $\int X^4Y^6Z^{-11}dT = \beta(4, 6, -11)$, since $4 + 6 - 11 + 1 = 0$.

To apply (6) we have a choice between the pairs of exponents (4, -11) and

(6, − 11). Taking the latter, we have

$$\beta(4,\,6,\,-\,11) = X^5\left[\frac{Y^5}{2\cdot 10\cdot Z^{10}} + \frac{3\cdot 5Y^3}{2^2\cdot 10\cdot 8Z^8} + \frac{3^2\cdot 5\cdot 3Y}{2^3\cdot 10\cdot 8\cdot 6Z^6}\right] + \frac{3^3\cdot 5\cdot 3\cdot 1}{2^3\cdot 10\cdot 8\cdot 6}\beta(4,\,0,\,-\,5).$$

For next reduction we have only the pair (4, − 5).

$$\beta(4,\,0,\,-\,5) = Y\left[\frac{X^3}{-\,7\cdot 4Z^4} + \frac{-\,3\cdot 3X}{7^2\cdot 4\cdot 2Z^2}\right] + \frac{3^2\cdot 3\cdot 1}{7^2\cdot 4\cdot 2}\beta(0,\,0,\,-\,1),$$

$$\beta(0,\,0,-\,1) = \int dT/Z = \int\frac{XdY - YdX}{-\,3Z^2} = \int\frac{XdY - YdX}{7X^2 + 2Y^2} = \frac{1}{\sqrt{14}}\tan^{-1}\frac{\sqrt{2}Y}{\sqrt{7}X}.$$

The solution is complete on substituting values found.

Note the following particular form:

$$\tfrac{1}{2}\int(a + bx)^{(L-1)/2}(a' + b'x)^{(M-1)/2}(a'' + b''x)^{(N-1)/2}dx = \beta(L,\,M,\,N), \qquad (L + M + N + 1 = 0).$$

$$X^2 = a + bx, \qquad Y^2 = a' + b'x, \qquad Z^2 = a'' + b''x,$$
$$(a'b'' - a''b')X^2 + (a''b - ab'')Y^2 + (ab' - a'b)Z^2 = 0.$$
$$dT = dx/(2XYZ).$$

Of other forms, note:

$$\int\sin^L x\cos^M x\,dx; \qquad X = \sin x, \qquad Y = \cos x, \qquad Z = 1,$$
$$X^2 + Y^2 - Z^2 = 0, \qquad dT = dx.$$
$$\int\tan^L x\sec^{M+1} x\,dx; \qquad X = \tan x, \qquad Y = \sec x, \qquad Z = 1.$$
$$-\,X^2 + Y^2 - Z^2 = 0, \qquad dT = \sec x\,dx.$$

In each of these the exponent of Z is $N = -\,L - M - 1$.

Note by the Editor.

In the application of the reduction formula, the problem is to recognize the four variables X, Y, Z, and T. It therefore suggests itself that we should look a little into the meaning of these variables in the integral. The equations (1) and (3) show that only two of the four are really independent, so that the formula in question ought to be transformable into a canonical form containing only one arbitrary function. But actually the situation is simpler still, on account of the homogeneity to which the author has alluded, which makes the ratios of X, Y, Z the only functions of importance.

In dealing with real quantities, we must be able to throw (1) into some such form as $(Z/c)^2 = (X/a)^2 + (Y/b)^2$. We may then use two variables Z and θ, so that $X = (a/c)Z\cos\theta$, and $Y = (b/c)Z\sin\theta$. (Geometrically, we are dealing with points (X, Y, Z) on a quadric cone, and θ is the eccentric angle in a principal elliptic section.) It follows that $dT = c^2(XdY - YdX)/Z = abZd\theta$.

Now it must be supposed that we integrate along some path lying on the cone and given by an equation $Z = F(\theta)$. But the homogeneity principle prepares us to see Z disappear from the integral, as indeed it does, the result of the transformation being $\int X^L Y^M Z^{-L-M-1} dT = a^{L+1} b^{M+1} c^{-L-M} \int \cos^L \theta \sin^M \theta \, d\theta$.

Thus every integral of the type in question is transformable into this one particular form. In the author's first example, the required transformation is

$$\cos \theta = \sqrt{\frac{7}{3}} \cdot x(2 + 5x^2)^{-1/2}.$$

It may be added that, if we possess the general reduction formula developed above, nothing is gained by this transformation from the point of view of integration. In fact the reduction formula is exactly the same before and after transformation.

RECENT PUBLICATIONS.

REVIEWS.

The Principles of Geometry. By H. F. BAKER. Volume 1, Cambridge University Press, 1922. 182 pp. Price 12 shillings.

That prince of teachers, the late Jules Tannery, once wrote in the introduction to a book by one of his former pupils: "Un petit livre est rassurant."

When a book is not only short but well printed, with wide margins and in a flowing style of the King's English, the reassuring impression is much strengthened. But the reader who takes up the work before us under these pleasant impressions, with the idea that in spite of a lack of any special preparation in the way of familiarity with the subject matter and point of view, he is going in a few hours to reach the real substance, this reader will have a very prompt chance to guess again.

It will take a much greater convulsion than the late World War to stop the output of books on Projective Geometry, especially from English writers. Yet so far, in England, they have clung to the tradition of Cremona and Stephen Smith, taking the metrical definition of cross ratios as fundamental, regardless of the fact that Klein proved half a century ago that the cross ratio can be reached by purely descriptive processes, and that Continental and American writers have been doing this in their texts for a generation. Sooner or later some Englishman was bound to fall into line, and it is a subject for satisfaction that the first should be such a distinguished scholar as Baker. The reason for the delay is doubtless this, that English boys are drilled in a subject of which Americans have scarcely heard the infelicitous name, Geometrical Conics. The metrico-projective treatment flows from this in the most natural manner. Moreover the present writer does not say that he is actually writing a book on projective geometry, and he starts his second chapter with a discussion of the descriptive properties of a limited region. But as soon as he has strengthened this to the point where it can stand on its legs, he adjoins to the universe of discourse ideal,

or, as he says, "postulated" points, until he has the whole projective domain once more available.

The book has one most unusual characteristic, informality. The writer says in the fourth line of the preface: "It (the book) assumes those relations of position for points, lines and planes which, furnished with a pencil, a ruler, some rods and some string, a student may learn by drawing diagrams and making models."

There is no question of the independence of axioms: the writer states what he presupposes in the following fashion (pp. 4, 5): "Through an arbitrary point there pass an infinite number of lines, of which one passes through any other arbitrary point. Thus a line is determined uniquely by any two points. The line contains an infinite number of points besides these two determining ones, and is determined by any two of these. Through any given line there pass an infinite number of planes, of which one passes through any point not lying on the given line. Thus a plane is determined uniquely by any three points which do not lie in line. The plane contains an infinite number of points besides the determining ones, and is determined by any three of these which do not lie in line. The plane entirely contains the line which is determined by any two of its points. Thus if two lines have a common point, and we take two other points, one on each of the lines, the plane determined by the three points contains both the lines, that is, two lines with a common point determine a plane, containing both lines. It is also true that two lines which are in the same plane have a common point. We have said that there is an infinite number of planes, all containing the points of a given line; it is also true that any two planes have common points lying on a line, or that any two planes intersect in a line."

This casual method of statement is entirely characteristic. The writer seems to have an ingrained dislike for type display. He will head a long paragraph "So and So's Theorem" and put the actual statement, with or without italics, at such point in the argument as pleases him. The usual assortment of labels, Axiom, Postulate, Assumption, Restriction, Theorem, he calmly ignores. At times one is tempted to resent this, and exclaim irritably: "Why in the name of common honesty doesn't the good man tell us clearly once for all what he assumes, and what he proves?"

Fair and softly, dear reader, fair and softly, you won't get anywhere by storming at him. If you realize that this is a thoroughly scholarly work, and read every word, you will find that everything necessary is there. If you mistake it for a newspaper, and try to read nothing but the headlines, that is not his fault.

Here is a case in point. On p. 11 we find the usual quadrilateral construction for the harmonic point-set. A wicked gleam will come into the eye of the malevolent reader, as he exclaims: "Aha, how do we know that this yields anything new? How do we know that the diagonals of a complete quadrilateral are not concurrent? Does the writer expect us to use pencils and rods and string?"

Not in the least; when he gets good and ready, namely three pages later, he calmly says: "But we shall, in accordance with a general assumption referred to in the beginning, assume provisionally that the point D does not coincide with the point C."

The first chapter deals with the general projective geometry, where no mention is made of order, or any equivalent characteristic of the real domain. We have the usual concurrence, collinearity and coplanarity theorems, which the writer well describes as "theorems of incidence." A good deal of attention is paid to "related ranges" of points. This means two ranges that are in perspective with the same third range. It is shown that two ranges which are related to a third are related to one another. Thus, when two one-dimensional forms are the first and last in a long string of projections and intersections, they may be connected by a single piece of machinery with only two joints. Next, of course, comes the fundamental theorem of projective geometry, whereby the projective relation between two one-dimensional forms is completely determined by the fate of three elements. The writer turns this over in every possible way, showing that it is equivalent to any one of four different theorems. The one which he prefers, and which constitutes the central assumption for this whole chapter, is Pappus's theorem, or Pascal's, where the conic is two lines. The preference for this particular form of statement is characteristic of the present work. This discussion of the reduction of perspectivities and the fundamental theorem is very careful; truth compels us to add that it is not short; the two require twenty-one pages.

All writers on the foundations of geometry, sooner or later, draw on arithmetic for aid; the present writer does so at the point we have now reached. Large letters are used to denote points, small letters denote a system of numbers about which we make all of the usual assumptions for addition and multiplication, *except the commutative law for the latter*. If the point P be defined as the same as the point rP, then the points of the line determined by two given points will be the system of points linearly dependent on them in the present symbolism, and in fact, it is easily shown that the present system of linear dependence completely covers the former theorems of incidence.

The correlation of arithmetical and geometrical operations is next developed, very satisfactorily, on the following lines:

(*a*) The relation between the numbers r and $-r$. The points $A + rB$ and $A - rB$ shall be harmonic conjugates with regard to the points A and B.

(*b*) The relation between the numbers r, s, and $r + s$. The point B shall be one vertex of a complete quadrangle, $A + rB$ and $A + sB$ shall be the intersections of a transversal through B with one pair of opposite sides, A and $A + (r + s)B$ shall be the intersections of this transversal with the remaining pair of sides. It is easy to show that this construction is independent of the choice of the quadrangle, and that addition is commutative and associative.

(*c*) The relation between the numbers r, s, and rs. A transversal shall meet one pair of opposite sides of a complete quadrangle in A and B, a second pair in $A + rB$ and $A + sB$, while it meets a third pair in $A + B$ and $A + rsB$. This will make $rs = sr$, when, and only when, the construction yields a unique result, and this unicity of result is shown, in turn, to depend on Pappus's theorem. We thus reach in a new form a result, first exhibited by Hilbert, namely, that the

independence of Pappus's theorem from the other theorems of incidence may be shown by the existence of number systems where multiplication is not commutative, the calculus of quaternions being the classical example.

The second chapter of our book takes a fresh start, dealing with a much more restricted domain, namely, a limited region of real space. Peano's axiom which requires that a line that meets one side of a triangle, while it meets a second side produced, shall meet the third side, is very fundamental. This is really a most useful assumption for it accomplishes two distinct ends, it gives sufficient conditions for the intersection of coplanar lines, and for the projection of betweenness into betweenness. Betweenness is, in fact, a very fundamental relation in this sort of development, and we have at the proper moment a continuity axiom based, as usual, on a Dedekind cut among the points of a line-segment. With continuity we can do many things: prove Pappus's theorem, establish connection with the algebra of real numbers, etc. But our author does not remain content with his limited space but presently fills it out to the complete space of projective geometry, introducing ideal elements by a device first employed, we believe, by Pasch. Suppose, in fact, that in this limited space we have a system of lines of which at least one goes through every point, and of which each two are coplanar. An example of such a system is the assemblage of all lines through a point of the region. When the system is not so constituted, *i.e.*, when no two intersect in the region, then we say that they determine an ideal or postulated point which lies on all of them. Ideal lines and planes are then introduced in a similar fashion, and the whole of projective space is presently available. One rather wonders why Baker did things in just this way, why he did not follow in the real domain the plan already pursued in the general domain of assuming at the outset the intersection of coplanar lines, and postulating the laws of separation of four elements of a one-dimensional fundamental form. The explanation is, perhaps, to be found on p. 2: "In geometry, as applied to the external world, we cannot but be conscious for instance, in dealing with the points of a line, of the difference between the points which we regard as accessible, and those which we regard as the others." This would seem to mean that historically, and psychologically, geometry begins with real accessible points, and that it is natural to base the logical structure on these also. Yet the other method of approach is very neat.

The third chapter is in the nature of reconciliation of the two that preceded it. It is shown how the assumption of complex points, *i.e.*, points corresponding to complex numbers, rounds out and completes the real space of the second chapter. There is, here, a real obscurity arising from the writer's reluctance to state his assumption explicitly. We are told on p. 141 that Ch. III is a continuation of Ch. I. That suggests that the assumptions made in Ch. I and not those in Ch. II are valid. Then we read on p. 152: "Conversely, two spaces of a like number of dimensions, with a point-to-point correspondence of such a kind that to every incidence of elements in one space corresponds a precisely similar incidence of elements in the other, are necessarily related (projective) as we see by recalling the construction of related ranges."

This amounts to saying that every transformation that carries a point to a point and a plane to a plane, or a line to a line, is a projective transformation; true in the real domain, but untrue in the complex one. Witness the transformation which consists in interchanging conjugate imaginary elements.

The final section shows how complex geometry may be built out of real geometry by defining complex elements as groups of real ones. The first successful development of this kind was given by Von Staudt, his method being to define a complex point as an elliptic involution of real collinear points, together with a sense of description. Baker, however, follows a later method devised by Klein, where complex points are defined by cyclic projectivities among real ones. His treatment is based very directly on the exposition of Vahlen, and does not seem to us as clear as the original, but it is fair to say that he explicitly announces that all he says on this subject should be regarded as provisional.

What shall we say of the book as a whole? Is it as brilliant and stimulating as Hilbert's *Grundlagen der Geometrie?* No, it is not. Is it as searching and profound as Veblen and Young's *Projective Geometry?* No one could make such a claim. Is it as crisp and clear-cut as Vahlen's *Abstrakte Geometrie?* Such is certainly not the case. Does it afford as direct and natural an approach to the standard theorems as does Enriques' *Geometria Proiettiva?* Such is not our candid opinion. What then? We must judge a book finally by what the author sets out to do. We turn back to the preface to see what he expects of his book: "It seeks to set these relations (of points and lines and planes) in an ordered framework of deduction, gradually rendered comprehensive and precise enough to include all subsequent theory. To this end it puts aside at first most of the intricate details which make up the burden of what is generally called geometry. . . . Experience has shown that many students, especially of the class who look forward to becoming Engineers or Physicists, to whom the geometry of the usual text-books is tiresome, find such a course stimulating and easy, when the matter is properly presented to them. The mathematician who has followed such a course will find that he has no cause to think that the wrong things have been presented to him."

So the author wishes us to judge his book as an introduction, especially suited to those students who do not intend to specialize in mathematics. Our judgment must finally depend on the extent to which we agree with his educational theory. If he be correct in his opinion that the usual subject matter should be replaced by something more generalized, more abstract, capable of opening wider horizons, why then he deserves the great credit that should be awarded to every real pioneer in education. On the other hand, if his first assumption be essentially unsound, if truth be on the side of the older theory, which believed that the human mind proceeded naturally from the concrete to the abstract, and insisted that the natural approach to geometry is through the clean-cut, definite theorems of elementary metrics, why then we still welcome him as the first of his nation to write a scholarly text on a topic of recognized importance and value. May he speedily be followed by others, anxious to continue the good work.

JULIAN L. COOLIDGE.

NOTES.

In *Oversigt over det Kongelige Danske Videnskabernes Selskabs Forhandlinger, Juni 1919–Maj 1920,* Copenhagen, 1920, there is a sketch of H. G. ZEUTHEN (1839–1920) by A. C. Juel, pp. 67–77, accompanied by a splendid portrait-plate.

G. H. HARDY's remarkably interesting biography of Ramanujan, to which we have already referred more than once (*1921,* 458; *1922,* 19), is reprinted in *The Journal of the Indian Mathematical Society,* June, 1922.

In *The Cyclopædia of American Biography, new enlarged edition of Appleton's Cyclopædia of American Biography,* volume 9 (New York, The Press Association Compilers, Inc., 1922), there are biographies of EMORY MCCLINTOCK (see *1916,* 271; *1918,* 228), page 15, E. C. PICKERING (*1919,* 134), 329, and T. J. J. SEE, 363.

In *The Encyclopaedia and Dictionary of Education* edited by Foster Watson, 4 volumes (paged consecutively), London, Pitman, 1921–1922, the following mathematical topics, among others, are treated: Teaching of algebra, Alligation, Approximation, Arabic notation, Arithmetic, Averages, Axiom, Binomial theorem, Calculating machines by G. H. BRYAN, Casting out nines, Checks on computation, Cocker's arithmetic, Teaching of commercial arithmetic, Teaching of conic sections, Convergence, Decimal coinage, Finger manipulation, Non-Euclidean geometry, Teaching of geometry, Cartesian geometry, Graphs and graphic curves, The Grube method, Logical and historical schools of mathematics (modern) by P. E. B. JOURDAIN, Teaching of machine drawing, Significance of mathematical theory by G. H. BRYAN, History of mathematics by W. W. R. BALL, Metric system, Napier's rods (or Napier's bones), Teaching of projective geometry by J. L. S. HATTON, Drawing of sections, Tillich's bricks, and Trigonometry.

The following quotation from an article by Professor JAMES HENRY BREASTED of Chicago University is of interest to students of the history of mathematics. Professor Breasted refers indirectly to an article by Professor L. C. KARPINSKI, "Algebraical developments among the Egyptians and Babylonians", which appeared in this MONTHLY, *1917,* 257–265. The quotation is taken from an article on The Edwin Smith Papyrus which appears in the current number of the *Bulletin of the New York Historical Society.*

> "The current conclusion regarding the mind of the ancient Egyptian, a conclusion in which I have myself heretofore shared, has been that he was interested in scientific principles, if at all, solely because of the unavoidable necessity of applying them in practical life; that if he discussed the superficial content of a many-sided geometrical figure or the cubical content of a hemisphere it was because he was obliged to measure fields for taxation purposes and to compute the content of granaries. In the field of Egyptian mathematics Professor Karpinski of the University of Michigan has long insisted that the surviving mathematical papyri clearly demonstrate the Egyptians' scientific interest in pure mathematics for its own sake. I have now no doubt that Professor Karpinski is right, for the evidence of interest in pure science, as such, is perfectly conclusive in the Edwin Smith Medical Papyrus."

The first edition of the first volume of ARNE FISHER, *The Mathematical Theory of Probabilities and its Application to Frequency Curves and Statistical Methods,* appeared in 1915 (New York, Macmillan). The second edition of this volume (1922), enlarged by more than one hundred pages of new material, treats

of Mathematical probabilities, frequency curves, homograde and heterograde statistics. Mr. Fisher proposes to treat the methods of Pearson, Edgeworth, Kapteyn, Bachelier, and Knibbs, and to show their relation to Laplace's theory, in the second volume. Macmillan announced for publication in August, 1922, Arne Fisher, *An Elementary Treatise on Frequency Curves and their Application to the Construction of Mortality Tables* (225 + 15 pages).

We have already listed (*1920*, 218; *1921*, 269) fifteen of the volumes of the great edition of Euler's *Opera Omnia* now being issued under the auspices of the Swiss Society of Naturalists. Three more volumes have recently been published: (*a*) series I, volume 6: *Commentationes algebraicae ad theoriam aequationum pertinentes*, edited by F. RUDIO, A. KRAZER and P. STÄCKEL, 1921; (*b*) series I, volume 7: *Introductio in Analysin Infinitorum*, tome primus, edited by F. RUDIO, 1922; (*c*) series II, volume 14: *Neue Grundsätze der Artillerie . . . aus dem Englischen des . . . Benjamin Robins übersetzt und mit den nöthigen Erläuterungen und vielen Anmerkungen versehen von Leonhard Euler*, edited by F. R. SCHERRER, 1922.

Many readers will recall *The First Book of Geometry* (London, Dent, 1905; 16 + 222 pp.) written by GRACE CHISHOLM YOUNG, with the assistance of her husband W. H. YOUNG, for the use of her children. An attractive German edition by S. and F. Bernstein was issued by Teubner in 1908 (16 + 239 pp.). In 1921 a Hebrew translation by Dr. Elias Olschwanger was published by Wostock of Dresden (14 + 187 pp.; price, in Germany, bound, 67 marks).

This work is often referred to in connection with another publication appearing about the same time, namely C. A. LAISANT'S notable little book: *L'Initiation mathématique, ouvrage étranger á tout programme; dédié aux amis de l'enfance* (Paris, Hachette, 1906). Among various translations of it into other languages, an English edition appeared in 1913 (with the title *Mathematics*, London, Constable). Compare *1920*, 339, 498. W. H. Young is now professor of mathematics at University College, Aberystwyth, Wales.

The first half of the second edition of the Geometry volume of the admirable and most useful *Repertorium der höheren Mathematik*, edited by H. E. TIMERDING, was published by Teubner in 1910. This contained 24 chapters devoted to foundations and plane geometry. The second part on space geometry appeared in February, 1922 (chapters 25–42, pages 12 + 537–1165; price, in Germany, 96 marks, bound). O. STAUDE, of Rostock, is author of the first five chapters on surfaces of the second order and gauche curves of the third and fourth order. Six chapters are written by L. BERZOLARI, of Pavia, on the general theory and geometry of algebraic surfaces, surfaces of the third order, general theory of algebraic space curves, and special algebraic curves. TIMERDING, of Braunschweig, contributes the chapters on special surfaces of the fourth order, and on rational transformations of curves. K. ZINDLER, of Innsbruck, wrote the chapter on line geometry, and E. SALKOWSKI, of Hannover, contributed the chapters on space curves and developable surfaces, general theory of surfaces, and special classes of surfaces and systems of surfaces. The "Register" occupies pages 1134–1161, and "Berichtigungen und Zusätze," pages 1162–1165.

One of the most useful reference works for the mathematician is J. C. Poggendorff's *Biographisch-literarisches Handwörterbuch zur Geschichte der exacten Wissenschaften* of which volumes 1–2, covering the literature through 1857, appeared in 1863; the third volume, for the period 1858–1882, edited by B. W. Feddersen and A. J. von Oettingen was published in 1898; and the fourth volume, 1904, edited by Oettingen, 1883–1903. The manuscript of a fifth volume for 1904–1922 is now in course of preparation. Appeal has been made by the editors for information to be sent to them, at the earliest possible moment, addressed: Poggendorff-Büro, Beethovenstrasse 6 (Universitäts-Bibliothek), Leipzig, Germany. The information requested is as follows: surname and first names really used; "consanguineous relations of interest to science"; degrees and when and where obtained; address; year, month, day and place of birth; present and former positions with dates of appointment; "Emeritus? Retired? Since when?" Are you editor of a periodical? instruments invented or special achievements and year of publication or finishing; titles of books published by the author, either alone or in collaboration with other, stating year and place of publication of each work, its size as well as the total number of pages; titles of those articles and papers in the domain of the exact sciences which have appeared in periodicals or proceedings of Societies since 1903,—in each case indicating the periodical, the volume, the year of publication and total number of pages of the article.

ARTICLES IN CURRENT PERIODICALS.

Jahresbericht der deutschen Mathematiker-Vereinigung, volume 30, nos. 9–12, 1921: "Johannes Thomae" by H. Liebmann, 133–144; "Über paraboloidische Flächen" by P. Franck, 145–151; "Das Problem der 36 Offiziere" by H. F. MacNeish, 151–153; "Über stetige Funktionen mit überalldicht divergierender Fourierreihe" by L. Neder, 153–155; "Eine algebraische Behauptung von Gauss. II" by A. Loewy, 155–158; "Das Abelsche Gleichungsproblem bei Euler" by S. Breuer, 158–169; "Über die Hardy-Littlewoodschen Arbeiten zur additiven Zahlentheorie" by E. Landau, 179–185.

Journal of the Indian Mathematical Society, volume 14, February, 1922: "Focus-acnodal cubic" by P. Hemraj, 3–13; "On a particular type of similar triangles" by F. H. V. Gulasekharan, 14–18.

Mathematical Gazette, volume 11, March, 1922: "The structure of the atom" by J. W. Nicholson, 37–42; "Vectors" by C. Godfrey, 43–44—May: "Differential equations in mechanics and in physics" by A. R. Forsyth, 73–81 [Presidential address.]

Mathematics Teacher, volume 15, March, 1922: "The psychology of the equation" by E. L. Thorndike, 127–136; "Reaction vs. radicalism in the teaching of mathematics" by G. W. Myers, 137–146; "The definition of similarity" by G. W. Evans, 147–151; "The place of elementary calculus in senior high school mathematics" by N. B. Rosenberger, 152–156; "Experimental geometry" by G. A. Harper, 157–163; "How can I bring the soul of mathematics to my pupils?" by A. H. Huntington, 164–171; "Papers by pupils of the plane geometry classes of Fullerton Union High School" by Lena E. Reynold, 172–181; Discussion, 182–184; New books, 185–188; News and notes, 189–190—April: "Functionality in mathematical instruction in schools and colleges" by E. R. Hedrick, 191–207; "Calculus for schools" by W. H. Tyler, 208–211; "The psychology of problem solving" by E. L. Thorndike, 212–227 (to be concluded); "Cultural value of mathematics" by M. G. Kane, 228–236; Discussion, 237–240; News and notes, 241–245; "The Chicago meeting of the National Council of Teachers of Mathematics," 246–251; Research department, 251–252—May: "The psychology of problem solving" (conclusion) by E. L. Thorndike, 253–264; "The teaching of beginning geometry" by A. J. Schwartz, 265–282; "The origin of mathematics—a first lesson in secondary mathematics" by W. Betz, 283–293; "Robert Recorde" by F. Cajori, 294–302; "A list of reference books and magazines for teachers of mathematics" by W. D. Reeve, 303–307; Discussion, News and notes, etc., 307–316.

MATHEMATISCHE ZEITSCHRIFT, volume 12, nos. 3–4, 1922: "Some problems of 'Partitio Numerorum': IV. The singular series in Waring's problem and the value of the number $G(k)$" by G. H. Hardy and J. E. Littlewood, 161–188 (see *1921*, 219); "Über das Wachstum der Potenzreihen in ihrem Konvergenzkreise. I" by O. Töplitz, 189–200; "Zum Waringschen Problem" by E. Landau, 219–247—Volume 13, nos. 1–2, 1922: "Bemerkungen zu einem Satz von J. H. Grace über die Wurzeln algebraischer Gleichungen" by G. Szegö, 28–55; "Neue Lösungen der Einsteinschen Gravitationsgleichungen. A. Das Feld in der Umgebung eines langsam rotierenden kugelähnlichen Körpers von beliebiger Masse in 1. und 2. Annäherung" by R. Bach, 119–133; "Neue Lösungen der Einsteinschen Gravitationsgleichungen. B. Explizite Aufstellung statischer axialsymmetrischer Felder" by R. Bach (Mit einem Zusatz über das statische Zweikörperproblem by H. Weyl), 134–145; "Arithmetical equivalents for a remarkable identity between theta functions" by E. T. Bell, 146–152.

MATHESIS, volume 36, January, 1922: "Préface" by J. Neuberg and A. Mineur, 5–8 ["Il y a quarante et un ans, paraissait le premier numéro d'un modeste recueil mathématique à l'usage des écoles spéciales et des établissements d'instruction moyenne, dirigé par deux anciens élèves de l'École normale des sciences avec la collaboration de plusieurs professeurs belges et étrangers; recueil modeste par le nombre de pages, mais bientôt considérable par le nombre et la valeur des articles publiés. Son programme correspondait essentiellement à la partie élémentaire des mathématiques supérieures et à la partie supérieure des mathématiques élémentaires. . . . Le succès vint rapidement, dépassant les plus belles espérances des fondateurs; d'année en année, les collaborateurs furent plus nombreux; le programme de 1881 s'élargit de manière à comprendre tout l'enseignement mathématique des facultés des sciences et le nombre de pages que l'on avait promis de publier annuellement fut doublé. . . . dans la préface de 1911, l'histoire des trente premières années de *Mathesis* est relatée de cette manière: . . . Cinq ans après avoir écrit ces lignes, les épreuves infligées à la Belgique forcèrent les fondateurs de *Mathesis* à suspendre la publication du journal; . . . *Mathesis* restera fidèle à son passé; elle reprendra la tâche interrompue, contribuer au progrès de l'enseignement des mathématiques."]; "Paul Mansion (1844–1919)" by A. Demoulin, 9–12 [with portrait]; "Géométrie et mécanique" by J. Neuberg, 13–15; "Sur les contacts des sphères tangentes à quatre plans" by V. Thébault, 16–18; "Sur l'inversion et sur une surface cubique à quatre points doubles" by L. Godeaux, 19–23; "Archimède. À propos d'un ouvrage récent" by H. Bosmans, 24–27 [review of P. V. Eecke, *Les Œuvres complètes d'Archimède traduites du Grec en Français avec une Introduction et des Notes* (Paris and Bruxelles, 1921)]—February: "Sur les involutions cubiques *sibi* conjuguées" by C. Servais, 45–49; "Bibliographie du triangle et du tétraèdre" by J. Neuberg, 50–52 (to be continued); "Sur une généralisation d'un théorème de Droz Farny" by R. Goormaghtigh, 52–55; "Analysis situs. Une démonstration du théorème de Petersen" by A. Errera, 56–61—March: "Sur l'hypocycloïde à trois rebroussements" by P. De Lepiney, 77–81; "Relations remarquables entre cinq sphères" by R. Goormaghtigh, 81–86; "Sur les sphères podaires" by C. Servais, 87–89; "À propos d'une construction du point de Feuerbach" by A. M(ineux), 89–94—April: "Bibliographie du triangle et du tétraèdre" (continued) by J. Neuberg, 117–120; "Une surface réglée du huitième ordre" by M. Stuyvaert, 121–124; "Sur l'ellipse et le cercle de Nagel d'un triangle" by J. Neuberg, 125–128 (to be continued); "Formule relative aux combinaisons avec répétition" by M. Sterkens, 129–130—May: "Bibliographie du triangle et du tétraèdre" (continued) by J. Neuberg, 161–164; "Sur l'orthopole et sur la cubique des dix-sept points" by R. Goormaghtigh, 164–166; "Remarques sur l' 'Arithmétique' de Simon Stevin" by H. Bosmans, 167–174 [with facsimile of two pages]; "Sur l'ellipse et le cercle de Nagel d'un triangle" (continued) by J. Neuberg, 175–177; "Calcul d'un emprunt par obligations, avec lots" by M. Sapin, 178–179.

REVUE DE MATHÉMATIQUES SPÉCIALES, volume 32, April, 1922: "Note sur la transformation conforme" by M. Lapointe, 145–146.

REVUE SCIENTIFIQUE, volume 60, April 22, 1922: "Les professeurs de mathématiques du Collège de France: Humbert et Jordan; Roberval et Ramus" by H. Lebesgue, 249–262; "Débat sur la relativité" by M. Gandillot, 262–267 ["AVERTISSEMENT.—Dans ce débat sur la relativité, la théorie d'Einstein est defendue par quatre disciples, *Mathémate, Astronomian, Physician et Logomane*, dont les cultures sont différentes; elle est attaquée par *Quaerens* qui exprime ma propre opinion."].

SCHOOL SCIENCE AND MATHEMATICS, volume 22, May, 1922: "The uses of algebra in study and reading" by E. L. Thorndike and Ella Woodyard, 405–415 (to be continued); "Pro-

jective method in mathematics" by B. Cosby, 451–455; "Tangent lines among the Greeks" by F. Cajori, 463–464; "Trigonometry without similar triangles" by J. A. Nyberg, 467–470; "A new ellipsograph?" by J. A. van Groos, 471–472—June: "The uses of algebra in study and reading" by E. L. Thorndike and Ella Woodyard, 514–522; "Graphical trisection of an angle" by E. D. Pickering, 548–549.

SCIENCE, new series, volume 55, May 5, 1922: "The Edward C. Pickering memorial" by S. A. Mitchell, 467–469 [A fund for the Association of Variable Star Observers to be devoted to variable star research]—May 12: "Mathematical publications," 508–509; "The Eliakim Hastings Moore Fund" by A. Dresden, 510—May 26; "The Einstein equations of the solar field from the Newtonian point of view" by L. P. Eisenhart, 570–572.

AMERICAN DOCTORAL DISSERTATIONS.

J. D. BARTER, "The homogeneous vector function and determinants of the P-th class," *University of California Publications in Mathematics*, vol. 1, 1920, pp. 321–343. (University of California, 1917.)

TERESA COHEN, "Investigations on the plane quartic." Pp. 191–211. [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] (Johns Hopkins University, 1918.)

H. D. FRARY, "The Green's function for a plane contour." Pp. 11–25. [Reprinted from *American Journal of Mathematics*, vol. 42, 1920.] (University of Illinois, 1918.)

K. W. LAMSON, "A general implicit function theorem, with an application to problems of relative minima." Pp. 243–256. [Reprinted from *American Journal of Mathematics*, vol. 42, 1920.] (University of Chicago, 1917.)

FLORA E. LE STOURGEON, "Minima of functions of lines." Pp. 357–383. [Reprinted from *Transactions of the American Mathematical Society*, vol. 21, 1920.] (University of Chicago, 1917.)

WAYNE SENSENIG, "Concerning the invariant theory of involutions of conics." Pp. 111–122. [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] (University of Pennsylvania, 1919.)

G. W. SMITH, "Nilpotent algebras generated by two units, i and j, such that i^2 is not an independent unit." Pp. 143–164. [Reprinted from *American Journal of Mathematics*, vol. 41, 1919.] (University of Illinois, 1917.)

J. S. TAYLOR, "A set of five postulates for Boolean algebras in terms of the operation 'exception,'" *University of California Publications in Mathematics*, vol. 1, 1920, pp. 241–248. (University of California, 1918.)

L. E. WEAR, "On self-dual plane curves of the fourth order." Pp. 97–118. [Reprinted from *American Journal of Mathematics*, vol. 42, 1920.] (Johns Hopkins University, 1913.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2977. Proposed by FLORENCE P. LEWIS, Goucher College.

A point moves in such a way that its polars with respect to two given conics intersect at right angles. Prove that the locus of this intersection is a rational quartic curve through the circular points, and find its double points.

which is the required formula.[1]

(b) Since a and d are positive acute angles, the last term cannot become negative and is zero only when $x = 0$. Therefore d will have a minimum value when $x = 0$, so that $r = r' = \frac{1}{2}a$, and $i = i'$, and the complete ray is symmetrically situated with respect to the prism.

(c) For this minimum value of d, say d_0, we have the classical laboratory formula

$$n = \frac{\sin \frac{1}{2}(a + d_0)}{\sin \frac{1}{2}a}.$$

2879 [1921, 89]. Proposed by E. J. OGLESBY, Washington Square College.

Given the values of $U_{5:9}$, $U_{5:10}$, $U_{5:11}$, $U_{6:9}$, $U_{6:10}$, $U_{6:11}$, $U_{7:9}$, $U_{7:10}$, $U_{7:11}$ where $U_{h:k} = \sqrt{hk}$, find the value of $U_{6\cdot 2:9\cdot 3}$ by interpolation.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

From the first three given terms we find by the method of differences[2] $U_{5:9.3} = 6.8190$, from the next three $U_{6:9,3} = 7.4698$ and from the last three $U_{7:9.3} = 8.0685$. Using these three values then as a series by the same method we find $U_{6.2:9.3} = 7.5937$.

Also solved by H. N. CARLETON.

2881 [1921, 89]. Proposed by E. B. ESCOTT, Oak Park, Ill.

If, in the polynomial $X^3 - 2$, we substitute $x^2 + x - 4$ for X, the given expression can be factored, that is, $X^3 - 2 \equiv (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6)$. Find a substitution for X so that the polynomial $X^3 + pX^2 + qX + r$ may be factored.

SOLUTION BY THE PROPOSER.

Let

$$X^3 + pX^2 + qX + r = (X - b)(X + c)^2 - a^2(X - d)^2. \qquad (1)$$

Expanding the second member and equating coefficients of like powers of X, we get

$$a^2 + b - 2c + p = 0, \quad 2a^2d - 2bc + c^2 - q = 0, \quad \text{and} \quad a^2d^2 + bc^2 + r = 0.$$

Eliminating b and d and solving for a^2 we get

$$a^2 = \frac{(3c^2 - 2pc + q)^2}{4(c^3 - pc^2 + qc - r)}.$$

Therefore, it is necessary and sufficient that [3]

$$c^3 - pc^2 + qc - r = n^2,$$

and we get by substitution

$$a = \frac{3c^2 - 2pc + q}{2n}, \quad d = -\frac{c^3 - qc + 2r}{3c^2 - 2pc + q}, \quad \text{and} \quad b = -a^2 + 2c - p.$$

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[1] We might say

$$\frac{1}{n^2} = \frac{\sin^2 \frac{1}{2}a \cos^2 x}{\sin^2 \frac{1}{2}(a + d)} + \frac{\cos^2 \frac{1}{2}a \sin^2 x}{\cos^2 \frac{1}{2}(a + d)}$$
$$= \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} + \left[\frac{\cos^2 \frac{1}{2}a}{\cos^2 \frac{1}{2}(a + d)} - \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)}\right]\sin^2 x$$
$$= \frac{\sin^2 \frac{1}{2}a}{\sin^2 \frac{1}{2}(a + d)} + \frac{4 \sin \frac{1}{2}d \sin (a + \frac{1}{2}d) \sin^2 x}{\sin^2 (a + d)}$$

and use this formula for (b) and (c) instead of the formula in the text, d being a minimum when $[\sin^2 \frac{1}{2}a]/[\sin^2 \frac{1}{2}(a + d)]$ is a maximum.—EDITORS.

[2] See *1921*, 330.

[3] Thus it seems to be a necessary part of the hypothesis that there is a rational number $-c$ that will make the given polynomial equal to minus the square of a rational number.

Letting f denote the polynomial, if $f(-c) = -n^2$ we can write

$$f(X) = (X + c)^2\left[X - 2c + p + \left(\frac{f'(-c)}{2n}\right)^2\right] - \left[\frac{f'(-c)}{2n}(X + c) - n\right]^2,$$

which we can make the difference of two squares by putting $X - 2c + p + \left(\frac{f'(-c)}{2n}\right)^2 = (x + t)^2$.

—EDITORS.

Thus, a, b, c, and d are known. The right-hand member of (1) will be the difference of two squares, and may be factored if we put $X - b = (x + t)^2$; *i.e.*, $X = (x + t)^2 + b$.

In the example given, $p = q = 0$, $r = -2$. The value $c = -1$ gives $n = 1$; whence, $a = 3/2$, $d = 5/3$, $b = -17/4$. $X^3 - 2 = (X + \frac{17}{4})(X - 1)^2 - \frac{1}{4}(3X - 5)^2$. Letting $X + \frac{17}{4} = (x + \frac{1}{2})^2$; *i.e.*, $X = x^2 + x - 4$, and substituting, we have

$$X^3 - 2 = [(x + \tfrac{1}{2})(x^2 + x - 5) + \tfrac{1}{2}(3x^2 + 3x - 17)][(x + \tfrac{1}{2})(x^2 + x - 5) - \tfrac{1}{2}(3x^2 + 3x - 17)]$$
$$= (x^3 + 3x^2 - 3x - 11)(x^3 - 6x + 6).$$

2885 [1921, 139].

If A, B, C, X, Y are given collinear points, construct Z so that $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, where $\{ABCX\}$ denotes the cross-ratio of the points A, B, C, X. [From *the Math. Tripos Exam.*, Cambridge, Eng., 1905.]

I. Solution by Gertrude I. McCain, Westminster College, New Wilmington, Pa.

If $\{ABCX\} + \{ABCY\} = \{ABCZ\}$, then $\dfrac{AB}{BC} : \dfrac{AX}{XC} + \dfrac{AB}{BC} : \dfrac{AY}{YC} = \dfrac{AB}{BC} : \dfrac{AZ}{ZC}$. Dividing by $\dfrac{AB}{BC}$, $\dfrac{XC}{AX} + \dfrac{YC}{AY} = \dfrac{ZC}{AZ}$. But $XC = -CX = -(AX - AC)$, $YC = -CY = -(AY - AC)$, and $ZC = -CZ = -(AZ - AC)$. Making the substitutions and adding 2 to each side of the equation, we have $\dfrac{AC}{AX} + \dfrac{AC}{AY} = \dfrac{AC}{AZ} + 1$, or $\dfrac{1}{AX} + \dfrac{1}{AY} = \dfrac{1}{AZ} + \dfrac{1}{AC}$. Let $AX = \cot\theta$, $AY = \cot\varphi$, $AC = \cot\psi$, and $AZ = \cot\tau$. Considering AX, AY and AC as line functions, construct the angles θ, φ and ψ, using a circle with radius unity. Also construct the tangents of θ, φ, and ψ as line functions using the same circle.

It is now possible to lay off $\tan\theta$ and $\tan\varphi$ as lengths on a line and from their sum subtract $\tan\psi$.

But $\dfrac{1}{AX} + \dfrac{1}{AY} - \dfrac{1}{AC} = \dfrac{1}{AZ} = \tan\tau$. Cot τ or AZ may then be constructed as a line function, and the point Z located on the line with A, B, C, X, Y.

II. Solution by Otto Dunkel, Washington University.

After showing that $1/AX + 1/AY = 1/AZ + 1/AC$, it should be observed that this result shows that the two sets of points A, X, Y and A, Z, C have a common fourth harmonic point K. Hence K can be constructed from the first set and then Z as the harmonic conjugate of C with respect to A and K, by the quadrilateral construction, thus giving a non-metrical construction. This would then give a projective definition of the sum of two cross-ratios. This result may be obtained by use of abridged notation as follows:

If $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $\xi = 0$, $\eta = 0$ are the equations of the points A, B, C, X, Y, then the expressions to the left in these equations may be chosen so that

$$\beta = b\alpha + \gamma, \qquad \xi = x\alpha + \gamma, \qquad \eta = y\alpha + \gamma,$$

and then $\{ABCX\} = x/b$, $\{ABCY\} = y/b$. It follows that $(x - y)\alpha = \xi - \eta$, and if K is the harmonic conjugate of A with respect to X, Y, its equation may be written $(x - y)\kappa = \xi + \eta = 0$ or $(x - y)\kappa = (x + y)\alpha + 2\gamma$. Since $2\gamma = (x - y)\kappa - (x + y)\alpha$, Z, the harmonic conjugate of C with respect to K, A, may have its equation written $2\zeta = (x - y)\kappa + (x + y)\alpha = 0$ or $\zeta = (x + y)\alpha + \gamma$. Hence $\{ABCZ\} = (x + y)/b = \{ABCX\} + \{ABCY\}$.

Also solved by Nathan Altshiller-Court, J. W. Clawson, Arthur Pelletier, and F. L. Wilmer.

2887 [1921, 139]. Proposed by the late L. G. WELD.

A carpenter's square moves with its outer edges in contact with two round pegs of given equal diameters. Define the locus of the "heel" of the square.

Solution by J. B. Reynolds, Lehigh University.

Let the radius of either peg be r and the distance between their centers $\sqrt{2}a$. Let the axes be taken so that the centers of the circles are at the points $(a, 0)$ and $(0, a)$, and assume that the

square is placed above and to the right of the two circles, respectively. Also let θ be the angle which the side tangent to the former circle makes with the x-axis. The limiting positions of the square will be when the heel is just in contact with one of the circular pegs; that is, for example, when $a \cos \theta = r + a \sin \theta$, or $\theta = \frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}$, so that the physical limits of θ are $\pm \left(\frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}\right)$.

The points of tangency for any position of the square will be $(a - r \sin \theta,\ r \cos \theta)$ and $(r \cos \theta,\ a + r \sin \theta)$ and the equations of these tangents will be, respectively,

$$-(x - a) \sin \theta + y \cos \theta = r \quad \text{and} \quad x \cos \theta + (y - a) \sin \theta = r,$$

from which we get

$$x = -r(\sin \theta - \cos \theta) + a \sin \theta(\sin \theta + \cos \theta),$$
$$y = r(\sin \theta + \cos \theta) + a \sin \theta(\sin \theta - \cos \theta)$$

for the parametric equations defining the curve, with $-\left(\frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}\right) \leqq \theta \leqq \left(\frac{\pi}{4} - \sin^{-1} \frac{r}{\sqrt{2}a}\right)$ for that portion subject to the physical limitations of the problem. By elimination of θ, we get for the general equation of the curve[1]

$$2r^2[x^2 + y^2 - 2a(x + y) + 2a^2] = [x^2 + y^2 - ax - ay]^2.$$

If in this $r = 0$, we have the circle $x^2 + y^2 = ax + ay$, as we should. For the above limits of θ to be correct, the shorter arm (18 inches) of the square must be $> \sqrt{(2a^2 - r^2)} - r$.

2895 [1921, 184]. Proposed by R. M. MATHEWS, Wesleyan University.

To construct an equilateral triangle with its vertices lying on: (a) any three coplanar lines; (b) three parallels in space; and (c) any three lines in space.

SOLUTION BY F. L. WILMER, Omaha, Neb.

(a) Let the three lines be l_1, l_2 and l_3. When they are not parallel we may suppose that the first two intersect in O. We bisect one of the angles between them and let the bisector meet l_3 in C. Then draw two lines from C, making angles of 30 degrees with CO (or with OC produced) and cutting l_1 and l_2 in A and B, respectively. ABC will be an equilateral triangle and there will generally be four triangles that can be constructed in this way.

When the three lines are parallel, draw a secant cutting them in P_1, P_2, P_3, respectively, and construct an equilateral triangle with P_1P_3 as one side and Q, say, as the third vertex. Rotate this triangle about P_2 until Q falls on l_2 and let the triangle in the new position be $P_1'Q'P_3'$. If we denote by A and B the points where $P_1'P_3'$ cuts l_1 and l_3, respectively, we shall find that the equilateral triangle ABC, constructed on AB on the same side of AB as Q', will satisfy the conditions of the problem; for $P_2P_1'/P_2A = P_2P_3'/P_2B$. Therefore P_2 is a center of similitude for the triangles $P_1'Q'P_3'$ and ABC, and C lies on P_2Q' or l_2.

(b) Consider the prismatic surface with edges l_1, l_2, l_3 and suppose that an equilateral triangle of side L can be placed with its three vertices on these three lines, respectively. Let a, b, c be the lengths of the sides of a right section $a \leqq b \leqq c$. Then

$$\sqrt{L^2 - a^2} = \sqrt{L^2 - b^2} + \sqrt{L^2 - c^2} \tag{1}$$

or

$$3L^4 - 2(a^2 + b^2 + c^2)L^2 + 16A^2 = 0, \tag{2}$$

where A is the area of the right section.

If L^2 is replaced by c^2, the expression in the first member of (2) reduces to $-(b^2 - a^2)^2$ and this shows that the equation in L^2 has two positive roots, one less than c^2 and the other greater than c^2 (unless $a = b$). The root greater than c^2 makes the three terms of (1) all real and satisfies (1) as written, since the first term of (1) is greater than or equal to each of the others.

(c) Partial solution—Instead of the three given lines, we may consider two non-coplanar lines and a point. Take the common perpendicular of the two for the z-axis and the bisectors of the angles made by their projections on the mid-plane as the axes of x and y. Then we may let

[1] If we take the line joining the centers of the circles as the x-axis, with the origin at the mid-point of that line, the equation will be $(x^2 + y^2 - b^2)^2 = 2r^2[x^2 + (y - b)^2]$, where $2b$ is the distance between the centers.—EDITORS.

(x_1, px_1, h), $(x_2, -px_2, -h)$ and (α, β, γ) be the vertices of an equilateral triangle of side L and there will be three expressions for L^2 which give the three equations

$$(1 + p^2)(x_1^2 + x_2^2) - 2(1 - p^2)x_1x_2 = L^2 - 4h^2,$$
$$(1 + p^2)x_1^2 - 2(\alpha + p\beta)x_1 = L^2 - \alpha^2 - \beta^2 - (\gamma - h)^2,$$
$$(1 + p^2)x_2^2 - 2(\alpha - p\beta)x_2 = L^2 - \alpha^2 - \beta^2 - (\gamma + h)^2.$$

Eliminating x_1 and x_2 from these three equations, we have an equation of the fourth degree in L^2 with coefficients which are functions of α, β, γ.

NOTE BY THE EDITORS.—There are an infinite number of solutions in (*a*) when the lines are not parallel, as well as when they are parallel. This may be shown as follows: draw an equilateral triangle ABC with A on l_1 and B on l_2. With a given position of AB, the vertex C may lie on either side of AB. One at least of the straight lines OC will cut l_3 in the point C', unless these are both parallel to l_3 and so the same line, and this can be avoided by shifting the position of the triangle. The lines $C'A'$ and $C'B'$, drawn from C' to l_1 and l_2, parallel respectively to CA and CB, will be two sides of the triangle $A'B'C'$ which is equilateral and satisfies the conditions of the problem. The triangles $A'B'C'$ for different triangles ABC will vary in size, for if they were all equal the locus of C' would be an ellipse and not the line l_2.

In (*c*) the construction is not always possible. If, for example, the given point is at the origin, $\alpha = \beta = \gamma = 0$, then $x_2 = \pm x_1$, and $L^2 = 4h^2/(1 - 3p^2)$ or $4p^2h^2/(p^2 - 3)$. It is necessary in this case, therefore, that the acute angles between the projections of the two lines shall be less than 60 degrees.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, 69 Weymouth St., Providence, R. I.

Miss MAY J. SPERRY, of Knox College, has been appointed instructor at the University of Syracuse.

Miss MARGARET C. PACKER, of Brown University, has been appointed instructor at Hood College, Fredericksburg, Md.

Miss FRANCES M. MERRIAM, of Brown University, has been appointed instructor of mathematics at Wellesley College.

Assistant professor J. W. HARRELL, of Baylor University, Waco, Texas, is on leave of absence studying at Brown University.

MARCUS SKARSTEDT, of Augustana College, has been appointed professor of mathematics in Whittier College.

Miss MINNIE W. CALDWELL has been appointed head of the department of mathematics in Chowan College, Murfreesboro, N. C.

Associate Professor E. L. DODD will teach at Williams College during 1922–1923 on leave of absence from the University of Texas.

C. G. SIMPSON, of the Carnegie Institute of Technology, has been appointed professor of mathematics in the Milwaukee College of Electrical Engineering.

J. G. FOWLKES has been made assistant professor of education in the University of Wisconsin.

R. L. CHARLES, associate professor of physics at Lehigh University, has been appointed professor of physics and electricity at Franklin and Marshall College.

At Brown University, W. R. BURWELL, of the University of Tennessee, has been appointed assistant professor of mathematics and dean of freshmen; J. H. FITHIAN, of Taft School, Watertown, Conn., and H. C. HICKS, of the University of Chicago, have been appointed instructors.

Dr. F. C. TOUTON, lecturer in secondary education at the University of California, has been appointed associate professor of education at the University of Southern California. Dr. Touton is joint author of the Hawkes, Luby and Touton mathematical text-books.

Professor GLENN JAMES, of Carnegie Institute of Technology, has been appointed assistant professor of mathematics in the Southern Branch of the University of California, and Dr. F. C. LEONARD has been appointed instructor of mathematics and astronomy, in charge of the work in astronomy.

Professor W. J. RISLEY, of James Millikin University, has been appointed professor of mathematics in the Colorado School of Mines, and Mr. J. R. EVERETT, of Carnegie Institute of Technology, has been appointed assistant professor (see *1921*, 286).

F. A. WELLS, instructor of mathematics at the United States Naval Academy, has been appointed instructor of mathematics at the University of Virginia.

J. E. DAVIS, instructor of mathematics at the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Arkansas.

CLAIR REID, instructor of mathematics at Purdue University, has been appointed instructor at the University of Michigan.

T. B. HENRY, instructor of mathematics at the University of Kansas, has been appointed head of the department of mathematics in Highland College, Kansas.

Professor J. M. HOWIE, for twenty-four years head of the department of mathematics at the Nebraska State Teachers College, has been appointed professor of mathematics at Alma College.

Miss MARTHA P. MCGAVOCK, of Sullins College, has been appointed instructor of mathematics at Wellesley College.

Professor OSWALD VEBLEN, of Princeton University, has been granted leave of absence for the second semester of 1922–1923 and is to spend it in Europe.

Dr. C. E. STROMQUIST, of the University of Wyoming, is on leave of absence in California, necessitated by his ill health. Professor H. C. GOSSARD, who during the past two years has been engaged with the Y. M. C. A., has been appointed acting chairman of the department of mathematics.

In the mathematics department of the State College of Iowa, leave of absence during 1922–1923 has been granted to Associate professor JULIA T. COLPITTS to study at Cornell University, and Assistant professor E. C. KIEFER to study at the University of Michigan.

At the University of Colorado, Assistant professor G. H. LIGHT has been promoted to a full professorship of mathematics and Dr. CLARIBEL KENDALL to an assistant professorship.

At the University of Maine N. R. BRYAN has been appointed associate professor of mathematics for the year 1922–1923 in place of Professor L. S. HILL, who has been granted leave of absence and will study and act as instructor at Yale University.

Associate Professor EMMA L. KONANTZ, of Ohio Wesleyan University, has been

granted an extended leave of absence and in January will resume her teaching in Peking University and her study of early Chinese mathematics (see *1921*, 402; *1922*, 195).

Dr. J. D. BOND, of Louisiana State University, has been appointed associate professor of mathematics in the University of Tennessee. (See *1921*, 403.)

Professor W. E. PATTEN, of the Government Institute of Technology of Shanghai, a charter member of this Association, has been made professor of hydraulic engineering in Tangshan College, Tangshan, North China.

Dr. H. C. M. MORSE and Dr. W. L. G. WILLIAMS have been promoted to assistant professorships at Cornell University.

Dr. E. B. WILSON, professor of mathematical physics at the Massachusetts Institute of Technology, has been appointed professor of vital statistics at Harvard School of Public Health.

At the University of Kansas, Dr. U. G. MITCHELL was advanced to a full professorship for the year 1920–1921, and Dr. E. B. STOUFFER for the year 1921–1922.

At the University of Minnesota, Professor A. L. UNDERHILL has been granted a sabbatical leave of absence for the year 1922–1923. He expects to study in France. Mr. H. W. CHANDLER, of the University of Iowa, has been appointed instructor of mathematics.

At the Western College for Women, Oxford, Ohio, Miss Hazel E. SCHOONMAKER, of Gulf Park College in Mississippi, a graduate of Wellesley and an M. A. of Radcliffe, has been appointed head of the department of mathematics. Compare *1921*, 332.

At the University of Washington, Dr. E. T. BELL has been promoted to a full professorship, and Dr. L. L. SMAIL to an assistant professorship. Dr. HERMANCE MULLEMEISTER has been granted leave of absence for the year 1922–1923, which he will spend in Holland in study and research. Miss ECHO D. PEPPER, of the University of Washington, will take his place.

At the University of Michigan, Assistant Professor R. B. ROBBINS has returned for the year 1922–1923 after an absence of two years, during which period he has served as assistant actuary in the Insurance Departments of the State of Missouri and the State of New York. Mr. R. V. CHURCHILL, of the University of Chicago, has been appointed instructor of mathematics in the Engineering Department. Mr. C. C. CRAIG, of Indiana University, has been appointed instructor of mathematics in the College of Literature, Science and the Arts.

Professor C. H. RICHARDSON, of Georgetown University, Kentucky, taught in the summer session of the University of Michigan for 1922.

At Yale University, Assistant Professor LEIGH PAGE, of the department of physics, has been promoted to a full professorship of the mathematical sciences, with assignment to the Sheffield Scientific School and Dr. W. L. CRUM has been promoted to an assistant professorship of mathematics.

On account of ill health, Professor E. W. BROWN, of Yale University, had to give up his mathematical work about November 1. His advanced courses are to be conducted during the present year by Professor G. D. BIRKHOFF, of Harvard University.

FREDERICK ANDEREGG, professor of mathematics, emeritus, at Oberlin College since 1920 (compare *1920*, 336), died October 9, 1922. He was born in Meiringen, Switzerland, June 11, 1852, and was brought to America when ten years of age. Graduating A.B. from Oberlin in 1885, he was tutoring mathematics there 1885–1888. During 1888–1890 he was at Harvard (A.M. 1889) as a graduate student, and as instructor in mathematics during the latter year. Appointed associate professor of mathematics at Oberlin in 1890, he became professor in 1892. During 1903–1904 he studied at the University of Berne. Professor Anderegg's mathematical publications were not extensive. His pamphlet, *Algebra Problems introductory to College Mathematics* (Boston and New York, 1891), contained graded exercises for freshmen followed by examination papers from Harvard and Oxford, 1884–1886. A little article, "A perfect magic square," appeared in this MONTHLY, *1905*, 195–196. In collaboration with E. D. Roe, Jr., he was the author of *Trigonometry for Schools and Colleges* (Boston, 1896; revised edition, 1913). He was a charter member of the Association, had been a member of the American Mathematical Society since 1911, and was a member of the Swiss Alpine Club.

Mrs. SOPHIE (WILLOCK) BRYANT, aged 72, met her death by accident near Chamonix, Switzerland, some time between August 15 and August 28, 1922, when her body was found by Alpine hunters. She was born in Dublin and educated privately and at Bedford College, London, where she graduated with mathematical and moral science honors in 1881. She took the degree of doctor of science in moral sciences, being the first woman in the British Isles to take that degree. She was married at the age of 19 to Dr. William Hicks Bryant, of Plymouth, and after his death a year later renewed her work as a student and became mathematical mistress at the North London Collegiate School for Girls. From 1885 she was head mistress of this school until 1918, when she retired. She wrote many educational, religious, and scientific articles and articles on Ireland and the Irish language. She collaborated with Charles Smith in preparing an edition of Euclid's *Elements*, Books I to IV, VI, and XI (London, 1901) and a key to the same (London, 1902).

PIERRE JEAN BAPTISTE HENRI BROCARD, whose constant and valuable contributions have long been a prominent feature of *L'Intermédiaire des Mathématiciens*, and many other periodicals, died at his home in Bar-le-Duc, France, January 16, 1922. He was born only a short distance away, at Vignot, on May 13, 1845. A student of the Ecole Polytechnique, 1865–1867, he rose to the rank of lieutenant colonel of the territorial engineers (Génie territorial). In 1874 he was appointed in charge of the government meteorological service in Algiers. The list of his publications relative to meteorology, climatology, rural economics and hygiene, and natural sciences, 1860–1894, occupies over 60 printed pages. For the same period the printed list of his publications in physics and chemistry occupies 26 pages, and in pure and applied mathematics, and astronomy, 72 pages. These lists, preceded by a synopsis of his scientific activities, were issued, in 1895, in one volume at Bar-le-Duc, where he spent the last thirty years of his life. His very numerous contributions to the literature of mathematics were mainly

in connection with the geometry of the triangle (Brocard point and Brocard circle are well-known terms), history, and bibliography. In 1897–1899 he published and distributed at his own expense the very valuable *Notes de Bibliographie des Courbes Géométriques* (344 + 243 pages), now being elaborated into *Courbes Géométriques Remarquables* (*Courbes Spéciales*) *Planes & Gauches* by H. Brocard and T. Lemoyne, of which the first volume was published in 1919 (see *1920*, 130; *1921*, 218); the other two volumes are ready in manuscript but publication has so far been delayed by prohibitive cost of the same. Brocard's library was left to his friend André Gérardin, the editor and publisher of *Sphinx Œdipe*.

JACOBUS CORNELIUS KAPTEYN, born at Barneveld, Holland, January 19, 1851, died June 18, 1922. After taking his doctorate at the University of Utrecht in 1875 he became an observer at the observatory in Leyden. Since 1878 he had been professor of astronomy and mechanics at the University of Groningen. Among his very numerous publications, possibly those of chief interest to the mathematician are on "Skew frequency curves in biology and statistics" (Groningen, 1903 and 1916). Kapteyn visited Mount Wilson Observatory in 1910, and later, and a number of his astronomical papers are published in *Contributions from the Mount Wilson Observatory*, 1909–1920. Professor P. J. van Rhijh has been appointed his successor.

ERNEST RUDOLF KÖTTER, born in Berlin, Germany, August 7, 1859, died January 26, 1922. He took his doctorate at the University of Berlin where he was privatdozent for a number of years. From 1897 to the time of his death he was professor of descriptive geometry and graphical statics at the polytechnic school in Aix-la-Chapelle (Aachen). He is probably best known to mathematicians by his admirable report (28 + 486 pages), *Die Entwickelung der synthetischen Geometrie von Monge bis auf Staudt* (1847), which first appeared in *Jahresbericht der deutschen Mathematiker Vereinigung*, volume 5.

AXEL THUE, born in Tönsberg, Norway, February 19, 1863, died March 7, 1922. He received his education at the University of Christiania and was appointed a teacher in the technical school at Dortheim in 1894. In 1903 he was appointed professor of applied mathematics at the University of Christiania. His many published papers have been almost wholly in connection with problems of Diophantine analysis and other topics in the theory of numbers. Many of his results are listed in Dickson's *History of the Theory of Numbers*, volumes 1 and 2. In particular he found a general theorem applying to a particular problem discussed in this MONTHLY (see *1921*, 124; and *1922*, 159), namely the solutions of the equation $x^2 - y^3 = 17$.

Professor G. A. MILLER, of the University of Illinois, has been made an honorary member of the Indian Mathematical Society, together with Professor G. H. HARDY, of New College, Oxford, and Dr. G. T. WALKER, director-general of observations in India.

In April 1922 Professors W. F. OSGOOD and G. D. BIRKHOFF were elected corresponding members of the Royal Society of Sciences at Göttingen,

mathematical physical class. The list of corresponding members published in 1920 contains 80 names, including the following Americans: L. A. BAUER, Washington, D. C.; J. M. CLARKE, Albany, N. Y.; WILLIAM HILLEBRAND, Washington, D. C.; and R. W. WOOD, Baltimore, Md.

The gold medal of the Royal Astronomical Society has been awarded to J. H. JEANS, sometime professor of applied mathematics at Princeton University, for his contributions to the theory of cosmogony. Professor Jeans has already received one medal, awarded to him in 1919 by the Royal Society of London (see *1920*, 46).

E. T. WHITE, mathematics master of the London, Ontario, Normal School, was awarded the degree of doctor of pedagogy by the senate of the University of Toronto at its meeting on March 10.

The honorary degree of doctor of science was conferred upon Professor A. G. WEBSTER, of Clark University, at the commencement exercises of Princeton University.

A mathematical meeting and dinner in honor of Professor CHARLOTTE ANGAS SCOTT, on the completion of her thirty-seventh year as head of the department of mathematics in Bryn Mawr College, was held April 18, 1922, by Professor Scott's former students. The exercises consisted of an address of welcome by President M. CAREY THOMAS, an introductory address by Miss MARION REILLY, 1901, and a lecture by Professor A. N. WHITEHEAD, professor of applied mathematics in the Imperial College of Science, South Kensington, on "Relativity and gravitation, Group tensors and their application to the formulation of physical laws." After the lecture a tea was served at the deanery to about 200 guests.

At the dinner there were present former students, members of the American Mathematical Society, and members of the Bryn Mawr College faculty. Miss MARION REILLY acted as toastmistress. The speakers were: Professors E. H. MOORE, FORENCE BASCOM of Bryn Mawr, JAMES HARKNESS of McGill, E. W. BROWN, FRANK MORLEY, and A. N. WHITEHEAD. Professor Scott gave a response with an expression of appreciation. In regard to Professor Scott's service to Bryn Mawr College, Professor Bascom said in part, "It is this wisdom impartial, rational, creative, articulate, that Dr. Scott possesses in a marked degree. This is the quality which makes her judgment the one sought on all important faculty matters."

The honorary committee consisted of Professors R. C. ARCHIBALD, G. D. BIRKHOFF, E. W. BROWN, F. N. COLE, J. A. EIESLAND, JAMES HARKNESS, E. R. HEDRICK, FLORENCE P. LEWIS, Dean ISABEL MADDISON, Professors EMILIE N. MARTIN, HELEN A. MERRILL, E. H. MOORE, FRANK MORLEY, L. W. REID, R. G. D. RICHARDSON, E. J. TOWNSEND, OSWALD VEBLEN, H. S. WHITE, and RUTH G. WOOD.

Published January 11, 1923.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the..

....Fund, and to be used[2]

for { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: Signature..

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December 28, 29, 1922

The following are dates of Section meetings of the Association in 1922 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29
IOWA, Des Moines, November 3; Cornell College, Mount Vernon, April 27-28, 1923
KANSAS, Topeka, January 21; January, 1923
KENTUCKY, Georgetown College, April 8; University of Kentucky, April, 1923
MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Annapolis, May 13; Washington, December 9
MINNESOTA, St. Paul, June 4, 1921; St. Paul, May 27
MISSOURI, Kansas City Junior College, November 18; University of Missouri, Columbia, November 30–December 1, 1923
OHIO, Columbus, April 14–15; March, 1923
ROCKY MOUNTAIN, Greeley, Colo., April 14–15; University of Colorado, April, 1923
SOUTHEASTERN, Atlanta, Ga., April 29; February, 1923
TEXAS, Dallas, November 25, 1921; Houston, December 1–2

SEVENTH SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

By invitation of the University of Rochester, the seventh summer meeting of the Association was held at this University on Wednesday and Thursday, September 6–7, 1922, in conjunction with, and immediately preceding, the summer meeting of The American Mathematical Society. There were 104 in attendance at the sessions, including the following 75 members of the Association:

N. H. Anning, University of Michigan.
R. C. Archibald, Brown University.
Florence L. Becker, High School, Webster, N. Y.
W. J. Berry, Brooklyn Polytechnic Institute.
William Betz, East High School, Rochester.
R. L. Borger, Ohio University.
W. G. Bullard, Syracuse University.
R. W. Burgess, Brown University.
W. D. Cairns, Oberlin College.
W. B. Carver, Cornell University.
E. H. Clarke, Hiram College.
H. E. Cobb, Lewis Institute.
A. B. Coble, University of Illinois.
C. E. Comstock, Bradley Polytechnic Institute.
Lennie P. Copeland, Wellesley College.
A. R. Crathorne, University of Illinois.
F. F. Decker, Syracuse University.
H. A. DoBell, Colgate University.
L. P. Eisenhart, Princeton University.
H. J. Ettlinger, University of Texas.
B. F. Finkel, Drury College.
C. A. Fischer, Trinity College.
W. B. Ford, University of Michigan.
R. E. Gaines, Richmond College.
A. S. Gale, University of Rochester.
R. E. Gilman, Brown University.
C. F. Gummer, Queen's University.
C. E. Harrington, University of Buffalo.
E. R. Hedrick, University of Missouri.
H. C. Hicks, Brown University.
T. H. Hildebrandt, University of Michigan.
W. A. Hurwitz, Cornell University.
Nelle L. Ingels, Interstate Commerce Commission.
L. C. Karpinski, University of Michigan.
O. D. Kellogg, Harvard University.
H. W. Kuhn, Ohio State University.
W. D. Lambert, U. S. Coast Survey.
Solomon Lefschetz, University of Kansas.
N. J. Lennes, University of Montana.
Anna D. Lewis, Lake Erie College.
T. R. Long, University of Rochester.
John Matheson, Queen's University.
G. A. Miller, University of Illinois.
Norman Miller, Queen's University.
G. R. Mirick, East High School, Rochester.
C. N. Moore, University of Cincinnati.
C. C. Morris, Ohio State University.
F. W. Owens, Cornell University.
Mrs. F. W. Owens, Cornell University.
L. R. Perkins, Middlebury College.
A. D. Pitcher, Western Reserve University.
L. C. Plant, Michigan Agricultural College.
V. E. Pound, University of Buffalo.
S. E. Rasor, Ohio State University.
F. W. Reed, Cornell University.
L. W. Reid, Haverford College.
Harris Rice, Worcester Polytechnic Institute.
R. G. D. Richardson, Brown University.
E. D. Roe, Syracuse University.
Hazel E. Schoonmaker, Western College.
W. H. Sherk, University of Buffalo.
W. G. Simon, Western Reserve University.
H. E. Slaught, University of Chicago.
Clara E. Smith, Wellesley College.
Virgil Snyder, Cornell University.
K. D. Swartzel, University of Pittsburgh.
W. A. Titsworth, Alfred College.
B. L. Waits, Florida A. and M. College.
C. W. Watkeys, University of Rochester.
J. H. Weaver, Ohio State University.
J. H. M. Wedderburn, Princeton University.
D. E. Whitford, University of Rochester.
F. B. Williams, Clark University.
C. H. Yeaton, Oberlin College.
J. W. Young, Dartmouth College.

From this list it will be noticed that a very good representation was afforded from the different regions covered by the Association. The joint dinner of the two organizations occurred on Thursday evening with Professor Eisenhart as presiding officer. Professor Archibald, the president of the Association, gave by

invitation an extended and informing account of various phases of his trip through Europe during the past spring and summer, and Professor Snyder gave an interesting description of the exercises at the seven hundredth anniversary of the founding of the University of Padua. Some extracts from Professor Archibald's address will be printed in one of the forthcoming issues of the MONTHLY. A hearty vote of thanks was given to the Board of Governors of the Oak Hill Country Club for their courtesy in extending the facilities for this dinner to the two organizations.

There was a notable variety of events in connection with the meetings at Rochester. On Wednesday afternoon at the close of the program, a short chamber concert was given for the members of the Association and the Society in Kilbourn Hall of the new Eastman School of Music, at which a program consisting of violin, piano, and vocal selections was given by the advanced students of the School of Music, and by Mr. Gleason, the head of the Organ Department. The visitors took advantage of the free evenings to attend the performances at the Eastman Theatre, which had been opened on Monday of that week; this theatre combines in notable fashion programs of motion pictures and of music. The tour of inspection through the Research Laboratory of the Eastman Kodak Company is referred to later. On Thursday afternoon, at the close of the Society program, the visiting members were taken in automobiles on a general tour of inspection of Rochester and its surroundings, particularly the Durand-Eastman Park near Lake Ontario, and the grounds of the Oak Hill Country Club where the joint dinner was held on Thursday evening. On Friday afternoon the opportunity was given for an inspection of the plant of the Bausch & Lomb Optical Company. An unusual amount of attention was paid to the mathematical meetings by the daily newspapers of Rochester which each day reported quite fully the details of the programs.

A considerable number of men were housed comfortably in Kendrick Hall, the other members of the Association staying at the hotels which were not distant from the University. Many provisions were made to enhance the comfort and pleasure of the visiting guests, and the Association recognized these courtesies in a formal vote of thanks which was offered at the meeting on Thursday morning, with particular mention of Professor Gale and his associates in the department of mathematics, the Eastman School of Music, the Eastman Kodak Company, and the Bausch & Lomb Company.

The following papers were given; abstracts of these papers follow, the numbers corresponding to the numbers in the lists of titles.

SESSION OF THE ASSOCIATION ON THE PRESENT STATUS OF UNIFIED MATHEMATICS.

(1) "The problem of organizing freshman college courses" by Professor J. W. YOUNG, Dartmouth College.

(2) "Historic consideration of unified mathematics" by Professor L. C. KARPINSKI, University of Michigan.

(3) "Some aspects of unified mathematics for freshmen" by Professor R. W. BURGESS, Brown University.

(4) "Internal reasons for unification" by Professor C. E. COMSTOCK, Brooklyn Polytechnic Institute.

(5) General discussion, led by Professors F. B. WILLIAMS, Clark University, K. D. SWARTZEL, University of Pittsburgh, and C. H. YEATON, Oberlin College.

1. The gist of Professor Young's paper will appear in an early issue of the MONTHLY.

2. The purpose of Professor Karpinski's paper was to show that the greatest teachers of the world's history employed methods, materials, and illustrations to enrich their instruction, as far as such material was available. Even such a text as Euclid's *Elements* contains, in addition to the seventh to tenth books which are strictly theoretical arithmetic, a wealth of algebraic material. Similarly Archimedes in his *Sphere and Cylinder* solves by the intersection of conics a cubic equation and gives what corresponds to analytical conditions for the existence of real roots of a given cubic. Ptolemy, the great geographer-astronomer, introduces into his astronomical treatise whole sections on geometry and on the computation of chords. The trigonometry in Ptolemy's *Almagest* is developed as completely as it would be in a treatise devoted strictly to mathematics rather than to astronomy. With the great Arabic text-book writers and scientists, with Al Khowarizmi and Omar al Khayyam, in particular, the treatises on algebra were enlivened and enriched by a mass of geometrical and practical illustrations. The same was true with Regiomontanus and with such writers as Euler and Newton. Our texts, prepared for a more immature audience, have made the great error of insistence upon strictly logical unity, rather than upon psychological adaptation to the readers for whom the texts are prepared. Historically, at least, good authority can be found for a broader treatment on a unified basis.

3. Taking as the key principle of unified mathematics for college freshmen the statement that subject matter and methods of instruction should be determined by the present and future needs of students who take no more mathematics, Professor Burgess urged that a searching examination of the various activities of life must be made to bring out just what mathematical attainments are really needed. Casual and uncritical statements, such as that of the graduate who says he never uses mathematics, are not to be accepted as evidence without scrutiny. Professor Burgess then suggested a division of the uses of mathematics into three main classes,—purely mathematical or geometrical, experimental, and statistical; and urged that increased attention be paid to the last type as the one which is most frequently needed subsequently by students. He urged that mathematicians recognize statistical mathematics as the mathematics of aggregates in which the individual elements differ from each other to a significant extent, and that they look upon even elementary statistical methods as worthy of serious attention. The statistical topics suggested for the freshman course are percentage analysis, frequency distributions, simple averages and diagrams, index numbers, and a rough determination of trend lines. Except

for the introduction of the statistical point of view, he joined with others in regarding the function concept as a unifying principle. In regard to the methods to be used, he suggested that extreme emphasis on securing expert facility is unwise, that proofs are of secondary importance in this course, that the needs of the individual class should modify the conduct of the course, and that problems should be selected from a wide field.

4. Professor Comstock discussed the four subjects, college algebra, trigonometry, analytic geometry and calculus. The central ideas of these subjects may be stated as follows: college algebra, a complex of somewhat unrelated topics, such as the solution of equations and the transformation of expressions containing the common functions of algebra; trigonometry, the algebra of the circular functions and the arithmetic of the triangle; analytic geometry, characterized by its three problems, finding the locus of an equation, finding the equation of a locus, and finding the properties of conics by algebraic methods; and calculus, concerned with the changes of functions due to changes in the independent variables. Now there has steadily developed an interpenetration of the fields which surround these central ideas, and this may be considered in two aspects: the more advanced subjects must build upon material from the lower courses; certain topics and methods of one subject are found helpful in presenting the concepts of another subject. For example, calculus calls on geometry for illustration; and geometrical concepts are found exceedingly helpful in presenting analytic processes; also the graph has invaded algebra and trigonometry. In fact, a rudimentary unification of all these subjects is actually going on.

The combination of elementary algebra and plane geometry in one course is not desirable, since they are two cultures made up of two sets of distinctive habits which the student must acquire; the exposure to each must be long enough and sufficiently free from interruption for the ideas to set firm as distinct habits. College students, however, are more mature and have these two lines of thought fixed. In separate college courses much time is lost on dislocated topics and unnecessary methods. Partial fractions should be left until they are needed in integration. A more adequate treatment of series can be given when calculus has been acquired. The introduction of the derivative in analytic geometry improves the treatment of many topics, notably the determination of tangents, the plotting of curves and the study of their properties. The student should be introduced to the most fruitful concepts of mathematics as early as possible. There should be a whole-hearted willingness to use any idea or method that will be most helpful at the time, whether the order is traditional or not.

The function concept runs through all four of the subjects that we have considered and may well be used as a unifying bond, the more so because of its three modes of expression, arithmetical, algebraic and geometric. Sufficient time must be given each topic to afford a deep infusion, and the connection between topics should be made evident. A two-year course is desirable, but students who take one year will have acquaintance with the most valuable methods.

5. Professor Williams said that the whole course should be developed in a

logical way, that it is necessary in all colleges to give the students some review of their previous material. It is not so much a question of time; we need to take them where, on the whole, they are found, and give them as much as can be done during the year. He outlined the course which he has given each year since 1907, a course comprising such topics as: the location of points by coördinates, trigonometric functions and applications, motion of a point in a plane, the intersection of two lines, the different approaches to determinants, the motion of three points on a line, and three lines through a point, with a suggestion of the principle of duality, tangents, limits, introduction to calculus, including its first applications. There should be proper attention paid to symbolic representation so as to avoid confusion in the student's mind on this score. The result of his experience of fifteen years is that this kind of a course is not merely possible but is very valuable.

Professor Swartzel believes that it is possible to write a satisfactory text in uniform mathematics for freshmen, but that this task is still to be done. The teaching of functions in this way does undoubtedly give the students a meaning of the term, but not the general meaning of "function" which we desire, such as is done in the traditional course. There should be a further discussion not merely of the topics which have been dropped in making up this course, but also of the educational side of the question,—a development of the mind toward clear, accurate thinking. The new book for this course should pay attention to the best development of the mind and the incorporating of the new relations which may well be added to the course for freshmen. Professor Swartzel has found it difficult in a course in uniform mathematics to get a sort of examination that shall be satisfactory to the teacher and to the pupils, examinations that will readily and fairly test the student's mastery. Further it is difficult to make the subject-matter coherent, and there is a lack of desirable drill in this course. Do not these difficulties make it evident that we are trying to do too much in a one-year course? He feels that the new style of course is best suited to the mediocre students, who at least appear to do a little better under this plan. A great opportunity lies at hand in the line of segregation of students according to their ability; this will incidentally help in this trial of uniform mathematics.

Professor Yeaton has found that much of the success of the uniform course depends on the enthusiasm of the particular teacher giving the course. He personally feels that this course, including the applications of calculus, is good for students who are sure that they will not continue after the first year, and that it arouses their interest and imagination. The course, however, has the fault of including too many topics, a great difficulty unless we plan a two-year course in its entirety. For the good students it seems quite unfortunate to crowd out the analytic geometry of the first year inclusive of some good serious study of conic sections and the necessary drill in such work.

Mr. Lambert of the Coast Survey raised the question whether the free combination of mathematical methods by the ancients was meant for students of the same grades as those in our colleges. He has found in setting examination ques-

tions for government examinations that questions of a mixed sort have been too difficult and that he had to resort to questions of the traditional sort.

Professor Lennes has seen no way of getting at the questions except by actual tests, and he almost despairs of a result for the reason that one's predisposition toward, say, uniform mathematics, as well as other lines of work, will warp one's judgment even with the best of intentions to be unprejudiced. He has found it hard to get the reaction of the pupil to the three or four pills separately administered in comparison with all given together in one dose.

Professor Plant has felt in opposition to Professor Lennes's point of view that we are getting back to what was the original method of one pill, not four pills in one, and that in his observation the use of various of the present-day texts in uniform mathematics has undoubtedly brought about a satisfactory mental development in the pupils.

Professor Berry has found that a difficulty arises in the case of men training for a special purpose who hold mathematics as a tool, and he concludes that for these the proper course in uniform mathematics has not yet appeared. He has used with success a sort of infiltration process, transferring topics from the traditional order to the point in a course where these are needed.

Second Session of the Association.

(6) "Contradictions in the literature of the group theory," retiring presidential address by Professor G. A. Miller, University of Illinois.

(7) "An English text of mathematics written about 1810" by Professor Elizabeth B. Cowley, Vassar College.

(8) "Impressions of mathematics and mathematical instruction in Italian universities" by Professor Virgil Snyder, Cornell University.

6. The address of the retiring president, Professor Miller, will shortly appear in this Monthly.

7. Miss Cowley's paper concerned a three-volume mathematical work in manuscript and will be published in a later issue of this Monthly.

8. Professor Snyder described mathematical instruction in Italy. In the preparatory course of twelve years in Italian schools, the range and scope are fairly similar to our own, but the pupils are better trained than ours in mathematics and the languages. In the university, the possible range of studies for a beginner is large, but after the initial selection, the subsequent range is very limited. Those selecting pure mathematics, for example, may have one course in chemistry, or in experimental physics, but with this exception, the entire college course is spent on pure mathematics. During the first year the average student takes fifteen hours, in the second he takes twelve, and in the third and fourth nine each and a thesis. If the term examinations are all passed, and the thesis accepted, and defended in a brief formal examination, the student graduates with the degree of doctor of mathematics. All instruction is by the formal lecture method, frequently not accompanied by any exercise period; there is no

control over attendance; a brief oral examination (twenty minutes) at the end of the year is the only criterion upon which to base the estimate of the quality of the year's work. Projective geometry is begun the first year, and the student starts analytic geometry at the same time; both continue throughout the first year. The calculus is not begun until the second year. Everywhere great emphasis is put on reasoning, and the vehicle most frequently and extensively employed for this purpose is synthetic geometry, in which the students become remarkably skillful. While the Italians have contributed essentially to the development of almost all general branches of mathematics, the present theory of algebraic geometry is preëminently an Italian product, and is perhaps the most important Italian contribution to modern mathematics.

SESSION AT THE RESEARCH LABORATORY OF THE EASTMAN KODAK COMPANY.

(9) "The physical problems involved in photographic research" by Mr. L. A. JONES, of the Research Laboratory.

(10) "Mathematics puzzles as an introduction to investigation" by Professor W. B. CARVER, Cornell University.

(11) "The present status of the formal discipline controversy" by Professor N. J. LENNES, University of Montana.

(12) Inspection of the Research Laboratory.

9. Mr. Jones described the general nature of the problems treated in the Research Laboratory, covering such topics as the following: the amount of energy to be used in producing the photo-chemical effect, the instability of the nitro-cellulose base, the use of dyes in making colored lights, sensitizing agents, etc. As an example of the sort of research carried on in the Laboratory, he showed by graphical representation the nature of the study of the sensitiveness of the ordinary portrait photographic plate, the bichromatic and the panchromatic plates. He made evident, in what at once appeared to those present as a functional dependence, the various factors which determine how perfectly the photographic process reproduces the impression gained by the direct vision of an object.

10. The paper by Professor Carver will appear in a later issue of this MONTHLY.

11. This paper contained a resumé and an evaluation of the arguments and experimental investigations which were supposed to have exploded the doctrine of formal discipline, a doctrine which for practical purposes may be said to deal with the effect of training in one ability upon other abilities. Professor Lennes pointed out that the doctrine of the "faculties of the mind" which was used earlier as the basis for the doctrine of formal discipline is not necessary for the latter. Moreover, the experimental investigations that have been made deal only with peripheral activities in which reflection and judgment have little or no part. Finally it is pointed out that the very simple (necessarily so) experiments that have been made show a considerable carrying over into other activities of the effects of training. Hence it is concluded that for practical purposes the

older doctrine of formal discipline so far from being exploded has not even been disturbed. It does require and is in fact in the process of receiving restatement to make it conform to the newer psychology, but its essential content need not be greatly different from the old doctrine.

12. Following the Thursday morning program and business meeting, members of the Association and other visitors were conducted by various members of the research staff through the Research Laboratory, where they were given some notion of the manifold activities involved in this side of the work of the Eastman Kodak Company's business. Costly apparatus of the most elaborate sort, a large range of chemical, optical and other physical equipment, fills the rooms in what seems to the tyro a crowded manner, but what the Laboratory's results show to be a systematic working plan. Many contacts of interest were made by the visitors on this tour of inspection which was finished all too quickly.

Meeting of the Trustees of the Association.

The following 26 persons and one institution, on applications duly certified, were elected to membership.

To individual membership.

P. L. Armstrong, A.M. (Southwestern Presb. Univ.). Instr., Georgia School of Tech., Atlanta Ga.
Sister M. Bernadita, A. B. (Univ. of Colorado). Head of dept. of math., Loretto Heights Coll., Loretto, Colo.
W. H. Boerckel, A.B. (Pennsylvania). Instr., Georgia School of Tech., Atlanta, Ga.
R. W. Brink, Ph.D. (Harvard). Asso. prof., Univ. of Minnesota, Minneapolis, Minn.
A. H. Cowling, A.M. (Texas). Prof., East Texas State Normal Coll., Commerce, Tex.
Elizabeth S. Dice, A.B. (Texas). Teacher, High School, Dallas, Tex.
H. A. DoBell, A.B. (Syracuse). Instr., Colgate Univ., Hamilton, N. Y.
J. L. Driscoll, A.B. (Washington and Lee). Instr., Georgia School of Tech., Atlanta, Ga.
E. F. Freier, A.B. (Minnesota). Eveleth, Minn.
J. C. Hinton, A.M. (Georgia). Dean, Wesleyan Coll., Macon, Ga.
D. N. Kingery, A.M. Prof., Macalester Coll., St. Paul, Minn.
Sister Laurentine Marie, A.B. (Trinity Coll., Washington). Instr., Emmanuel Coll., Boston, Mass.
O. B. Loewen, A.B. (Bethel Coll.). Head of dept. of math., Ottawa Univ., Ottawa, Kans.
T. R. Long, A.B. (Rochester). Asst., Univ. of Rochester, Rochester, N. Y.
R. L. Newlin, B.S. (Guilford). Instr., Guilford Coll., Guilford College, N. C.
R. L. O'Quinn, A.B. (Louisiana State). Instr., Louisiana State Univ., Baton Rouge, La.
Elsie M. Plapp, M.S. (Chicago). Asst. prof., Hollins Coll., Hollins, Va.
V. E. Pound, Ph.D. (Toronto). Instr., Univ. of Buffalo, Buffalo, N. Y.
R. S. Sheppard, M.A. (Alberta). Prin., High School, Strathcona, Alb., Canada.
G. S. Smith, A.M. (Texas). Head of dept. of math., High School, Port Arthur, Tex.
Virgil Snyder, Ph.D. (Göttingen). Prof., Cornell Univ., Ithaca, N. Y.
G. T. Trawick, B.S. in E.E. (Georgia School of Tech.). Instr., Georgia School of Tech., Atlanta, Ga.
Ann Van Oel, B.S. (Drake Univ.). Teacher, High School, Greene, Ia.
V. H. Wells, Ph.D. (Michigan). Prof., Carleton Coll., Northfield, Minn.
R. I. White, A.B. (Pennsylvania). Instr., Georgia School of Tech., Atlanta, Ga.
E. B. Wilson, B.S. (Chicago). Instr., Georgia School of Tech., Atlanta, Ga.

To institutional membership.

Providence College, Providence, R. I. Pres. William Noon, Official representative.

These names, together with those of all others elected to membership during the past three years, have been printed in the *Register of Officers and Members*, 1922.

The Trustees voted to recommend to the business meeting of the Association the draft of the proposed By-Laws which was presented on Thursday morning.

The resignation of Professor A. A. Bennett as Editor-in-Chief of the MONTHLY was received and accepted, it being understood that his duties terminate at the close of the volume for 1922. Professor W. B. FORD of the University of Michigan was appointed Editor-in-Chief beginning with the issue for January, 1923. Some of the Associate Editors for the year 1923 were appointed, the full list to be completed at the December meeting. Professor C. H. YEATON was appointed Assistant Secretary of the Association. Professor L. C. KARPINSKI was appointed Librarian of the Association, and Professor MARY EMILY SINCLAIR, Assistant Librarian.

The Trustees approved the formation of a Southeastern Section to include Alabama, Florida, Georgia, North Carolina and South Carolina, and possibly Tennessee.

The Trustees also transacted various matters of business concerning the Library of the Association, and the extension of the Association's subvention to the ANNALS.

The President was requested to send the greetings of the Association to Professor Neuberg of the University of Liège, and to the Belgian Mathematical Society whose official organ is *Mathesis*.

BUSINESS MEETING OF THE ASSOCIATION.

A draft of the proposed By-Laws of the Association, which had been considered at great length by the Trustees, was recommended by them to the Association; because of the delay of five months in the printing of the MONTHLY, printed notice of the proposed amendments and of the change in the Articles of Association was sent to the members of the Association one month before the summer meeting, the purpose being to comply with the spirit of the requirement of the By-Laws to give such notice through the official journal. The By-Laws as amended were unanimously adopted at the meeting on Thursday morning, September 7th, 1922, as well as the modifications of the Articles of Association which changed the number of Trustees from nineteen to twenty so as to include a Librarian.[1]

W. D. CAIRNS, *Secretary-Treasurer*.

[1] These By-Laws were printed with the *Register*, 1922, 44-46.

INFINITE AND IMAGINARY ELEMENTS IN ALGEBRA AND GEOMETRY.

By R. M. WINGER, University of Washington.

1. Introduction. Imaginary Points in Analytic Geometry. The traditional treatment of imaginary and infinite elements in algebra and geometry has curiously resisted the reform movement in American textbook writing. While imaginary numbers are receiving perhaps more attention in the algebra books, the imaginary point is still an outlaw.[1] The equation $x^2 + y^2 = 0$ "represents a single point or point circle"[2] while the equation $x^2 + y^2 + 1 = 0$ "represents no locus whatever." This is all the more remarkable in view of the historical fact that it was the geometric representation of imaginaries that gave these numbers their algebraic standing. Probably these same writers would not hesitate to speak of a point in space of four dimensions as a geometric image of a tetrad of numbers although such a point is as impossible to plot as one whose Cartesian coördinates in the plane are imaginary. Now that modern pure geometry has found a way to introduce imaginary points independent of algebra, it would seem absurd not to utilize the algebraic approach which is more elementary.

A reciprocal custom prevails with respect to infinity. Modern geometry has found a line at infinity indispensable as a bond of union between projective and metric properties, to render the principle of duality universally valid and to preserve a (1, 1) correspondence between a figure and its projection. Geometry has also found the number ∞ serviceable in the analytic theory of collineations, the parametric representation of curves and in studying the behavior of curves at infinity. But algebra steadfastly refuses to avail itself of these conveniences. True in the theory of equations in one variable the conditions for infinite roots are frequently given, but no mention is made of infinite roots of simultaneous equations where the idea is most valuable.

Authors of college algebras may hesitate to increase the formidable multiplicity of topics or they may regard the difficulties insuperable to freshmen. But these authors do not scruple to encroach on the fields of analytic geometry and even calculus to an extent that important algebraic material is crowded out altogether.[3] On the other hand, projective geometry has usually been presented synthetically so that any algebraic discussion would appear artificial. Careful writers on both subjects follow the sound practice—under the limitations they

[1] Exception must be made of the excellent book by Snyder and Sisam, *Analytic Geometry of Space*. Even here after discussing imaginary elements the authors lapse into the old form of statement,—that a tangent plane cuts an ellipsoid in a "point ellipse," *e.g.*, rather than in a pair of imaginary lines. Crawley and Evans, *Analytic Geometry*, is the only freshman text that has come to my notice which recognizes imaginary points.

[2] This statement always recalls a bizarre work on *Algeometry* in which the author declares that while $x^2 + y^2 = 0$ is the equation of a *round* point no one has been able to write the equation of a *square* point!

[3] One recent book of 250 pages devotes nearly a third of its space to calculus, both differential and integral.

have prescribed—of telling the truth and nothing but the truth even if it is impossible to tell the whole truth. An occasional author however fails to tell the strict truth, as for example when he says that a linear and a quadratic equation in two variables *always* have two solutions or that two quadratics always have four. And I am not sure that the writer who says that the locus of the equation $x^2 + y^2 = 0$ is a single point does not convey the impression that he is telling a general truth rather than a truth restricted to the real domain. For he invariably uses "point" and "locus" to signify "real point" and "real locus" without troubling himself to explain his meaning.

Certainly in analytic geometry where the streams of algebra and geometry merge there would seem to be the least excuse for these half-truths. For nowhere is the interplay between algebra and geometry more beautifully exemplified than in the theories of infinite and imaginary elements. The college student should come to the study of analytic geometry with a knowledge of imaginary numbers which, thanks to such books as Wilczynski's,[1] have been freed of their stigma and made genuinely real in his experience. Imaginary solutions of equations have been accepted on the same terms as real solutions. He has been taught that a quadratic equation in one variable always has two roots, real or imaginary, distinct or equal. Why then should he be told that certain equations have no loci and that equations with imaginary solutions represent curves which do not intersect, if indeed they represent curves at all? Why not take advantage of the ground already gained in algebra and say that a line and a circle in the plane always meet in two points, real and distinct, real and coincident or conjugate imaginary? The new statement not only serves every purpose of the old "two points, one point, no point" terminology but is actually more descriptive of the relation of the line and circle. A generator of a circular cone is a true example of a line that meets the circle of the base in but one point while the axis of the cone meets the circle in no point.

Absolute coördinates. It will be instructive to consider briefly a system of metrical coördinates in which the conventional rôles of real and imaginary are varied. We refer to the absolute coördinates used with such signal success by Professor Morley and his students in discussing metrical properties. In this system the bilinear axes are the circular rays through the Cartesian origin. If X and Y are the rectangular coördinates of a point, the absolute coördinates x, $\bar{x}$ are defined to be

$$x = X + iY, \qquad \bar{x} = X - iY, \qquad i = \sqrt{-1}. \tag{1}$$

The absolute coördinates, being complex, are represented geometrically in the usual manner on the complex plane. But if plotted separately x and $\bar{x}$ would in general determine two points. To avoid ambiguity we plot only the x, although the $\bar{x}$ cannot be ignored. When the Cartesian coördinates are real the absolute coördinates are in general imaginary. The converse is however not generally true. Neither does the reality of the absolute coördinates imply

[1] *College Algebra with Applications*, Boston, 1916.

reality of the corresponding point. Thus on the line

$$x + \bar{x} = 2, \qquad \text{or} \qquad X = 1, \tag{2}$$

those points are real whose x-coördinates are of the form $1 + iY$. The corresponding $\bar{x}$-coördinates are then $1 - iY$. A single real point of the line has real absolute coördinates, viz., the point (1, 1). But while the number pairs (2, 0), (3, $-$ 1), ($1 + \sqrt{2}$, $1 - \sqrt{2}$), (i, $2 - i$) satisfy the equation of the line, they cannot be represented. For if we plot the x alone the first three points would all lie on the line $x - \bar{x} = 0$ whereas this line cuts (2) only in the point (1, 1). If the algebra and geometry are to correspond, we must say that these pairs of numbers are coördinates of *imaginary* points on the line. In this system we have the following criterion:

A point is real or imaginary according as its absolute coördinates are or are not conjugate complex numbers.[1]

Again the parametric representation of curves in absolute coördinates differs from that in Cartesian coördinates. In the latter system real parameters are assigned to real points. In absolute coördinates, on the other hand, if the coefficients in the equations are real, the parameters of real points will be imaginary in general. Instead of thinking of the parameter as running along a line and ranging through all values, we may suppose the parameter to run around the unit circle in the complex plane, *i.e.*, the parameter t is a complex number of absolute value 1. The conjugate of t is then $1/t$. Thus the parametric equations of the unit circle are

$$x = t, \qquad \bar{x} = 1/t, \tag{3}$$

for the x-coördinate of a point is the same as the parameter. The only real parameters attached to real points of the circle are $\pm$ 1 which are cut out by the line $x - \bar{x} = 0$.

We need go no farther in this development here. Enough has been said perhaps to indicate the danger of dismissing imaginary coördinates as representing no points. The introduction of imaginary points into analytic geometry obviates a multitude of exceptions which are distasteful to a geometer who believes that his subject is every whit as good as algebra.[2]

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[1] Complex refers to either real or imaginary numbers, a real number being self-conjugate.

[2] One has but to glance through the recent book of Osgood and Graustein, *Plane and Solid Analytic Geometry*, to see how many such exceptions must be noted by an exacting writer. A conspicuous example is the important theorem (pp. 167–8) relating to a pencil of curves $u + kv = 0$. They divide the theorem into two parts according as $u = 0$ and $v = 0$ do or do not intersect. The first part has an exception when $u + kv = 0$ represents a single point (or a finite number of points). The second part reads: "Let $u = 0$ and $v = 0$ be the equations of two non-intersecting curves. Then the equation $u + kv = 0$ ($k \neq 0$) represents, in general, a curve not meeting either of the two curves." The beautiful positive theorem (part one) is thus replaced by a weak and purely negative theorem devoid of its chief interest. Moreover two exceptions are given (in addition to $k \neq 0$), viz., $u + kv = 0$ may represent (a) a single point, (b) no locus. Even then (b) must stand for such diverse cases as constant $= 0$ and $x^2 + y^2 + 2 = 0$. Finally a special case is given when $u = 0$ and $v = 0$ represent parallel lines. How much more satisfactory, having annexed the infinite and imaginary domain, to be able to say: $u + kv = 0$ (*always*) represents a curve through all the common points of $u = 0$ and $v = 0$, and, if $k \neq 0$ or ∞, through no other

2. The Number Infinity in Algebra. On the other hand, the algebraist who believes his subject is as good as geometry will be distressed at the failure of his fraternity to appropriate the advantage won by the geometer in the introduction of infinite elements. The geometer has found it convenient to postulate a point at infinity on every line or a line at infinity in every plane.[1] The most immediate consequence is that parallel lines may then be defined as lines which meet at infinity.[2] How can we make use of this definition and postulate in algebra? Perhaps the best approach is through homogeneous equations when infinite roots are placed on the same basis as zero roots. But we need not resort to homogeneous equations. We might begin by postulating a number ∞—call it an improper number if you like—which shall be the coördinate, in a Cartesian coördinate system along a line, of the "improper" point at infinity on the line.[3] This number is now clothed with properties to conform to those of its geometric counterpart. Thus the x-intercept of the line

$$y = mx - k \tag{4}$$

is

$$x_0 = k/m. \tag{5}$$

Now if $m = 0$, the line is parallel to the x-axis and cuts it therefore at infinity. Accordingly in virtue of (5) we attribute to ∞ the property

$$k/0 = \infty, \qquad k \neq 0, \tag{6}$$

for if $k = 0$ at the same time, the line (1) coincides with the x-axis and intersects it at every point.[4] It is then but a step to the theorem that the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_rx^{n-r} + \cdots + a_{n-1}x + a_n = 0 \tag{7}$$

has r infinite roots if $a_0 = a_1 = \cdots = a_{r-1} = 0$, so that an equation of apparent degree r may be regarded as an equation of degree n with $n - r$ roots infinite.

Although some writers explain that repeated and imaginary roots must be counted if an equation of degree n is to have n roots, few authors include infinite roots in the enumeration on the ground that the degree of the equation has been

point of either curve. The trivial exception here applies only to the parameter whereas the exceptions in the text restrict the base curves themselves. Again one objects to a definition (p. 312) which confines the polar of a point outside a conic to the line segment cut off by the conic—especially when its equation is given, for the equation certainly represents the whole line. These and kindred exceptions due to the refusal of the authors to countenance imaginary and infinite elements and to assign values to indeterminates mar, in my judgement, an otherwise admirable book.

[1] I wish to insist that this is not projective geometry. Any geometry that isolates a line at infinity and invests it with exceptional properties is not projective. The line at infinity is considered in projective geometry because it serves as a bridge between projective and Euclidean metric properties.

[2] This definition suffices for either the plane or space since two intersecting lines necessarily lie in the same plane.

[3] This is precisely what is done in Veblen and Young, *Projective Geometry*, vol. 1, chapter 7—EDITOR.

[4] Other properties may be assigned by considering the intercepts of the parabola $y = ax^2 + bx + c$, but this will suffice for our present purpose.

reduced. The recognition of infinite roots is however no less essential. For example, the student has learned to expect two solutions of a linear and a quadratic equation even though the roots be coincident or imaginary. Then he encounters two equations like

$$x^2 - 9y^2 = 7, \quad (8) \qquad\qquad x = 3y + 1, \quad (9)$$

which on combination lead to

$$9y^2 - 9y^2 + 6y = 6, \tag{10}$$

whence $y = 1$, $x = 4$. Again if he attempts to solve (8) with

$$x = 3y, \tag{11}$$

he will get

$$9y^2 - 9y^2 = 0 = 7, \tag{12}$$

a traditional absurdity. Do (8) and (9) then have a single solution while (8) and (11) have none? This will be the view of his textbook. The orthodox algebraist must say—if he does not avoid the cases in question—that pairs of linear and quadratic equations in two variables and with real coefficients fall into five classes as follows: those with two solutions which are (*a*) real and distinct, (*b*) real and equal, (*c*) conjugate imaginary, (*d*) those with one real solution, and (*e*) those with no solution whatever,[1] while his hyper-orthodox geometrical brother would say that a line cuts a conic in two points, one point or no point. In a sense the algebraic statement is above criticism—it is largely a question of the point of view. But the geometrical statement is not adequate since it affords no criterion for distinguishing a tangent from a line parallel to an asymptote, nor an asymptote from other lines which "do not cut the conic." Furthermore, it places tangent and asymptote in entirely different categories whereas an asymptote might at least be defined as the limit of a tangent. The heterodox but progressive analytic geometer would say that *a linear equation and a quadratic always have two solutions and that a line always cuts a conic in two points*—real or imaginary, distinct or coincident, finite or infinite—thus placing algebra and geometry on an equal footing. Only by recognizing infinite as well as imaginary roots and both imaginary and infinite intersections can this theorem be generalized to two equations (or curves) of degrees m and n.

But we need not go to simultaneous equations and the geometry of two dimensions nor to the infinite region of the plane to find a sound reason for introducing infinite elements. The recognition of infinite elements in the geometry on a curve, which is essentially a geometry of one dimension, is extremely useful. For example, the equation of a circle with radius 1 and center at the origin can be written parametrically in rectangular coördinates

$$\begin{aligned} x &= \cos\theta = (1 - t^2)/(1 + t^2), \\ y &= \sin\theta = 2t/(1 + t^2), \end{aligned} \tag{13}$$

where $t = \tan(\theta/2)$. To find the intersections with the x-axis, we set $y = 0$

[1] If imaginary coefficients are taken into account, he would have even more varieties.

whence $t = 0$, $x = 1$. But obviously the x-axis cuts the circle in a second point $(-1, 0)$. We must say then (a) there is one point on the circle with no parameter, (b) equations (13) represent a circle exclusive of the point $(-1, 0)$ or (c) the x-axis cuts the circle in a second point with parameter $t = \infty$. Is there anyone who will not admit that on considerations of continuity, generality and elegance the last statement is superior? Isn't it more in harmony with the spirit of modern mathematics and is it any less intelligible to the student?

It frequently happens in Cartesian geometry when curves are written parametrically that the parameter ∞ is attached to a point in the finite part of the plane. Another familiar instance is the folium of Descartes,

$$x = \frac{at}{t^3 + 1}, \qquad y = \frac{at^2}{t^3 + 1},$$

the double point of which is located at the origin but has the parameters 0 and ∞. If θ is the parameter on a curve and x and y are rational functions of $\sin\theta$, $\cos\theta$, $\tan\theta$, then the coördinates of the points on the curve can be expressed as rational functions of $\tan\theta/2$ by means of the relations in (13). There will be one point on the curve with the parameter ∞ and this point is as likely to be in the finite as in the infinite region.

3. Infinite Roots and Inconsistent Equations. The solution of the simultaneous equations:

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \end{aligned} \tag{14}$$

may be written

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{A}{C}, \qquad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{B}{C}\,. \tag{15}$$

Obviously the solution is unique unless $C = 0$. If we examine the possibilities when $C = 0$, we recognize two cases: (I) A and B not both $= 0$ and (II) $A = B = 0$.

Case I. $C = 0$, $A \neq 0$. The equations are commonly said to have no solution. There are three characteristic hypotheses depending on the way in which C becomes zero.

(a) $a_1 = b_1 = 0$. Then $A = -b_2c_1$, $B = c_1a_2$. The intercepts of the first line are $-c_1/a_1$ and $-c_1/b_1$ which under our hypothesis become infinite and the equation of the first line takes the form

$$0x + 0y + 1 = 0. \tag{16}$$

In other words the line has moved off to infinity and (16) may be regarded as the equation of the line at infinity. Since A is different from zero b_2, $c_1 \neq 0$ and the other line remains finite. The two lines thus meet at infinity,—at the infinitely distant point on the second line.

(*b*) $a_1 = a_2 = 0$. Then $A = b_1c_2 - b_2c_1$, $B = 0$ and the two lines reduce to $b_1y + c_1 = 0$ and $b_2y + c_2 = 0$. Since $A \neq 0$ the two lines exist and remain distinct. They are manifestly parallel to the x-axis (or one is parallel to the x-axis and the other coincident with it) and they meet therefore at the infinitely distant point of that axis.

(*c*) $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. Barring hypotheses (*a*) and (*b*) the equations assume the form

$$a_1x + b_1y + c_1 = 0, \qquad a_1x + b_1y + c_2 = 0,$$

whose solution is $x = \infty$, $y = \infty$. The lines are parallel and meet at infinity.

Under each hypothesis of case I we have found that the two equations may be interpreted as representing distinct lines which meet in a point at infinity. The coördinates of this point will be a solution of the equations and we no longer have pairs of inconsistent linear equations. Or we may retain the term "inconsistent," introducing the definition: The two linear equations (14) are inconsistent when and only when they represent lines which meet at infinity.

Under case II, $A = B = C = 0$ when the lines coincide and the equations have a single infinity of solutions. If however $a_1 = a_2 = b_1 = b_2 = 0$, the lines coincide with the line at infinity and the equations would then be "inconsistent."

Three linear equations. Similarly the solutions of the three equations

$$a_ix + b_iy + c_iz + d_i = 0, \qquad i = 1, 2, 3, \tag{17}$$

may be written $x = A/D$, $y = B/D$, $z = C/D$. If $D = 0$ but not all the numerators are zero, the values of one, two, or all of the unknowns will be infinite. The equations will then be "inconsistent" as before. Geometrically, the three planes meet in a unique point on the plane at infinity.[1]

If $A = B = C = D = 0$, the equations are dependent and have ∞^1 or ∞^2 solutions according as the planes have a common line or coincide. Such equations however may have "no solution" according to the orthodox view, as, *e.g.*, the three equations

$$ax + by + cz = d_i, \qquad i = 1, 2, 3, \qquad d_1 \neq d_2 \neq d_3. \tag{18}$$

These three planes are parallel and hence have their infinitely distant lines in common. We are led to extend our definition of inconsistent equations as follows: Three non-homogeneous linear equations in three variables, whether dependent or independent, are inconsistent when and only when they represent planes which meet only at infinity.

A general definition of inconsistent equations. These results may be generalized for a set of n linear equations in n variables.[2] Indeed the definition is valid when there are fewer equations than variables. Thus the equations of two parallel planes would be inconsistent. Furthermore any set of equations in

[1] There are three possibilities: (1) one plane cuts the other two and is parallel to their line of intersection, (2) two of the planes are parallel and the third a finite plane, (3) two planes meet in a finite line while the third is the plane at infinity.

[2] This demands of course the assumption that the locus of points at infinity in a linear space S_n of n dimensions is an S_{n-1}.

linear form will be inconsistent under similar conditions. For example, the equations

$$\begin{aligned} ax^2 + by^2 + c_1 &= 0, \\ ax^2 + by^2 + c_2 &= 0, \end{aligned} \qquad c_1 \neq c_2, \tag{19}$$

are linear in x^2 and y^2. Subtracting we should get constant $= 0$ so that the equations are inconsistent in the usual sense. But using formula (15) we find $x^2 = \infty$, $y^2 = \infty$ and we may say that the equations have two pairs of equal roots, *i.e.*, the curves have double contact at infinity.

Two equations need not be of the same degree to be inconsistent. An instance of this already noticed is the equation of a hyperbola and that of its asymptote. Another example is furnished by the equations

$$\begin{aligned} y^3 - xy^2 - x^2 + 2xy &= 0, \\ y^2 - y + x + 1 &= 0, \end{aligned} \tag{20}$$

which, on elimination of x, yield

$$0y^4 + 0y^3 + 0y^2 + 0y - 1 = 0. \tag{21}$$

The equations which should have six solutions thus have no finite solutions and the corresponding curves meet wholly at infinity. We may now formulate a general criterion for inconsistent equations:

r non-homogeneous (dependent or independent) equations in n variables, $r \leqq n$, are inconsistent when and only when the loci of the equations intersect, whether in real or imaginary points, wholly at infinity.

The purpose of this discussion has been not merely to criticize the shortcomings of our current elementary textbooks in respect to imaginary and infinite elements but to illustrate how these fruitful ideas can be employed to enrich both algebra and geometry. The suggestions here embodied are consonant with the spirit that pervades modern mathematics and with sound European tradition. It is my conviction that they could be incorporated into our elementary texts without sacrificing either clearness or rigor and at no greater cost of space than is required to detail the exceptions which they eliminate.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

26. Burckhardt on Modern Teaching.

The number of times the teaching of mathematics has been reformed and the general similarity of view of the reformers are always interesting to the student of the history of the subject. Every day, in educational circles, theories are launched forth that have been common property so long that it is not to be wondered that, in certain schools, the history of education is frowned upon,

ostensibly because "we must face the future" but really because the makers of these theories fear the revelations of the past. One of our perennial discoveries is that mathematics needs to have the waste material eliminated, that the pupil should study only those parts that he will use, and that the best place for this is in some form of laboratory.

Among my autographs is a letter on this subject, written somewhat over a century ago, by Johann Karl Burckhardt.[1] Since he became (1799) a naturalized French citizen, his given name more often appears as Jean Charles. He was a member of the Académie des Sciences and of the Institut (Classe des Sciences physiques et mathématiques) and, although primarily an astronomer,[2] is known for his factor tables and for his considerable interest in pure mathematics. M. Jean-Denis Barbié Du Bocage,[3] a well-known geographer and a member of the Institut, had asked Burckhardt for a plan for the mathematical training of his son, and the result was a letter which, in view of its modern ideas, might well form an appendix to any of the recent reports on the improvement of the teaching of the subject. The following is a translation:

Monday, August 30, 1813

My celebrated Colleague,

I have thought a great deal since yesterday about the best thing to do for your son. It seems to me that there are two points which ought to be considered: First, that it is frequently the case that there are taught for long periods many things which are useless from the practical standpoint and for the purpose which I have in mind for your son, the result being that there is not left sufficient time for the necessary features. Second, that the pupil should not be discouraged in his mathematical studies by any difficulties which he may encounter.

Here, then, is the plan which seems to me the most practicable:

(1) The teacher should present only those features of plane geometry which are necessary for the complete understanding of the theorems which form the foundation of surveying. He should therefore omit such non-essentials as the actual computation of the value of the number $3.14159\cdots$, a thing for which your son would have neither the time nor the need.

(2) After this the instructor should teach algebra up to and including equations of the second degree. The first chapters of Euler[4] form a good model for this kind of work and also include the theory of logarithms. I believe that this study is necessary, if only for the purpose of knowing how to read and use those algebraic formulas which are so necessary in practical mathematics.

(3) Plane trigonometry.

(4) Spherical trigonometry, which should be preceded by certain necessary theorems of geometry which were not taken up in the earlier course. The treatment which M. Delambre has given in his *Abrégé*[5] seems to me to merit preference.

(5) I have not mentioned solid geometry, but it is easy to determine a geographic position[6] even if ignorant that the cone is a third of a cylinder, &c, &c. The teacher may therefore give whatever the time permits.

Perhaps the difficulty will lie in finding a teacher who can carry out this idea. The plan requires more work on his part, fewer [private] lessons and [hence] a diminished financial return. Perhaps, also, he may hesitate to make himself known through fear that his colleagues will decry

[1] Born at Leipzig, April 30, 1773; died at Paris, June 21, 1825.

[2] I have, for example, the original manuscripts of his papers "Sur la comète actuelle" and "Tables de la Lune" (introduction), presented to the Institut in 1811, each with Delambre's signed memorandum.

[3] Born at Paris in 1760; died in 1825. He founded the Société de Géographie in 1821.

[4] His algebra had appeared at Petrograd (Petersburg) in 1770 and had later been translated into French and English.

[5] *Abrégé d'astronomie* (Paris, 1813) which had just appeared.

[6] Referring to M. Barbié Du Bocage's standing as a leading geographer.

him as disloyal to his profession.[1] On the other hand it is quite possible to lead the teacher to see that this work, once done, will serve to assist others, since circumstances will often force parents to demand more expeditious teaching and such as is directed towards practical ends.

With respect to the necessary equipment, it should be ordered early, since the makers are often engaged for a long time ahead.

Finally, on Nov. 8 and Nov. 9 certain stars will be in eclipse about 6 o'clock in the evening. If your son wishes to observe them with me, he may come about 5 o'clock. If the weather should be unsettled, he would not need to come, for other opportunities will offer themselves.

With sentiments of high consideration, I am,

Your servant and colleague
BURCKHARDT.

It would be interesting to know, but exceedingly wearisome to attempt to find, the number of times that these same ideas have been advanced in the hundred and ten years which have elapsed since Burckhardt "thought a great deal since yesterday" on the great problem of mathematical education. I often wonder if some of the reformers give any more time to the subject than Burckhardt did on this occasion.

27. BURCKHARDT AND THE ETERNAL PROBLEM OF PUBLICATION.

The World War has emphasized the difficulty of publishing scientific works of every kind, both in this country and abroad. The gift of the Hegeler Trust to the Mathematical Association of America is affording relief in one important line at present, the National Research Council has secured funds to enable it to make a beginning in another line, and the Carnegie Institution of Washington has lent a hand, but there still remains the great difficulty of finding some way of publishing existing manuscripts of undoubted mathematical value and of financing the preparation of others.

In view of this state of affairs it is interesting to see occasional evidences of the fact that the problem is not merely one of the present day. It was inadequately solved in earlier times by finding a patron among the old nobility and dedicating to him a work of which he could rarely comprehend even the general nature, but after the French Revolution this method passed rapidly into disuse and the favorite method came to be by resort to communications to learned societies.

Among my autographs is a letter written by Burckhardt[2] in 1814 showing the trouble he was having with his factor table, and it reads not unlike dozens of communications of the present day. The address is mostly obliterated, but it was that of a member of the Institut. The translation is as follows:

Friday, Sept. 30, 1814

Monsieur,

I propose to send, next Monday, a letter to M. Pfaff,[3] professor at Halle, for the purpose of placing in his hands a list of books which the Institut wishes to buy at the sale of the library of M. Klügel.[4] If you wish to purchase anything from this collection, I wish you would send me a note on Monday, at the Institut.

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[1] In the original, as a *gate-metier,* one who debases his profession.

[2] See page 298, footnote 1.

[3] This was Johann Friedrich Pfaff (1765–1825), who had gone to Halle four years before this.

[4] This was Georg Simon Klügel (1739–1812), whose dictionary of mathematics appeared at Leipzig 1803–1808 (3 vols. with later fourth and fifth volumes).

During the period in which we were so anxious to carry on our scientific pursuits, I worked out the factors of the third and fourth millions.[1] Madame Courcier has offered to print them if the Institut will contribute a thousand francs towards the expenses. This is only 500 francs for each million, although the second million alone has required a sacrifice of 1000 francs. I have a little repugnance at proposing this request; nevertheless I shall gain nothing except the uncomfortable labor of correcting the proofs, which probably no one after my death would ever attempt. On the other hand, I have seen in the reports of the past year that 1500 francs was given for the printing of botanical works, and yet that the mathematical section did not make any expenditure of a similar nature.

Madame Courcier has a good opinion of the sale of this work, since the English have bought several copies of the second million; if this hope is not maintained, she will probably increase her demand to such an extent that it would seem perhaps advisable to profit by the offer which she now makes.

I have the honor of being, with respect,

Monsieur,

V. T. H. & T. O. S,[2]
BURCKHARDT

Below there is a memorandum in Delambre's hand:

"Granted 1000 fr. 7 November 1814."

The letter reads so much like several that have come to my attention in recent years that I feel that it may contain some suggestions of value and some words of encouragement to those who have to meet the problem at the present time.

QUESTIONS AND DISCUSSIONS.

Edited by C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

NEW QUESTION.

The following question was asked by a member of the Ohio Section of the Association, at the meeting of April 20, 1922:

47. Can anyone teaching the theory of investment furnish references for bond valuation formulas when the income rate is a varying function of the time?

DISCUSSIONS.

In the discussion concerning definitions, Professor Allen draws attention to the inexcusable carelessness of many writers on algebra in the terminology of complex numbers. It seems a reasonable demand that an author should use the terms "real," "imaginary" and "complex" in such a way that it may be definitely determined whether or not a given number belongs to one of these classes; and it is rather surprising to learn how few writers of texts have taken the trouble to do this consistently.

The vague and unsettled use of the word "imaginary" in former times can scarcely be blamed for the present situation. It is true that negative numbers

[1] That is, of his *Table des diviseurs pour tous le nombres des premier, deuxième et troisième millions, avec les nombres premiers qui s'y trouvent,* which was published in complete form, at Paris, in 1817.

[2] Votre toute honoré et tout obeissant serviteur.

were at one time called imaginary, and that infinity is sometimes apologetically placed in that category. No doubt also, in popular speech, a cash balance becomes imaginary when it attains the value zero. Still, in careful mathematical speech, a certain definite usage has become established, and departures from this are for the most part the result of negligence.

It is, for instance, generally understood that "imaginary" is equivalent to "unreal," and that "complex" includes the real. When anyone admits these things on one page, and on the next writes "complex" for "imaginary," he is simply following the bad tradition that mars so many older text-books, of *forgetting to allow for the extreme cases.*

It would be easy, but unnecessary, to name some books of this type in which we can never be certain that the positive number system is not going to include zero, and where the inequality $x > y$ is or is not consistent with $x = y$ according to the caprice of the author. In geometry too a great deal of confusion commonly prevails about the limiting cases. Of the many writers who entreat the reader to accept as a triangle one that has a zero angle, how many take the trouble to warn him of the necessity of re-stating some elementary theorems? For instance, a triangle may have two angles equal, but no two sides equal.

The reader will of course not take too literally the "definition" of i as one of the roots of $x^2 + 1 = 0$. How do we know that this equation has two roots, and not more (as in quaternion theory)? The answer is that we have already at this stage decided that a root i exists, and that it shall obey certain formal algebraic laws. It is a consequence of these assumptions that $x^2 + 1 = 0$ only when $x = \pm i$. It is therefore a matter of choice whether we take i to be a one- or a two valued number.[1]

It may be noted that Professor Allen's last suggestion is exactly the opposite of one that is sometimes followed. For example, Harkness and Morley[2] denote by $\sqrt[n]{a}$ *any* nth root of a, and by a^x the exponential function of $x \log a$, where $\log a$ is the *principal* logarithm and hence a^x is one-valued. This plan has one inconvenient consequence: it makes an odd root of a negative number imaginary. It agrees with the general practice of making e^x one-valued.

The discussion by Professors Cajori and Miller arises from Professor Miller's former article on the same subject, and requires no comment.

I. Definitions of Imaginary and Complex Numbers.[3]

By Edward S. Allen, Iowa State College.

In reading certain parts of about 60 texts on algebra—those of Chrystal, Serret, and Weber among them[4]—I have discovered with surprise that the

[1] For the synthetic treatment of complex numbers as pairs of real numbers, see L. E. Dickson, *Elementary Theory of Equations*, p. 21.

[2] *Introduction to Analytic Functions*, London, 1898, p. 24, 168.

[3] Read at the April meeting of the Ohio Section of the Association.

[4] The list includes also books by all members of that subcommittee of the National Committee on Mathematical Requirements which made the admirable report on "Terms and symbols in elementary mathematics." Dickson's *First Course in the Theory of Equations*, published since this discussion was submitted to the editors of the Monthly, conforms to the policy here recommended in its elegant, brief exposition of the complex number system.

treatment of imaginary and complex numbers is in no instance free from logical inconsistency. A discrepancy is usually found before the following four items have been passed:

1. Definition of imaginary numbers,
2. Definition of complex numbers,
3. If a, b, and c are real, and $b^2 - 4ac$ is negative, then the roots of the equation $ax^2 + bx + c = 0$ are imaginary,
4. The sum of two complex numbers is a complex number.

Probably the most common error, one which persisted in books published as late as 1916, is the definition of an imaginary number as an even root of a negative number. A class of numbers which includes $1 + i$ and $\sqrt{3} + i$, but excludes $2 + i$ and $1 + \sqrt{3}\, i$, is of no value in connection with the usual theorems.

Some authors, realizing the strength of this argument, define imaginary (or pure imaginary) numbers as those whose squares are negative. A complex number s then defined as the sum of a real number and an imaginary (or pure imaginary) one. Real numbers are then not complex; yet the authors using these definitions do not balk at the statement that the sum of two complex numbers is always a complex number. $(2 + 3i) + (4 - 3i) = 6$.

Another group of books handles the subject in the following way. A pure imaginary number is defined as the product of i and of a real number (so that 0 is included). Imaginary and complex numbers are then declared to be identical —sums of real and pure imaginary numbers. The theorem about imaginary (or complex) roots of a quadratic equation loses its meaning, real numbers being, by definition, imaginary. This is Weber's procedure, for instance. Serret, in his *Algèbre Supérieure*, argues in a similar way. He uses the word "imaginaire" on page 86, volume 1, to indicate any number in the complex plane, but on page 269, for instance, to indicate one not on the real axis. Likewise Chrystal, on page 222 of his first volume, intends complex numbers to cover the whole plane; on page 134 ("the coefficients in the factors are complex numbers") he wishes them to avoid the axis of reals. And he uses the word "imaginary" with never a definition.

Two text-books I found, which were so careful as to miss rigor but slightly; perhaps their comparative freedom from error should be specifically mentioned. In Wilczynski and Slaught's *College Algebra* I detected only this flaw—that on page 35 complex numbers must avoid both axes, on page 105 they are excluded only from the real axis, while on page 189 they are allowed to occupy any position on the plane. In Eiesland's *Advanced Algebra* there is but this—on pages 64 and 66 it was forgotten that, according to the definitions used, 0 is real, pure imaginary, and complex, but not imaginary.

Now it is time for "constructive suggestions,"—suggestions which will in no case be new. In the first place let i be defined as one of the roots of the equation $x^2 + 1 = 0$. A pure imaginary number is then the product of i and of

any real number (including 0). A complex number is the sum of a real number and of a pure imaginary one. Finally, an imaginary number is a complex number which is not real. Linguistically, we don't like to have imaginaries come later than pure imaginaries. Very well, say "neomonic," if you wish. The important thing is that there must be a name to cover all numbers of the complex plane, a second one for those on the vertical axis, and a third for all numbers not on the horizontal axis.

Now that I am offering suggestions to writers of text-books and dictionaries, I will venture on another remark. $\sqrt{x}$ is always called a single-valued function when x is positive or zero, it is often regarded as such when x is negative, but it is undoubtedly double-valued when x is—if I may use the definition in the last paragraph—imaginary. $\sqrt[3]{x}$ is single-valued if and only if x is real, $\sqrt[4]{x}$ only if x is positive. It is perhaps inevitable, it is surely bewildering, that the same symbol should indicate, now a single-valued, now a multiple-valued function. There is a need for a symbol which shall always indicate that we may take our choice among all the nth roots of x. Should we not agree that $x^{1/n}$ shall be that symbol? Should not future books say that $4^{1/2} = \pm \sqrt{4} = \pm 2$?

II. The Formula $\frac{1}{2}a\ (a + 1)$ for the Area of an Equilateral Triangle.

A Reply to Professor Miller by Florian Cajori, University of California.

In this Monthly (*1921*, 257) Professor G. A. Miller writes on Gerbert's explanation of the question why $\frac{1}{2}a(a + 1)$ gives too large a value for the area of an equilateral triangle; Professor Miller claims that Cantor's figure is "inaccurate" and then states:

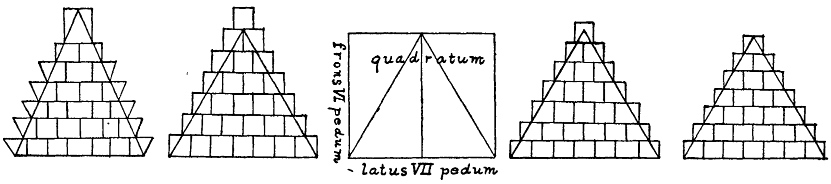
"What is more important is the fact that the corresponding figure found in various histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude, while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics*, 1896 or 1917, p. 132."

Professor Miller is in error; the figure in the 1917 edition does not represent "according to the explanations in the text an isosceles triangle whose base is equal to the altitude"; my figure, as well as my explanation, fit (as they should) the case of an equilateral triangle.

One is astonished at Professor Miller's declaration that Bubnov in his edition of Gerbert's *Opera mathematica* gives "a correct figure," and that those of Cantor, Günther and Cajori are all inaccurate or misleading. In the first place, Bubnov gives *two* figures. In the second place, Professor Miller misses completely the essential fact that we do not possess Gerbert's own drawing, and that the drawings in our histories are necessarily conjectural. Neither of Bubnov's two figures agrees with the figure given in Pez's edition of Gerbert's geometry and in the reprint of Pez by Migne in 1880. Pez's figure is due to some unskilful scribe of the Middle Ages whose copy of that geometry contains errors both in the text

and in the figures.[1] In the scribe's drawing (which we here reproduce) the 28 small areas ("XXVIII pedes") are not all alike either in shape or size; hence that drawing is obviously wrong and does not correctly represent Gerbert's reasoning.

Gerbert's text is not sufficiently explicit to enable us to reconstruct the figure with certainty. Three somewhat different attempts have been made, one by Cantor, a second by Günther, a third by Bubnov. We reproduce all three.



From MS. used by Pez — Bubnov's conjectures — Cantor's conjecture — Günther's conjecture

Bubnov's first figure represents Gerbert's rough approximation of 6 feet as the height of the equilateral triangle whose sides are 7 feet. Gerbert knew that 6 was only approximate. Nevertheless Bubnov exhibits the impossible figure in which the sides are 7 and the height is exactly 6.

Cantor's figure shows that the altitude really exceeds 6. For any equilateral triangle whose side is an integer exceeding 7, some of the squares will be wholly outside of the triangle and a very slight but obvious alteration of phraseology in Gerbert's letter would be the only change necessary to make the argument general.

In Günther's drawing, 28 rectangles are used instead of 28 squares. With Gerbert "quadratus" sometimes meant "rectangle."[2] These rectangles are together less in area than $\frac{1}{2}a(a+1)$, nevertheless they more than cover the triangle. The advantage of this construction is that it gives elegance to Gerbert's argument by being at once applicable to any equilateral triangle, be its sides 7 or 30 or a still larger integer. Professor Miller's declaration that Gerbert's argument is "trivial" because it cannot be extended to cases in which $a > 7$ is unwarranted by the facts.

Professor Miller misinterprets Hankel. When Hankel estimates Gerbert's letter as "the first mathematical paper of the Middle Ages which deserves this name," he means the Middle Ages *in Europe*. Hankel had considered Hindu and Arabic research in previous chapters. His statement is explicit; his "Mittelalter" refers, as he says, to the "abendländischen Nationen."

COMMENTS ON THE "REPLY" OF PROFESSOR CAJORI BY G. A. MILLER, University of Illinois.

The main question involved in the preceding "reply" seems to be whether square feet, or feet which are rectangles having unequal sides, were used in

[1] H. Weissenborn, *Gerbert*, Berlin, 1888, p. 42.

[2] See Gerbert's "Geometria" in J. P. Migne, editor, *Patrologiae cursus completus ecclesiae latinae*, vol. 139, Paris, 1880, p. 137. Gerbert, *Opera mathematica*, ed. by N. Bubnov, Berlin, 1899, p. 344.

measuring the areas of the particular equilateral triangles under consideration, since the figures in question relate to the following statement by Gerbert: "Behold in this little diagram there are included 28 feet, not all of which are integral."[1] A little earlier in the same letter Gerbert said: "I believe it is known to you what feet are linear, what square, and what cubic, and that in measuring areas we take only square feet."

One of the two rules given in this letter by Gerbert to find the area of an equilateral triangle was to multiply one half of the numerical measure of the base by the numerical measure of the altitude. There seems to be no question here as regards the kind of feet used, but, as I understand it, the author of the preceding "reply" aims to convey the impression that the figure in the 1917 edition of his *History of Elementary Mathematics* is correct because he used here the term *rectangle,* instead of *square,* to describe one of the 28 feet of which Gerbert spoke. On page 116 of his later second edition of "*A History of Mathematics*" (as well as in a part of the above "reply"), he retained the more definite term "square" for the same purpose, and it seemed only fair to give him credit for concluding finally that the latter term is the better in this connection. This credit seems to have been undue.

If one would like to assume that Gerbert used rectangles of unequal sides as units of measure, one meets with the difficulty of assigning numerical values to these sides as well as to the three concurrent edges of the corresponding rectangular parallelopiped used in measuring solids, to which Gerbert refers in the same letter. It is singular that while the present writer has claimed that Gerbert's letter is of much less scientific value than the references to it in various mathematical histories imply, he does not want to make this letter appear so ridiculous as, it seems to him, the author of the preceding "reply" does, who seems to try to uphold the favorable statements relating to this letter but also to attribute to it by implication a gross want of clearness as regards units of measure.

The foregoing remarks may suffice to make it clear that it is practically certain that Gerbert meant 28 *square feet* when he referred to the diagram to which the figures noted in the first part of the above "reply" relate and hence they seem to justify my claim that most of these figures are inaccurate. It should be noted that by an accurate figure I do not mean that it must be drawn to scale. The sides of the supposed equilateral triangle in the *History of Elementary Mathematics* noted above are not equal, and the squares, or what should be squares, have unequal sides, but I did not find fault with these points. What one seems to have a right to demand of a figure is that if one makes allowances for slight inaccuracies in construction and if one interprets the figure in the light of the subject matter and the text, it should actually illustrate the points in question. In the figure given by Cantor the vertex of the equilateral triangle is about .6 of a unit above the base of the uppermost square while it should be about .06 of a unit above this base. This seemed to me to be too much of a discrepancy to

[1] G. A. Miller, *School Science and Mathematics*, vol. 21, 1921, p. 650.

be regarded as correct while the discrepancy in Bubnov's figure seemed to be too slight to be noticed in a small drawing.

I fail to see why it is assumed in the above "reply" that I missed "completely the essential fact that we do not possess Gerbert's own drawing." It is true that I said that Bubnov gave a correct figure (I did not say that he gave only one figure), but it is to be assumed that the authors of histories in question thought they also gave correct figures, and this assumption is supported by the above "reply." No one, as far as I know, has accused these authors of being ignorant of the fact that their figures are not reproductions of Gerbert's figure. The present writer has never said in public or in private that he thought that we possessed Gerbert's drawing.

The main object that the present writer had in view in making the preceding remarks was to throw some additional light on the nature of the questions involved in the above "reply" to his article. It was there stated that the article "deals admittedly with questions whose satisfactory treatment would require a large amount of space." It can scarcely be supposed or desired that different writers on historical questions should agree as regards all details, since interpretation of language such as the one noted in the last paragraph of the preceding "reply" is often involved, but I hope that enough has now been said to make the main points in question clear to all who may be interested therein.

Final Note on the Discussion by Florian Cajori, University of California.

Professor Miller's original contentions are (1) that Bubnov has a "correct" figure, (2) that Cantor's and Günther's are "inaccurate" and "misleading," (3) that Cajori's represents "an isosceles triangle whose base is equal to the altitude," (4) that Gerbert's explanation is "trivial," (5) that Hankel's appreciation of Gerbert's letter is invalid.

Of these only the first two require further comment. My contention that, from Gerbert's text, one cannot tell with certainty what Gerbert's own figure really was, seems borne out not only by the different conjectures made by Cantor, Günther and Bubnov, but also by the fact that Professor Miller, after pronouncing Bubnov's figure "correct," suggested in *School Science and Mathematics*, XXI, p. 652, a different figure of his own. This, however, *does not fit Gerbert's equilateral triangle*. To me, Bubnov's figure is impossible, because it represents to my eye the height as exactly 6. In my judgment, Cantor's figure conveys the truth even to a casual reader by rendering the existence of a fractional excess over 6 strikingly visible. Cantor's and Günther's, though different, appear to me satisfactory illustrations to Gerbert's ingenious letter. In my *History of Mathematics*, 1919, I follow Cantor's explanation; in my *History of Elementary Mathematics*, 1917, I follow Günther's.

Professor Miller says that the main issue is whether the unit of area in Gerbert's letter is the square foot or a rectangle that is not a square. But I have not claimed nor implied that it was other than a square foot. We agree further that "quadratus" sometimes meant "rectangle," for in his own translation of

Gerbert's letter this rendering is given once. His criticism of Günther's figure is valid only if other interpretations of the Latin involved are invalid. Günther's argument, I take it, is simply this, that $\frac{1}{2}a(a+1)$ squares exceed the $\frac{1}{2}a(a+1)$ rectangles drawn, which in turn exceed the area of the given triangle. Finally, I predict that, if Gerbert's own figure is ever brought to light, it will be found to be not Bubnov's nor Miller's, but Cantor's or Günther's.

RECENT PUBLICATIONS.

REVIEWS.

New Mathematical Pastimes. By P. A. MacMahon. Cambridge, University Press, 1921. 8vo. 7 + 116 pages. Price 12 Shillings.

This is an interesting little volume filled with strange and bizarre figures,[1] and punctuated with quaint quotations in verse.[2] The notions are extremely simple (which may surprise readers of some of the author's more serious works) and the language is clear almost to verbiage, as for example (page 50): "The question now before us is the transformation of the set of pieces so that they will no longer be of the same shape and differently coloured or numbered. It proves to be possible to effect this so that the pieces are of different shapes, and are not differently coloured, numbered or otherwise distinguished. In fact, instead of having a set of pieces of the same shape and differently coloured we construct a set of different shapes but of the same colour. The boundary of the assembled pieces now varies in shape with each type and variety but is not otherwise distinguished." The matter is as a whole fresh and pleasingly consecutive. It is far removed in content and treatment from the advanced texts usually issued by the Cambridge University Press.

The author describes his purpose in the following words: "The author of this book has, of recent years, devoted much time and thought to the development of the subject of 'Permutations and Combinations' with which all students are familiar. He has been led, during that time, to construct, for use in the home circle, various sets of pieces, of elementary geometrical shapes based upon these ideas, and he now for the first time brings them together with the object of introducing, in a wider sphere, what he believes to be a pleasant by-path of mathematics which has almost entirely escaped the attention of the well-known writers upon Mathematical Recreations and Amusements. The book differs *in toto* from their works because everything that it contains, with scarcely an exception, is the invention of the author. It is not a bringing together of materials derived from wholly different ideas. From beginning to end it proceeds along one defined

[1] In its few small pages are one hundred thirty-five separate cuts, averaging some ten distinct designs to a cut.

[2] There are fifty-seven poetic quotations from a wide range of classical sources, scattered through the book. Most of these in their original setting could have had but the most tenuous connection with the thought of this text. They serve merely to entertain the reader by their casual verbal allusions.

path from which it never diverges. One continuous thread of thought runs through it from cover to cover."

The book is divided into three parts. The first two are intimately related and yield a systematic method of constructing certain types of what in America we call "jig-saw puzzles." Those here treated have all of their pieces distinct in shape and possess the peculiar advantage of having a minimum number of distinct types of edges. The age-old and ever fresh popularity among children of picture cubes and jig-saw puzzles is ample excuse for the interesting and novel study here presented. A map dissection, in which no two pieces will fit together unless they match in the completed mosaic, will amuse youthful minds but gives training and practice in little more than the simplest sort of visual perception. A puzzle offering the maximum number of possible pairings other than the correct one has an added interest proportional to the ingenuity required. Probably most readers of the MONTHLY have seen the celebrated "Chinese" checkerboard puzzle, for which the checkerboard is sectioned into many differently shaped pieces each of which contains several of the original component squares. This deservedly popular incitement to hours of fruitless patience is not mentioned explicitly by the author. The checkerboard puzzle is exasperatingly baffling, but the reviewer feels convinced that for a given number of pieces the systematic treatment by the author bids fair to yield the most ingenious designs. Perhaps some enterprising toy-maker will forthwith attempt to copyright a large number of these and the present generation may yet greet a whole class of fresh puzzles, new in detail but in general character nearly as old as the human race. The author refrains from touching upon related but diverging problems, such as the dissection of a given figure into an assigned number of pieces which may be refitted to form another given figure.

Part III while following readily upon the previous portions of the book has a wider mathematical appeal, and does not in fact require the elaboration of the first two parts for its understanding. It is concerned with the principles of repeating designs. The treatment is synthetic in that it proceeds from the elementary bases of triangles, parallelograms, squares and hexagons to more elaborate figures. In this third part, one meets many interesting and novel designs but the reviewer is personally disappointed at the lack of suggestive references. The subject is obviously identical in content with the analysis of repeated designs in the Euclidean plane. Far from being new, this is one of the classic lines of investigation in connection with elliptic functions. A possible fundamental region for an elliptic function is no other than a figure whose duplicates under translation may be made to fill the plane without overlapping. The celebrated investigations which in a vastly wider domain have applied group theory to transcendental functions (typified for example by Klein's work on the elliptic modular function) have been left unacknowledged and worse still apparently unutilized by the author. A more definite reduction to the standard types of fundamental triangles might have systematized the discussion. The only excuse although it is perhaps an adequate one for this proofless synthetic treatment is the

fact that the work merely claims to be a pastime. The combinatorial character of the first two parts is suggestive of the analogous problem of the magic square. There one does not expect a completely analytic treatment except under favorable restrictions. The third part deals rather with systematic material affected by the presence of arbitrary arcs. It becomes a problem of analysis situs combined perhaps with certain non-scientific esthetic demands of minor interest. Possibly in the more extensive discussion that the author promises the objections here raised may be no longer applicable.

A useful but limited chronological bibliography of works on mathematical recreations is given in the appendix.

The casual reader may be disappointed in finding no explicit statement of puzzles to tax his ingenuity and arouse his interest. The "pastimes" are suggested rather than directly announced. But any one with a little imagination and with a healthy taste for numerical and tactical puzzles will find a fruitful source of amusement in this book. A far more instructive book might perhaps have been developed upon the same basis of original material but no one should demand instruction in any book claiming to offer nothing more than new mathematical pastimes.

ALBERT A. BENNETT.

NOTES.

The excellent *Revue de l'Enseignement des Sciences*, published by Felix Alcan, Paris, 1909–1920, has ceased publication. Its place is taken by *Bulletin Scientifique des Professeurs de l'Enseignement du 2e degrê* (*B.S. 2*) which has been published twice a month since October, 1921, at Chaussenueil, France; P. Martin, director; 12 francs a year.

Mr. ROGER S. HOAR, a member of the Association, has printed privately a discussion of the mechanics of a new design of gasoline power shovel. It is in a simple form, convenient for students of elementary mechanics. Any number of copies gratis are to be had upon application to him, care of Bucyrus Company, South Milwaukee, Wis. This suggestion may appeal to some teacher looking for "practical" applications.

Mr. HARRY B. MARSH, head of the department of mathematics in the Technical High School, and Springfield Junior College, Springfield, Mass., has published a pamphlet: *Elementary Algebra Outline based upon College Entrance Requirements and Examination Papers* (New York, Newson and Co., 1922, 48 pages).

A twenty-nine page printed report to the College Entrance Examination Board upon Elementary Algebra, Advanced Algebra, and Plane Trigonometry by the Commission on College Entrance Requirements in Mathematics, 431 West 117th Street, New York City, has recently been distributed. The Commission's Report on Plane Geometry and Solid Geometry is being printed separately. The report is elaborate and specific and should come to the attention of all interested in the ground covered in sub-freshman work in mathematics. Professor W. F. OSGOOD, of Harvard University, is chairman of the Commission.

We have already indicated (*1921*, 267) the contents of the first edition of

Suggestions for Students of Mathematics. Mathematics and Life Activities (7 pages), issued in March, 1921, by the Department of Mathematics of Brown University. An extended review appeared in *Mathematics Teacher*, volume 14, pages 349–350. A third revised and enlarged edition was published in October, 1922. In this edition a new section is devoted to a "Literature List"; that is, a list of 23 books, in English, and in the mathematical library of the University, "which may be read with profit—though not in every case with complete understanding—by those who are taking, or have credit for, freshman mathematics." A limited number of these pamphlets are available for distribution to those interested.

Two new parts of the *Encyklopädie der Mathematischen Wissenschaften* have been published April 1 and May 15, 1922, respectively. The first part, V-2-5, "Elektronentheorie der Metalle" by Rudolf Seeliger, 15+777-878 pages, completes volume V-2. The second part, III-1-9, "abgeschlossen 31 Aug. 1916", is "Polyeder und Raumeinteilungen" by Ernst Steinmetz, 1-139, special paging in italics. It is announced that III-1-8 with register for III-1 will appear later. There are now 10 complete part volumes of the *Encyklopädie*, namely: I-1, I-2, II-1-1, II-1-2, II-2, IV-1, IV-3, IV-4, V-1, and V-2.

For many years, 1900-1915, an important feature of *Bibliotheca Mathematica* was a section devoted to "Kleine Bemerkungen" on the latest edition of Cantor's *Vorlesungen über Geschichte der Mathematik*. In this way a great many corrections, and much valuable information, were brought together. In *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 31, 1922, pages *73-77*, a section of this kind has been started, and will continue so long as the publication of *Bibliotheca Mathematica* is suspended. In this section there are 11 notes, by Ferdinand Rudio, on various passages in Cantor's volume 3, pages 549-624.

ARTICLES IN CURRENT PERIODICALS.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 28, January–February, 1922: "The twenty-eighth summer meeting of the American Mathematical Society" by R. G. D. Richardson, 1–15; A letter from the President of the Society, G. A. Bliss, 16; "Report on topics in the theory of divergent series" by W. A. Hurwitz, 17–36; "Note on an irregular expansion problem" by D. Jackson, 37–41; "The Bôcher Memorial Prize" by E. B. Van Vleck, 42–44; "On Kakeya's minimum area problem" by W. B. Ford, 45–53; "Convergence of sequences of linear operations" by T. H. Hildebrandt, 53–58; Review by D. Jackson of C. de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle* (Paris, 1919), 59–61; Reviews by D. E. Smith of B. Lefebvre, *Notes d'Histoire des Mathématiques* (*Antiquité et Moyen Âge*) (Bruxelles, 1920), of *Bibliotheca Chemico-Mathematica* (2 volumes, London, 1921), and of W. W. Bryant, *Kepler* (London, 1920), by P. Field of P. Burgatti, *Lezioni di Meccanica Razionale* (2d edition, Bologna, 1919), and of C. Burali-Forti and I. Boggio, *Meccanica Razionale* (Turin and Genoa, 1921), by A. R. Crathorne of A. Loewy, *Mathematik des Geld- und Zahlungsverkehrs* (Leipzig and Berlin, 1920), by V. Snyder of T. Schmid, *Darstellende Geometrie*, Vol. II (Berlin and Leipzig, 1921), by H. B. Phillips of F. Reiche, *Die Quantentheorie, ihr Ursprung und ihre Entwicklung* (Berlin, 1921), and by D. N. Lehmer of L. Poletti, *Tavole di Numeri Primi entro Limite Diversi, e Tavole Affini* (Milan, 1920), 62–71; Notes, 72–82; New Publications, 83–88—March: "The October meeting of the American Mathematical Society" by R. G. D. Richardson, 89–94; "The October meeting of the San Francisco Section" by B. A. Bernstein, 94–97; "The simple group of order 2520" by G. A. Miller, 98–102; "A theorem of oscillation" by W. E. Milne, 102–104; "A two-way infinite series for Lebesgue integrals" by M. B. Porter, 105–108; "Note on Euler's

φ-function" by R. D. Carmichael, 109–110; "Extensions of Dirichlet multiplication and Dedekind inversion" by E. T. Bell, 111–122; "Charles Leonard Bouton" by W. F. Osgood, J. L. Coolidge, G. H. Chase, 123–124 [A Minute read before the Faculty of Harvard University, March 28, 1922]; Review by V. Snyder of *Felix Klein: Gesammelte mathematische Abhandlungen.* Vol. 1. *Liniengeometrie, Grundlegung der Geometrie, Zum Erlanger Programm* (Berlin, 1921), 125–129; Reviews by A. Emch of G. Scheffers, *Lehrbuch der darstellenden Geometrie* (2 volumes, Berlin, 1919, 1920), and of G. B. Halsted, *Girolamo Saccheri's Euclides Vindicatus* (Chicago, 1920), by L. W. Dowling of E. Bally, *Géométrie Synthétique des Unicursales de Troisième Classe et de Quatrième Ordre* (Paris, 1920), and by F. Cajori of A-C. Clairaut, *Éléments de Géométrie* (Paris, 1920), 130–133; Notes, 134–138; New Publications, 139–144—April–May: "The fourteenth regular meeting of the Southwestern Section" by E. B. Stouffer, 145–147; "The twenty-eighth annual meeting of the American Mathematical Society" by R. G. D. Richardson, 148–164; "The moment of inertia in the problem of n bodies" by W. D. MacMillan, 165–168; "Substitutions commutative with every substitution of an intransitive group" by G. A. Miller, 168–170; "Summable infinite determinants" by W. L. Hart, 171–178; "Algebraic guides to transcendental problems" by R. D. Carmichael, 179–210; Review by J. P. C. Southall of *Oeuvres complètes de Christiaan Huygens publiées par la Société Hollandaise des Sciences.* Tome treizième. *Dioptrique* (La Haye, 1916), 211–214; "Books on relativity" by G. D. Birkhoff, 215–221 [Review of *Das Relativitätsprinzip. Lorentz. Einstein. Minkowski* (Leipzig and Berlin, 1920), H. Weyl, *Raum. Zeit. Materie* (Berlin, 1921), A. Einstein, *Relativity. The special and the general Theory* (Translated by R. W. Lawson, New York, 1921), R. D. Carmichael, *The Theory of Relativity* (New York, 1920), A. Angerbach, *Das Relativitätsprinzip* (Leipzig and Berlin, 1920), A. N. Whitehead, *The Concept of Nature* (Cambridge, England, 1920), and L. E. J. Brouwer, *Wiskunde, Waarheid, Werkelijkheid* (Groningen, 1919)]; Reviews by E. B. Wilson of C. Schaefer, *Einführung in die theoretische Physik* (Berlin, 1921), by E. S. Allen of D. M. Y. Sommerville, *The Elements of Non-Euclidean Geometry* (Chicago and London, 1919), by E. W. Brown of *Annuaire du Bureau des Longitudes pour 1921* (Paris, 1921), by J. W. Young of L. Heffter, *Die Grundlagen der Geometrie als Unterbau für die Analytische Geometrie* (Leipzig and Berlin, 1921), by C. N. Reynolds of L. Fabre, *Les Théories d'Einstein* (Paris, 1921), 222–224; Notes, 225–227; New Publications, 228–232.

Bulletin des Sciences Mathématiques, volume 46, May, 1922: "Sur l'itération de certaines fonctions algébriques" by M. P. Fatou, 188–198; "Sur un invariant cinématique et le théorème de la composition des vitesses" by M. G. Koenigs, 198–199; "Sur un théorème de M. Fatou" by M. G. Valiron, 200–208.

Journal de Mathématiques pures et appliquées, ninth series, volume 1, fasc. 2, 1922: "Les théories einsteiniennes et les principes du calcul intégral" by A. Buhl, 95–104; "La théorie de la relativité et les faits observés" by S. Zaremba, 105–139; "Sur les équations de la gravitation d'Einstein" by E. Cartan, 141–203; "Relativité restreinte et géométrie des systèmes ondulatoires" by J. Le Roux, 205–253.

Journal of the Indian Mathematical Society, volume 14, April, 1922: "Progress report" by P. V. Seshu Aiyar, 41–42; "A statistical study of some examination marks" by P. V. Seshu Aiyar and S. R. Ranganathan, 43–54; "Determinants whose elements are Eulerian, prepared Bernoullian, and other numbers" by C. Krishnamachary and M. Bheemasena Rao, 55–62; "Mathematics in India, then and now" (conclusion) by V. Sankaran, 63–67; "Some criteria for divisibility" by P. K. Das, 68–69; "Perpetual calendar for every year of the Christian era" by C. Ganapati Aiyar, 70–71; Questions and Solutions, 72–79.

Mathesis, volume 36, June, 1922: "Bibliographie du triangle et du tétraèdre" (to be continued) by J. Neuberg, 209–211; "Sur une transformation homographique d'une équation" by F. Simonart, 212–218; "Sur deux groupes de trois coniques" by J. Degueldre, 218–219; "Sur l'ellipse et le cercle de Nagel d'un triangle" (conclusion) by J. Neuberg, 220; "Application du calcul vectoriel à la détermination du couple gyroscopique" by F. Bouny, 221–225; "Remarques sur l' 'Arithmétique' de Simon Stevin" (to be continued) by H. Bosmans, 226–231; "Sur un théorème de M. d'Ocagne et son extension aux courbes gauches" by R. Goormaghtigh, 232–233; "Un théorème de M. Appell" by Ad. M., 234–236; "Nécrologie," 236–237; "Notes mathématiques," 237–241; Questions and Solutions, 242–256.

Nature, volume 109, June 17, 1922: "More books on relativity" by E. Cunningham, 770–772 [Review of W. H. V. Reade, *A Criticism of Einstein and his Problem* (Oxford, 1922), H. Dingle, *Relativity for All* (London, 1922), C. Nordmann, *Einstein and the Universe: A Popular Exposition of the Famous Theory* (London, 1922), J. Becquerel, *Le Principe de Relativité et la Théorie de la*

Gravitation (Paris, 1922), G. Mie, *La Théorie einsteinienne de la Gravitation: Essai de vulgarisation de la théorie* (Paris, 1922), E. M. Lémeray, *L'Éther actuel et ses précurseurs* (*simple récit*) (Paris, 1922), C. Von Horvath, *Raum und Zeit im Lichte der speziellen Relativitätstheorie* (Berlin, 1921). Chiefly a criticism of Nordmann's book. "The book is certainly readable. The language is not only clear, but also picturesque. 'Einstein may be a treasure, but there is a fearsome troop of mathematical reptiles keeping inquisitive folk away from it. Let us drive them off with the whip of simple terminology, and approach the splendour of Einstein's theory.' This is the author's intention, like that of many others; how does he succeed? . . . Now comes the task of describing in 'simple terminology' the work of reconstruction. The reader must judge whether the following is simpler than a brief algebraic equation: 'The distance in time and the distance in space are numerically to each other as the hypotenuse and another side of a rectangular triangle are to the third side, which remains invariable. Taking this third side for base, the other two will describe above it, a triangle more or less elevated according as the velocity of the observer is more or less reduced. This fixed base is a quantity independent of the velocity. It is this which Einstein has called the "*Interval*" of events.' . . . M. Nordmann is well equipped in many ways for it. But he has fallen into the very common error of supposing that the essential truth can be given while omitting the demand for concentrated thinking on vital details. The world suffers far too much from loose thinking already. Vague generalisations, misleading analogies, superficial manifestations are made to do duty for precise statements, logical reasoning, and fundamental principles. It is not necessarily true that mathematical skill is the only way of approach to an understanding of Einstein's fundamental ideas. But it is certain that if such an understanding is to be reached it can only be by going down to a patient analysis of our own preconceived notions until we find them insufficient."]; Review of L. E. Dickson, *First Course in the Theory of Equations*, 773–774 ["Prof. Dickson's book possesses all the merits of an excellent text-book, and it is to be hoped that its circulation will be a wide one."]; "Immediate solution of dynamical problems" by G. Greenhill, 778 ["A discussion is submitted here in the manner called elementary, where the theorems of the gravitation of a sphere are proved for any portion of a spherical surface, such as a bowl, before proceeding to the result for a complete sphere."]—June 24: "A century of astronomy" by A. S. Eddington, 815–817 [From the presidential address delivered before the Royal Astronomical Society on May 30.]—Volume 110, July 1: "Advanced mathematical study and research at Cambridge" by H. S. Carslaw, 8—July 8: "Metric and British measures" by R. J. T., 29–30 ["In view of the vigorous and sustained efforts of the exponents of the metric system, and the eminent names that are to be found among them, it is perhaps not a little surprising that it makes so little progress towards general acceptance in Great Britain. . . . A fundamental distinction must, at the very outset, be drawn between the importance of stability in the units of quantity and of dimension respectively. The units of mass and capacity, speaking generally, serve simply for determining a certain quantity of goods, and the margin of tolerance is usually fairly large; . . . The unit of length is of a different character. Size, which determines the interchangeability of parts and fittings, is not capable of ready adjustment, and an error in dimensions often involves the waste of the whole article. . . . The view is quite widely held that the Imperial units are, as magnitudes, more suitable for commercial purposes than the metric. There is, therefore, at least a possibility that the solution of the metric controversy may be found in the development of a system based upon the British units, but so modified as to be capable of treatment on pure decimal lines. The Report of the Committee of the Conjoint Board suggests that the possibilities of such a solution should be explored, and one experiment in this direction has already been tried with success. The troy pound was abolished in 1878, but the troy ounce was too firmly established to be dismissed entirely. Trade in the precious metals, however, is now carried on in terms of troy ounces only, and bullion weights are made up solely in decimal multiples and sub-multiples of that unit. There has certainly been some activity in this direction in recent years, and should a really logical system upon a decimal basis be devised and secure general acceptance in the countries now using Imperial units, it may be found that these units are, after all, destined to survive."]; "Absolute measurements of sound" by A. G. Webster, 42–45; "Prof. J. C. Kapteyn, For. Mem., R. S." by A. C. D. Crommelin, 48–49.

Revue Générale des Sciences, volume 33, May 30, 1922: "A propos de la conception einsteinienne de l'espace fini" by R. Paucot, 289–290—June 15: "Une famille d'astronomes: les Herschel. A propos du centenaire de la mort de William Herschel" by E. Doublet, 326–330.

Revue de Mathématiques Spéciales, volume 32, July, 1922: "Démonstration géométrique des formules de Serret-Frenet" by L. Pomey, 217–219.

Science, volume 55, June 23, 1922: "Are scientists encouraging popular ignorance?" by E. C. Bingham, 664–667 [Considers the objections to the metric system in the report of the National Industrial Conference Board.]—Volume 56, July 14: "Methods of German publishers" by R. C. Archibald, 45 ["In Germany the chief publishers of mathematical books and periodicals are Springer, Teubner, and Vereinigung wissenschaftlicher Verleger (a combination of the firms: Göschen, Guttentag, Reimer, Trübner and Veit). They have decided that for their mathematical publications of 1922 Americans shall, in general, be required to pay at least as much as $2.40 per 100 marks of the price for Germany. Of *Jahrbuch über die Fortschritte der Mathematik*, volume 45 . . . part 1 . . . and part 2 . . . are sold in Germany for 73 and 190 marks respectively. The corresponding prices for America are $4.65 and $9.00! Such extortion ought appreciably to hasten the appearance of an American abstract journal, the establishment of which has been already approved by the National Research Council. But again, *Journal für die reine und angewandte Mathematik* (Crelle), volume 151 (1920–21), is sold in Germany for 96 marks; the price to America is $6.00! . . . The above facts, obtained from the publishers themselves on May 26 and May 31, 1922, will probably suggest to mathematicians the immediate cancellation of all contemplated orders for the publications of Vereinigung wissenschaftlicher Verleger—at least."]

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. P. Manning.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

2981. Proposed by H. P. MANNING, Providence, R. I.

Find the envelope of the circle on which two diametrically opposite points divide in a given ratio the focal radii of a variable point on an ellipse or hyperbola.

2982. Proposed by H. P. MANNING, Providence, R. I.

Solve the following problem without using analytic geometry: A triangular yard has a post at each corner. Given the lengths of the sides of the yard and of the posts, find the length and the position of the foot of a ladder that will just reach the top of each post without changing this position.

2983. Proposed by R. S. UNDERWOOD, Alabama Polytechnic Institute.

Prove that the following terminating series have the sums indicated, where by 0! is meant 1:

$$(1)\quad \frac{1}{n!\,1!}+\frac{1}{(n-2)!\,3!}+\frac{1}{(n-4)!\,5!}+\cdots=\frac{2^n}{(n+1)!},$$

$$(2)\quad \frac{1}{(n+1)!\,0!}+\frac{1}{(n-1)!\,2!}+\frac{1}{(n-3)!\,4!}+\cdots=\frac{2^n}{(n+1)!}.$$

(These coincide for n even but are distinct for n odd),

$$(3)\quad \frac{1}{n!\,1!}-\frac{1}{(n-2)!\,3!}+\frac{1}{(n-4)!\,5!}-\cdots=\frac{\alpha(-4)^{[n/4]}}{(n+1)!}$$

where $[n/4]$ denotes the greatest integer not exceeding $n/4$, and where $\alpha = 1, 2, 2, 0$ according as n is congruent to 0, 1, 2, 3 (mod 4).

Obtain a similar form for the series

$$(4)\quad \frac{1}{(n+1)!\,0!}-\frac{1}{(n-1)!\,2!}+\frac{1}{(n-3)!\,4!}-\cdots.$$

Obtain also the sums of the following terminating series and of the four series obtained from them by changing the signs of the alternate terms:

(5) $\frac{1}{n!\,1!} + \frac{1}{(n-4)!\,5!} + \frac{1}{(n-8)!\,9!} + \cdots,$

(6) $\frac{1}{(n+1)!\,0!} + \frac{1}{(n-3)!\,4!} + \frac{1}{(n-7)!\,8!} + \cdots,$

(7) $\frac{1}{(n-2)!\,3!} + \frac{1}{(n-6)!\,7!} + \frac{1}{(n-10)!\,11!} + \cdots,$

(8) $\frac{1}{(n-1)!\,2!} + \frac{1}{(n-5)!\,6!} + \frac{1}{(n-9)!\,10!} + \cdots.$

2984. Proposed by A. L. CANDY, University of Nebraska.

Find the number of numbers of n digits each that can be written with n consecutive digits, allowing all possible repetitions, such that the sum of the digits in each number is a multiple of n.

2985. Proposed by A. L. CANDY, University of Nebraska.

Find the number of combinations of n digits each that can be made with the first n consecutive digits, allowing repetitions, and such that the sum of the digits in each combination is a multiple of n.

SOLUTIONS.

2890 [1921, 184]. Proposed by B. F. FINKEL, Drury College.

Having given a triangle whose base is $2c$ and (a) the sum of whose other two sides is $2a$, (b) the difference of whose other two sides is $2a$, determine the envelope of the perpendicular bisectors of the variable sides.

I. Solution by the Proposer.

(a) Letting the vertices of the triangle be $(-c, 0)$, $(c, 0)$ and (α, β), we have

$$\alpha^2/a^2 + \beta^2/b^2 = 1, \tag{1}$$

where $b^2 = a^2 - c^2$.

The equation of the perpendicular bisector of the line connecting $(c, 0)$ and (α, β) may be written[1]

$$(\alpha - x)^2 + (\beta - y)^2 = (x - c)^2 + y^2. \tag{2}$$

Taking α as a variable parameter and differentiating with respect to α, we have $d\beta/d\alpha = -b^2\alpha/a^2\beta = -(\alpha - x)/(\beta - y)$, and our problem is to eliminate α and β.

The elimination may be carried out as follows:

$$\beta = b^2\alpha y/[b^2\alpha - a^2(\alpha - x)],$$

and if we write e for c/a, we have

$$\beta = \frac{(1 - e^2)\alpha y}{x - e^2\alpha} \quad \text{and} \quad \beta - y = \frac{(\alpha - x)y}{x - e^2\alpha},$$

and equations (1) and (2) become

$$a^2 - \alpha^2 = \frac{(1 - e^2)\alpha^2 y^2}{(x - e^2\alpha)^2}, \tag{3}$$

and

$$(x - ae)^2 + y^2 - (\alpha - x)^2 = \frac{(\alpha - x)^2 y^2}{(x - e^2\alpha)^2}; \tag{4}$$

whence

$$\frac{(x - ae)^2 + y^2 - (\alpha - x)^2}{a^2 - \alpha^2} = \frac{(\alpha - x)^2}{(1 - e^2)\alpha^2},$$

and therefore

$$(a^2 - e^2\alpha^2)(\alpha - x)^2 = (1 - e^2)[(x - ae)^2 + y^2]\alpha^2. \tag{5}$$

Now equation (3) is the same as

$$(a^2 - \alpha^2)(e^2\alpha - x)^2 = (1 - e^2)\alpha^2 y^2, \tag{6}$$

[1] It should be noticed that this solution, as well as the next, gives only one part of the locus, the two parts being symmetrical with respect to the y-axis—EDITORS.

and if we subtract and divide by $1 - e^2$ we get

$$(1 + e^2)a^2\alpha^2 - 2a^2x\alpha - e^2\alpha^4 + x^2\alpha^2 = (x - ae)^2\alpha^2,$$

or simplifying further and dividing also by $e\alpha - a$,

$$e\alpha^2 + a\alpha - 2ax = 0;$$

whence

$$x = \frac{(a + e\alpha)\alpha}{2a}. \tag{7}$$

Equation (6) then gives

$$y = \pm \frac{e\alpha + a - 2ae^2}{2a}\sqrt{\frac{a^2 - \alpha^2}{1 - e^2}}. \tag{8}$$

These are the parametric equations of the locus.

(*b*) The parametric equations for this part of the problem may be obtained in the same way.

II. Solution by J. K. Whittemore, Yale University.

The fixed vertices of the triangle are the foci of (*a*) an ellipse or (*b*) a hyperbola described by the variable vertex; the variable sides are the focal radii of the ellipse or hyperbola; the middle point of each focal radius describes a similar conic, the center of similitude being the corresponding focus. The problem stated is a special case of the following:

Given any curve whose polar equation is $r = f(\varphi)$ at each point P of the curve a line PQ is constructed perpendicular to OP, the radius vector of P. The envelope of PQ is required.

If the length PQ is denoted by p the rectangular coördinates of Q are

$$x = r\cos\varphi - p\sin\varphi, \qquad y = r\sin\varphi + p\cos\varphi.$$

In order that the locus of Q be the envelope of PQ it is necessary and sufficient that

$$\frac{dy}{dx} = -\cot\varphi$$

or

$$\frac{(r' - p)\sin\varphi + (r + p')\cos\varphi}{(r' - p)\cos\varphi - (r + p')\sin\varphi} = -\frac{\cos\varphi}{\sin\varphi},$$

where accents denote differentiation with respect to φ. From the last equation $p = r'$, so that the equations of the envelope in terms of the parameter φ are

$$x = r\cos\varphi - r'\sin\varphi, \qquad y = r\sin\varphi + r'\cos\varphi.$$

In the problem stated the envelope is given by substituting

$$r = f(\varphi) = \frac{1}{2}\frac{me}{1 + e\cos\varphi}, \qquad e = \frac{c}{a}, \qquad m = \pm\frac{a^2 - c^2}{c},$$

where the upper and lower signs in m correspond to the ellipse and hyperbola respectively.[1]

We may also give the intrinsic equation of the envelope in the general case. From its equations

$$dx = -(r'' + r)\sin\varphi d\varphi, \qquad dy = (r'' + r)\cos\varphi d\varphi,$$

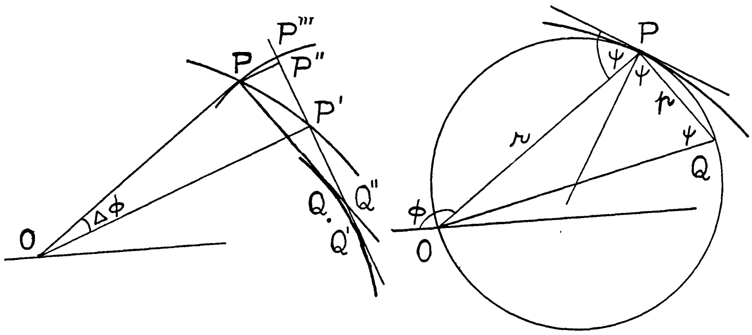
$$\rho = \frac{ds}{d\varphi} = r'' + r.$$

If the envelope is given in intrinsic form, $\rho = F(\varphi)$, the original or pedal curve is given by solving the differential equation, $r'' + r = F(\varphi)$. For example, (1) if the envelope is a point $F(\varphi) = 0$, $r = A\cos(\varphi - \alpha)$, and the pedal curve is a circle passing through the origin; (2) if the envelope is a circle $F(\varphi) = A$, $r = A + B\cos(\varphi - \alpha)$, and the pedal curve is a limaçon. Both of these results are easily verified.

Note by Otto Dunkel, Washington University—The results obtained by Professor

[1] The equations obtained in this way reduce to those given under I if we express $\cos\varphi$ and $\sin\varphi$ in terms of α and transfer the origin to the center—Editors.

Whittemore may be obtained geometrically as follows: Consider two neighboring points, P and P', on the given curve and let O be the pole, Q and Q' the points of tangency of the perpendiculars with their envelope. Then the intersection Q'' of PQ and $P'Q'$ lies on the circle through



O, P and P'. When P' approaches P as a limit the circle has for its limit a circle tangent to the given curve at P and passing through O. This gives a geometrical construction for Q. If ψ is the angle between OP and the tangent to the given curve then $\angle\, OQP = \psi$ and $\tan\psi = r/p$. But the well-known formula of polar coördnaites gives $\tan\psi = r/(dr/d\varphi)$ and hence $p = dr/d\varphi$. Consider that involute of the envelope (Q) which passes through P and let it cut $Q'P'$ in P'''; let P'' be the foot of the perpendicular from P upon $Q'P'$. Then $P'''Q' - PQ = \Delta s$ and $P'''Q' = P'''P'' + r\sin\Delta\varphi + p + \Delta p$ and hence

$$\frac{P'''P''}{\Delta\varphi} + \frac{r\sin\Delta\varphi}{\Delta\varphi} + \frac{\Delta p}{\Delta\varphi} = \frac{\Delta s}{\Delta\varphi}.$$

On taking the limit we have

$$r + \frac{dp}{d\varphi} = \rho,$$

since $P'''P''$ is equal to the length of the perpendicular from P to the tangent to the involute at P''', and hence it is an infinitesimal of the second order.

2892 [1921, 184]. Proposed by R. T. McGregor, Bangor, Calif.

Two parabolas have parallel axes. Prove that their common chord bisects their common tangent.

I. Solution by Marcia L. Latham, Hunter College.

Let the axes be rectangular, with the x-axis parallel to the axes of the parabolas; let P_1 and P_2, respectively, be the points of contact of the common tangent with the two parabolas, and P_3, the midpoint of P_1P_2.

Then the equations of the parabolas will be

$$y^2 + 2B_1x + 2C_1y + D_1 = 0, \tag{1}$$
$$y^2 + 2B_2x + 2C_2y + D_2 = 0. \tag{2}$$

The tangent to (1) at P_1 is

$$yy_1 + B_1(x + x_1) + C_1(y + y_1) + D_1 = 0.$$

But this passes through P_2; therefore,

$$y_1y_2 + B_1(x_1 + x_2) + C_1(y_1 + y_2) + D_1 = 0. \tag{3}$$

Again the tangent to (2) at P_2 passes through P_1; therefore,

$$y_1y_2 + B_2(x_1 + x_2) + C_2(y_1 + y_2) + D_2 = 0. \tag{4}$$

Subtracting (4) from (3), we can write

$$2(B_1 - B_2)\frac{x_1 + x_2}{2} + 2(C_1 - C_2)\frac{y_1 + y_2}{2} + (D_1 - D_2) = 0. \tag{5}$$

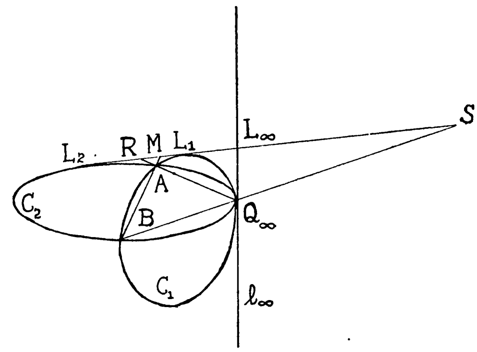
But the equation of the common chord is

$$2(B_1 - B_2)x + 2(C_1 - C_2)y + (D_1 - D_2) = 0, \tag{6}$$

and thus (5) states that the coördinates of P_3 satisfy (6) and the common chord bisects the common tangent.[1]

II. Solution by W. E. Cleland, Princeton University.

Let the two conics C_1 and C_2 be tangent to the line at infinity of a Euclidean plane at Q_∞. Then C_1 and C_2 are parabolas whose axes are parallel. Let a common tangent be L_1L_2, tangent to C_1 at L_1 and to C_2 at L_2, and intersecting the line at infinity in L_∞. The common chord is AB, intersecting L_1L_2 in M. Let the intersections of AQ_∞ and BQ_∞ with L_1L_2 be R and S, respectively.



Since the line L_1L_2 is tangent to C_1 at L_1, L_1 is a double point in the involution determined on L_1L_2 by the pairs of points $R\ S$, and $M\ L_\infty$. Similarly, L_2 is the other double point of the same involution. Hence M is the harmonic conjugate of L_∞ with respect to L_1 and L_2. M is therefore the midpoint of L_1L_2. (For proofs of these statements see Veblen and Young, *Projective Geometry.*)

Also solved by William Hoover, L. C. Mathewson, J. B. Reynolds, J. K. Whittemore, and F. L. Wilmer.

NOTES AND NEWS.

It is hoped that readers of the Monthly will coöperate in contributing to the general interest of this department by sending items to H. P. Manning, 69 Weymouth St., Providence, R. I.

Charles Albert Fischer, Seabury professor of mathematics at Trinity College, Hartford, Conn., died December 7,[2] 1922. He was born at Wheaton, Ill., October 3, 1884, and received the degree of A.B. from Wheaton (Ill.) College in 1905. He spent 1909–1912 in graduate study in mathematics and astronomy at the University of Illinois (A.M., 1910) and University of Chicago (Ph.D., 1912). His thesis, published in the *American Journal of Mathematics*, 1913, was entitled: "A generalization of Volterra's derivative of a function of a curve." At the meeting of the American Mathematical Society at the University of Rochester last September he gave, by invitation, an hour lecture on "Functions of lines." He was a charter member of the Association.

Arnold Sommerfeld, professor of mathematical physics at the University of Munich, will be in residence at the University of Wisconsin for the first semester of the academic year 1922–1923, holding the Karl Schurz Memorial Professorship in the University for that period. Professor Sommerfeld is expected to give a

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[1] The solution will have a particularly simple form if we take the common tangent and the line through its mid-point parallel to the axes of the parabolas for coördinate axes, giving us a system of oblique coördinates. The common chord will then appear as a line through the origin —Editors.

[2] The date is given incorrectly as December 9 in *Science*, December 15.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

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[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

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BUSINESS CORRESPONDENCE should be addressed to the Secretary-Treasurer of the Association, W. D. Cairns, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922

Seventh Annual Meeting, Harvard University, December 28, 29, 1922

The following are dates of Section meetings of the Association in 1922 (unless otherwise specified):

Illinois, Rockford, Ill., April 28–29
Iowa, Des Moines, November 3; Cornell College, Mount Vernon, April 27-28, 1923
Kansas, Topeka, January 21; January 20, 1923
Kentucky, Georgetown College, April 8; University of Kentucky, April, 1923
Maryland-Virginia-District of Columbia, Washington, December 9; Baltimore, May, 1923
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Missouri, Kansas City Junior College, November 18; University of Missouri, Columbia, November 30–December 1, 1923
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The REGISTER OF OFFICERS AND MEMBERS of the Association issued as Number 6, June–July, was sent only to those specially subscribing for it. The paging of the MONTHLY for August continues from that for May, so that the 1922 volume will be complete aside from the REGISTER.

CONTRADICTIONS IN THE LITERATURE OF GROUP THEORY.

By G. A. MILLER, University of Illinois.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE MATHEMATICAL ASSOCIATION OF AMERICA, SEPTEMBER 6, 1922.

The ancient Greeks, even before the time of Euclid, assumed that if two hypotheses lead to contradictory results they cannot both be true, and in Euclid's *Elements* frequent use is made of a method of proof known as *reductio ad absurdum,* called by the Greeks "reduction to the impossible." The "impossible" here often meant a result which contradicted a result which was assumed to be well established. While contradictions are so distasteful to the normal mind they have not been completely avoided in the mathematical literature. For instance, even Euclid used the term circle both for a curve and also for the area inclosed by this curve, and the double meaning of this particular term has persisted to our day. The seriousness of contradictions in the literature of our subject is illustrated by following well-known dictum due to Poincaré: "Mathematics is the art of giving the same name to different things." [1]

Many contradictions have been so mellowed by age that they are scarcely noticed by the teacher while the student is greatly perplexed by them. As teachers we cannot be too careful to point out emphatically where such contradictions appear. While this might seem to be a sufficient reason for the choice of the present subject for this extraordinary occasion, it may be admitted that the real reason was much more personal. A referee of a paper which I offered for publication did not agree with me in view of the fact that we used different definitions of the same term. Both of these definitions were sanctioned by long usage but it was only after considerable correspondence that the real source of the difficulty became clear. Hence I address you on this subject in somewhat the same state of mind as that exhibited by the rich man in the Bible who wanted Abraham to send Lazarus to testify to his five brethren "lest they also come into this place of torment."

The source of the particular difficulty to which I have just referred proved to be the fact that both in abstract group theory and in substitution group theory we commonly say that a given finite group is transformed into itself only by the operators or the substitutions of its holomorph. Hence some operators or substitutions transform such a group into itself while other operators or substitutions do not have this property. On the other hand, every possible operator or substitution transforms a given group into a simply isomorphic group, and all the simply isomorphic groups are usually regarded as the same group when abstract groups or regular substitution groups are listed. Hence we also say, at least implicitly, that a given group is transformed into itself by every possible group operator or by every possible substitution.

[1] *Bulletin des Sciences Mathématiques,* vol. 32, 1908, p. 174.

While the statements to which I have just referred seem to me to involve a contradiction, yet the facts involved in these statements are so well known that the student familiar with group theory is usually not apt to be disturbed thereby. Difficulties are more apt to arise in the newer developments where the contradictions to which we have just referred present themselves in a new form. For instance, a characteristic operator of a group is defined as an operator which must correspond to itself in every automorphism of the group. In particular, the identity is a characteristic operator of every possible group of finite order.

If all the simply isomorphic groups are regarded as the same, the identity is the only characteristic operator that can appear in a group, for every other operator can be transformed into an operator which is not itself by means of some group operator. Hence such an operator cannot be characteristic under the one of the two given definitions of the same group for this operator can be transformed into a different operator by means of an operator which transforms this group into a simply isomorphic group. In this simple isomorphism, or automorphism, the operator in question would therefore not correspond to itself.

It is evidently desirable to call some operators characteristic which are not the identity. In fact, this desirability is clearly reflected in a well-established custom. Moreover, it is not difficult to present the matter in such a way that no ambiguity arises as a result of this custom. It is only necessary to say that a characteristic operator is one which corresponds to itself in every possible automorphism arising under the holomorph of a group; if other automorphisms are regarded as possible, as is sometimes done, a characteristic operator does not necessarily correspond to itself in these automorphisms.

When I first studied this question it seemed to me best to say that a characteristic operator must correspond to itself in every possible automorphism in accord with the common definition. If this view is adopted it is necessary to say that if we establish a (1, 1) correspondence between the operators of two groups in such a way that a characteristic operator does not correspond to itself, the two groups cannot be regarded as the same group. This was done in an article by the present writer published in the *Proceedings of the National Academy*, volume 7, 1921, page 146.

It seems to me now wiser to modify the definition of characteristic operator as suggested above and to state explicitly that in group theory we employ two extreme definitions of what is meant by the same group. According to one of these definitions a characteristic operator must correspond to itself in every possible automorphism of the group while this is not the case when the other definition is under consideration. If this is done the corrections to which the article noted at the end of the preceding paragraph is devoted are uncalled for.

In addition to the two extreme definitions of the expression *the same group* there is a third which is still older but relates only to non-regular substitution groups of finite order. This was implicitly employed by A. L. Cauchy in his enumeration of the substitution groups whose degrees do not exceed 6 (*Comptes Rendus*, Paris, volume 21, 1845, pp. 1363 and 1401). A necessary and sufficient

condition that two substitution groups are the same, according to this definition, is that one of them can be transformed into the other by means of some substitution. The most general definition of what is meant by the same group can be obtained from this oldest definition by adding the condition that the two substitution groups in question shall be regular and therefore implies that a necessary and sufficient condition that two groups are the same is that a simple isomorphism can be established between their substitutions or operators. The other extreme definition implies that two groups are the same if and only if they involve the same substitutions or operators irrespective of the arrangement of these substitutions or operators within the group.

Just as the two different definitions of the term circle have persisted for at least two and a half millenniums, so these three different definitions of the expression *the same group* may reasonably be expected to persist. When new terms relating to these definitions are introduced, it should be made clear which definition is to be understood. Sometimes this can readily be inferred from the context but at other times it may be necessary to state the matter explicitly in order to avoid confusion. At any rate, the beginner should be informed early as regards the significance of these various definitions and that contradictions are likely to arise therefrom.

Another fundamental question which may give rise to contradictory results on the part of the student of group theory has recently presented itself to me in very definite form by the assertion in C. J. Keyser's *Mathematical Philosophy*, 1922, page 204, that when all the positive and negative rational integers, including zero, are combined according to subtraction they do *not* constitute a group. While I am not inclined to call this assertion incorrect I would have been inclined to agree also with Professor Keyser if he had stated that they *do* constitute a group. While these assertions are evidently contradictory, it depends entirely on what is understood by the rule of combination of the elements of a group whether the one or the other is correct. If each of the rational integers is understood to represent the operation of subtracting it from something; that is, if the integer n is understood to mean that n is to be subtracted, then these integers, together with zero, evidently do constitute the same group as when they are combined by addition.

On the other hand, if S_1 and S_2 represent any two of these integers and if when S_1 and S_2 are combined it means that the integer represented by S_2 is to be subtracted from that represented by S_1, or vice versa, it is evident that the associative law is not satisfied since $S_1 - S_2 - S_3$ is not generally equal to $S_1 - (S_2 - S_3)$. It should, however, be noted that when S_1, S_2, S_3 represent ordinary substitutions, and if by the product $S_1 \cdot S_2$ we should mean that the substitution S_2 were to be performed on S_1; that is, if $S_1 \cdot S_2$ should be understood to mean $S_2^{-1}S_1S_2$, then the associative law would also not be satisfied by the ordinary substitutions. By combining two substitutions we do not mean that the second substitution is to be applied on the first but we mean merely that one of these substitutions is to be followed by the other.

It would seem natural to say that the substitutions of a group are combined according to the rule of substitutions. In this case, we would mean thereby that the symbols of our group would represent substitutions and that the rule of combination of these symbols was determined by the properties of these substitutions. In any group in which the elements represent definite operations the law of combination of these elements is determined by the properties of the elements. The principal thing is the properties of the elements. The law of combination of the elements, being deduced from these properties, may be regarded as secondary.

I emphasize these points here because from some of the well-known definitions of an abstract group one might naturally conclude that the law of combination of the elements was of primary importance in group theory and that the properties of the elements themselves played a secondary rôle, while just the opposite seems to me to be the case. In the definition of an abstract group it would seem desirable to state explicitly that the law of combination of the elements is deduced from the properties of the elements. In particular, if we speak of groups whose elements are the rational positive and negative integers together with zero, and if we say that these integers are to be combined according to the rule of subtraction, it would seem to me natural to say that a symbol like 4 was to be construed to mean that the number represented by this symbol was to be subtracted, while the symbol -4 was to be regarded as meaning that the number -4 was to be subtracted, or that 4 was to be added.

If this is done, each of these numbers represents a certain operation which may be applied on something outside of the group. In fact, a group generally looks outside of itself for its main fields of usefulness and the law of combination of its elements is normally construed by this outward look as may be inferred from the fact that this law is commonly denoted by the single operation multiplication. In looking outward it seems to me that we must say that the operations of performing ordinary substitutions, as well as rational integers when combined by subtraction, obey the associative law. In confining our view to the system of elements themselves one may still construe the combination of these elements in such a way that the associative law is satisfied but in this case it may seem more natural to say that the associative law is not satisfied when the elements are combined by subtraction.

The fundamental nature of these remarks may be inferred from the fact that Professor Keyser was writing for the educated laymen and not mainly for the mathematicians. The above remarks therefore show that even those who are not professional mathematicians may come across contradictions in this field if they are so fortunate as to catch the spirit of the group concept, which Professor Keyser pictured for them not only in our commonplace mundane surroundings but also in the movements of the heavenly messengers of good tidings to those who are faithful to the end, as is evidenced by the group of angel flights.

He even raises the question: "Is mind a group?" and states that "it may be that a genius of the so-called universal type—an Aristotle, for example, or a

Leibniz or a Leonardo da Vinci—is one whose mind has the group property." This possible fact may serve to explain some of the extraordinary claims made for group theory which many would be inclined to say could be readily contradicted if they had not been made by men of such eminence. For instance, one of the last public utterances of H. Poincaré was that "the theory of groups is, so to say, entire mathematics, divested of its matter and reduced to a pure form."[1] I presume that those who are most familiar with the varied scientific activities of H. Poincaré would be inclined to call him a genius of the universal type.

At any rate, it is true that while some groups are such omnipresent objects that familiarity has blunted our appreciation of their significance there are others which exhibit themselves only to the most gifted or to those who have made a serious search for them. It is a singular fact that the former did not receive any explicit attention until some of the latter were discovered and their rôle in the theory of algebraic equations became apparent. How different mathematical history would be if Archimedes had emphasized the importance of the group concept in the development of our subject. In particular, the later Greeks would then not have been satisfied with one solution of a quadratic equation but they would have recognized that there is no such thing as an equation in one unknown and of degree $n > 1$, but that the so-called equation in one unknown is really one in n unknowns, n being the degree of the equation. They would have seen that in such an equation one has been led to call different things by the same name x in spite of the fact that one may have desired to do otherwise. Algebra is more resourceful than the inventors thereof.

Group theory as an autonomous subject was initiated by looking below the surface and noting that what is denoted by a single symbol x is really not a single number but a composite of n numbers in disguise. It is the mind not the eye that saw these groups, and ever since then it is the groups which the mind sees which have led to many sweeping remarks relating to the scope of their theory. To those of us whose intellectual insight is less keen some of these remarks may appear to be contradicted by facts just because we see these facts only dimly and fail to recognize some deep connections.

A contradiction of a very well-known elementary theorem on substitution groups is found in the 1900 edition of Pascal's *Repertorium der höheren Mathematik*, volume 1, page 32. It is here stated, as well as in the original Italian edition, that an invariant subgroup of a transitive substitution group does *not* involve all the marks or letters of the group. The following sentence is also evidently inaccurate, since the group in question does not necessarily include all the substitutions but only all the positive substitutions. On the same page it is also incorrectly stated that if all the substitutions of a k times transitive group transform a given group into itself, then this group must be at least $k - 1$ times transitive. It is very well known that the holomorph of the regular abelian group of order 2^n and of type $(1, 1, 1, \cdots)$ is triply transitive. In particular, the holomorph of the non-cyclic regular group of order 4 is the symmetric group of degree

[1] *Acta Mathematica*, vol. 38, 1921, p. 145.

4. It is singular that three such obvious errors should appear on the same page of the standard reference work in question.

On page 203 of Keyser's *Mathematical Philosophy* the term "system having the group property" is defined in practically the same way as the term "group in the most general sense" is defined in volume 2, page 243, of tome 1 of the *Encyclopédie des Sciences Mathématiques*. Hence his statement "that in the older literature of the subject they are called groups, or closed systems" is correct but misleading. On page 576 of volume 1 of the *Encyclopédie* there appears a much more restrictive definition of the term group, and no reference is given from either one of these definitions to the other. These two widely different definitions of this term in such a standard work of reference are a reflection of the fact that the word "group" as a technical mathematical term has been employed by good authorities with widely different meanings. It is only natural that this wide difference in meaning has led to contradictions, but it has not produced chaos in the field of group theory, since one can usually determine from the context what meaning is to be assigned to the term in question.

Perhaps many of us would be inclined to think that such an authoritative work as the *Encyclopédie* should have used its influence to secure uniformity as regards the use of the term "group" with a technical meaning. In so far as uniformity reduces suggestiveness it is an evil, especially in the newer fields of our subject. It is much better to encounter contradictions than sterility. In so far as the use of the same term with widely different meanings suggests analogies and common ground which would otherwise escape the reader's notice it is an advantage. On the other hand, clear thinking demands that the technical terms in use have definite meaning in so far as this meaning relates to the main objects of thought under consideration. A man may work very successfully in the ordinary theory of substitution groups, for instance, without considering all the conditions which an abstract group is supposed to satisfy, or without having grasped the reason why the group of Euclidean geometry does not include the similarity transformations but does include the special homothetic transformations in which the coefficients are plus or minus unity.

The most conspicuous contradictions in the literature of group theory naturally appear in the history of this subject since the wide range of topics with which the mathematical historian is inclined to deal makes it very difficult to avoid misinterpretations. As an instance of this difficulty we may refer to the term *simple group*. This term has been used with at least three different meanings. P. Ruffini used its equivalent (permutazione semplice) in his *Teoria generale delle equazioni*, 1799, page 247, for a cyclic group. H. W. Tanner used the same term for a cyclic prime-power group in volume 20, 1888, of the *Proceedings of the London Mathematical Society*, page 70. This obsolete meaning unfortunately slipped into Dickson's *History of the Theory of Numbers*, volume 1 (1919), page 131. It is somewhat singular that the meaning of this term which is now universally adopted, and was introduced by C. Jordan in 1869, *Mathematische Annalen*, volume 1, page 142, is less in accord with the non-technical meaning

of simple than those noted above, for the modern simple groups are really the most difficult groups.

In a semi-serious manner we may refer here to the fact that one of my colleagues recently directed attention to my questionable use of the term *Lagrange's theorem* for the fundamental theorem that the order of a finite group is divisible by the order of each of its subgroups, since Lagrange did not use this theorem with its general meaning.[1] Such a use was made of it by Galois and one might be inclined to change the name of the theorem to *Galois' theorem* just as the name of *Pell's equation* has been changed by many to *Fermat's equation.* In the present case, there is, however, less substantial reason for making the change since Lagrange actually first used the theorem in a special but important case, while Pell had nothing to do with the equation which bears his name. The term *Lagrange's theorem* for the theorem in question was used already by J. Petersen, *De algebraiske Ligningers Theori,* 1877, and became widely known, especially through the German translation of this popular work under the title *Theorie der Algebraischen Gleichungen,* 1878. While it has been widely used it is not found in some of the standard works of reference, including the large German and French mathematical encyclopedias.

In view of the facts that the term *Lagrange's theorem* has appeared in some of the most frequently consulted group theory literature for such a long time, and that the workers in this field are naturally inclined to express their gratitude to Lagrange for his large share in bringing this theorem to the attention of the public, it seems to the present writer that the use of this term is appropriate. The student of the history of our subject is not likely to be misled thereby since he is constantly reminded of the fact that many mathematical terms have grown in richness of meaning with the development of our subject. Netto called the theorem in question "Lagrange's fundamental theorem of the theory of substitutions" in the opening sentence of an article published in the *Mathematische Annalen,* volume 13 (1878), page 249, and a similar term was used for it in the review of this article in the *Jahrbuch über die Fortschritte der Mathematik.* In Jordan's *Traité des substitutions,* 1870, page 25, there appears a heading entitled "Theorems of Lagrange and of Cauchy." It seems likely that this is the source of the term "Lagrange's theorem".

In closing we shall refer to a few conspicuous historical errors relating to group theory which appear in the *Fortschritte.* We do this not because it is unusually easy to find such errors here but because remarks relating to such a standard work are likely to be more effective than if they related to one less generally used. In an article published in the *Bulletin of the American Mathematical Society,* volume 27 (1921), page 459, we referred to a few mathematical errors relating to group theory and appearing in this work. The present remarks may be regarded as an addition thereto.[2]

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[1] R. D. Carmichael, *Bulletin of the American Mathematical Society,* vol. 27, 1921, p. 474.

[2] A considerable number of other errors relating to group theory were listed by the present writer in an article published in this MONTHLY, *1913,* 14-20.

We begin with a review found in the latest available volume at the time this address was prepared—viz., volume 45, relating to the publications of 1914–1915. On page 254 thereof appears a review of an article by G. Heussel (misspelled in the *Fortschritte*) entitled "Ueber Gruppen aus zwei Elementen gleicher Ordnung, b und c, die der Bedingung $b^2 = c^{2m}$ genügen." It is at once evident that the letter m in this title is superfluous for if m is odd it must be prime to the order of c and if it is even this order must be the double of an odd number to which m must again be prime. Hence the article in question is devoted to the well-known groups which are generated by two operators having a common square. Moreover, it does not contain any really new theorem; but neither the author of the article nor the reviewer in the *Fortschritte* indicated that we are here dealing with well-known results expressed in a form which differs only slightly from that in which they appeared earlier. This is the more singular because an article on these groups by the present writer appeared about ten years earlier in the same journal in which Heussel published his article.

The special case when each of the two generating operators which have a common square is of order 2 was fully considered by J. de Perott in the *Bulletin de la Société Mathématique de France*, volume 21, 1893, page 62. This case is so simple that others must have recognized it at an earlier date but the *Encyclopédie des Sciences Mathématiques* fails to give any reference relating thereto. It may be noted, in passing, that the references in this encyclopedia relating to the well-known simple group of order 168, tome 1, volume 1, page 563, are misleading. It would appear from these references that Hermite and Brioschi first directed attention to this group while this honor actually is due to Kronecker and Mathieu. The volume in which Mathieu published his results bears an earlier date (1858) than the volume in which Kronecker's results were published (1859), but Kronecker's paper was presented several months before that by Mathieu, and hence it may be just to say that Kronecker first directed attention to this group, as is done in the *Fortschritte*, volume 21, page 142.

The honor of Kronecker in this connection is beclouded by his remarkable display of ignorance in regard to the then available literature relating to substitution groups since he stated, in substance, that the group of order 168 and degree 7 constitutes the first example of a substitution group of degree n and containing more than one regular cyclic subgroup of this order without being either alternating or symmetric. Among the then known substitution groups satisfying this condition is the triply transitive group of degree 6 and of order 120, studied by Hermite and Cauchy more than a decade earlier. The subgroup of order 60 contained in this group is also an example of such a group. The history of the simple group of order 168 is perhaps especially interesting since this is the simple group of next to the lowest composite order.

Hence it may be of interest to note that Brioschi stated incorrectly that two groups of degree 7 and of order 168 exist, *Annali di Matematica*, volume 2 (1859), page 60. It may also be desirable to add that none of these early workers on the simple group of order 168 seems to have recognized that he was actually dealing

with a simple group. The simplicity of this group seems to have been first established in Jordan's *Traité des substitutions*, as a special case. It is well known that Galois first directed attention to the simplicity of the icosahedral group, which is the simple group of lowest composite order. The history of simple groups of composite order has its origin in the works of Galois but finds its first considerable development in the works of Jordan.

The student who is inclined to accept statements found in standard works of reference without attempting to test them as regards accuracy might be referred to page 160 of volume 34 of the *Fortschritte*, where it is stated that W. Burnside proved that a group which admits an automorphism of order 3 has the property that every pair of conjugate operators contained in it are commutative. This is the opening sentence of the review in question and hence it is the more striking since it is so obviously incorrect. To the thinking reader such obvious misstatements on the part of a reviewer are often refreshing since such a reader will readily see that some condition must have been omitted and in supplying this he may secure a deeper insight into the theorem under consideration.

It is evident that the order of the group of isomorphisms of any group G is equal to the number of different possible ordered sets of k independent generators of G such that the αth operator of each of these sets corresponds to the αth operator of a given one of these sets in some automorphism of G where α is any one of the numbers $1, 2, \cdots, k$. In the special case when G is an abelian prime-power group and its k independent generators are a reduced set, all the operators of the same order in such a set of generators may correspond, and the situation becomes unusually clear. It seems questionable whether such obvious facts should be noted in a review as if they were results found in the article under review, as is done in the *Fortschritte*, volume 40, page 192. What is, however, more to the point is that the term *invariant* is used in this particular article, and the review thereof, with an unusual meaning, and the reading of the review is made more difficult thereby as well as by other obscurities which are not removed in the review.

We venture to refer here to a somewhat amusing incident related to the subject under consideration. The able mathematician I. Schur of Berlin stated correctly on page 189 of volume 40 of the *Fortschritte* that the present writer attributed to him an assertion which he did not make and which, moreover, is false. On page 177 of the following volume of the same journal Schur attributed, in turn, to the present writer an assertion which he also did not make and which, moreover, is also false. It is stated here that the present writer asserted that when the Sylow subgroups of order p^m are transformed according to a triply transitive group then the prime number p must exceed 2. That this assertion is false results directly from the fact that in the alternating group of degree 5 the Sylow subgroups of order 4 are evidently transformed according to a triply transitive group. When a reviewer uses somewhat harsh language about a mistake it is perhaps natural for the one who made this mistake to feel a slight relief when he observes that the reviewer's harsh language applies equally to one of his own mistakes even if this does not remove the stain.

The preceding references to reviews in the *Fortschritte* do not imply that the present writer does not believe in critical mathematical reviews. On the contrary, he heartily believes in public criticism since such criticism is much more apt to be fair than private criticism, and most people form their judgment of the works of others from one or the other of these two sources. It is unfortunate but natural that the reviewers for the *Fortschritte* have exhibited wide differences of judgment as regards such reviews. Many of them were non-critical and practically contented themselves with an exhibition of the main results found in the work under review without even distinguishing between what was known earlier and what was then new. A few others were unduly critical as a result of their own ignorance of the subject under review and their youthful ambition to appear wise. On the whole these reviews are, however, very valuable as many of us know from experience and the *Fortschritte* has always been one of the most important mathematical periodicals for those seeking a broad knowledge of our subject.

It was not without misgivings that I ventured to appear before you on such an occasion as a critic of certain minor parts of the literature to which so much of my energy has been devoted. The normal attitude of mind should be to get all the good we can out of the writings of others and to make allowances for shortcomings, for we all admire generosity and many of us feel the need thereof. On the other hand, as teachers it behooves us also to strive hard to guide those who depend on us so that they may avoid the pitfalls which beset their ways. It is, therefore, necessary for us to become also familiar with this less attractive phase of our subject. Criticisms do not necessarily imply that the author thereof feels that he could have done better than the one that is being criticized but only that he feels that he can add another element which will tend to make the truth stand out more brightly. The preceding remarks are to be construed in this spirit.

THE APRIL MEETING OF THE IOWA SECTION.

The tenth regular meeting of the Iowa section of the Mathematical Association of America was held, in conjunction with the thirty-sixth annual meeting of the Iowa Academy of Science, at Drake University, Des Moines, Iowa, on April 29, 1922. The meeting consisted of one session with Professor C. W. EMMONS presiding. There were twenty-five in attendance, including the following twenty members of the Association:

O. W. Albert, E. W. Chittenden, Julia T. Colpitts, C. W. Emmons, Fay Farnum, C. Gouwens, E. C. Kiefer, G. E. King, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, I. F. Neff, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, E. R. Smith, G. W. Snedecor, C. W. Wester.

The following officers were elected for 1922–1923: Chairman, C. W. EMMONS, Simpson College; Vice-chairman, I. F. NEFF, Drake University; Secretary-

treasurer, J. F. Reilly, State University of Iowa. A committee consisting of Professors Wester, Colpitts and Neff was appointed to draw up resolutions expressing the feeling of the section on the death of Professor B. F. Simonson, and the illness of Professor W. J. Rusk.

The next meeting will be held at Des Moines in November.

The following papers were presented:

(1) "The method of averages" by Professor G. W. Snedecor;

(2) "Mathematical bulletins" by Professor T. M. Blakslee (by title);

(3) "Topics in general analysis" by Professor E. W. Chittenden;

(4) "The place of the equation in a scheme of education" by Professor C. W. Wester;

(5) "Persymmetric determinants whose elements are finite sums" by Professor E. R. Smith;

(6) "On the meaning of the classification of statistical series by means of the Lexis ratio" by Professor H. L. Rietz;

(7) "Definitions of imaginary and complex numbers" by Professor E. S. Allen (by invitation);

(8) "An extension of the figurate numbers" by Professor J. F. Reilly.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. The generalization of the King-Hardy method of fitting the mortality curve was indicated. The method of averages is the special case in which the function to be fitted is linear. This method was compared to the method of least squares. Professor Snedecor showed that the method of averages can be depended upon to give good results if, and only if, the departure of the data from linearity is small.

2. Professor Blakslee suggested a solution of a pedagogical problem. He thinks that in our teaching the earlier foundations of mathematics are usually omitted. He suggested that state bulletins be issued to remedy this, holding that under the plea of "saving time" time is now being lost.

3. Professor Chittenden presented, in addition to a discussion of the significance of the fundamental notions of neighborhood, limit and distance in general analysis, the solution of a problem proposed by Fréchet. If it is possible to define distance for a space S with a preassigned definition of elements of accumulation so that the elements of accumulation remain unchanged and every sequence of elements of S which satisfied the Cauchy condition has a limit in S, the space S is said to be complete. It is found, in answer to an inquiry of Fréchet, that the rational numbers form an incomplete space S.

(4) This paper asserts that the only equations needed to solve the usual problems of the grades are of the types, $n \pm a = b$, $an = b$, $x^2 \pm a = b$, $1/x + 1/a = 1/b$. The linear types can be thoroughly mastered for whole numbers in the second and third grades, and the extension to all rational numbers in the later grades is extremely easy. The work for teacher and pupil will be less than at present and with greatly increased efficiency. The equation will become,

for all educated people, the normal medium for thought about number relations and for expression of that thought. These statements were made by Professor Wester on the basis of tests given in the classroom.

5. In fitting parabolic curves by the method of least squares or by the method of moments a normal system of equations is formed whose determinant was given. Professor Smith showed that for certain useful sets of values this determinant may be reduced to a simple expression whose numerical value is easily computed.

6. In this paper Professor Rietz gave an exposition of the meaning of the Lexis ratio as it arises in the comparison of Poisson and Lexis distributions with corresponding Bernoulli distributions. He then showed how to apply this criterion to experiments with games of chance and to actual statistical data, and emphasized that we may obtain from this criterion important indications as to whether the dispersion has its foundation in a constant probability or in probabilities with certain types of variation.

7. Professor Allen's paper has been published in this MONTHLY, *1922*, 301.

8. In this paper Professor Reilly represented by ${}^aH_m{}^{a-k}$ the sum of all the products of degree m that can be formed from the quantities a, $a-1$, $a-2$, $\cdots$, $a-k$, putting ${}^aH_0{}^{a-k} \equiv 1$. Then he showed that if m and k be each given the values 0, 1, 2, 3, $\cdots$, a being arbitrary, a system of numbers is obtained similar to but more general than the figurate numbers. The law of formation of this system of numbers is

$${}^aH_{m+1}{}^{a-k} \equiv {}^aH_{m+1}{}^{a-k+1} + (a-k){}^aH_m{}^{a-k}.$$

These numbers were found useful in reducing a certain type of determinant.

JOHN F. REILLY, *Secretary-Treasurer.*

THE MAY MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION.

The eleventh regular meeting of the Maryland-Virginia-District of Columbia Section of the Association was held at the United States Naval Academy, Annapolis, Maryland, on Saturday, May 13, 1922. The Chairman of the Section, Mr. O. S. Adams, presided at both morning and afternoon sessions.

There were sixty-two in attendance, including the following forty-two members of the Association: O. S. Adams, J. J. Arnaud, R. N. Ashmun, H. G. Avers, E. A. Bailey, A. J. Barrett, Sarah Beall, G. A. Bingley, C. C. Bramble, J. A. Bullard, Paul Capron, G. R. Clements, A. Cohen, L. S. Dederick, A. Dillingham, J. B. Eppes, Harry English, J. N. Galloway, W. M. Hamilton, W. E. Heal, P. E. Hemke, Nelle L. Ingels, R. P. Johnson, L. S. Johnston, W. D. Lambert, J. J. Luck, E. S. Mayer, Frank Morley, C. A. Mourhess, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, M. R. Richardson, H. M. Robert, Jr., R. E. Root, W. F. Shenton, C. A. Shook, T. McN. Simpson, Jr.

Those in attendance at the meeting were guests of the Department of Mathematics of the Naval Academy for lunch. At the close of the afternoon session the program was varied by visits to points of interest about the Naval Academy, or to some of the athletic contests in progress there.

The officers elected for the year 1922–1923 are: FRANK MORLEY, Chairman; G. R. CLEMENTS, Secretary-Treasurer; W. D. LAMBERT and T. MCN. SIMPSON, Jr., additional members of the executive committee. The twelfth regular meeting of the Section will be held in December, 1922, probably in Washington, D. C.

The following papers were presented:

(1) "Self-inverse curves" by Professor FRANK MORLEY;

(2) "The significance of assumptions" by Professor T. MCN. SIMPSON, Jr.;

(3) "Remarks on the theory of relativity" by Professor F. D. MURNAGHAN;

(4) "On Græffe's method for the numerical solution of algebraic equations" (illustrated) by Professor L. B. TUCKERMAN, Bureau of Standards (by invitation).

(5) "The parallax of the moon and the ellipticity of the earth" by Mr. W. D. LAMBERT.

Abstracts of the papers follow:

1. Professor Morley's paper was, in brief, as follows: An equation of an algebraic plane curve, when written in conjugate coördinates, gives immediately the images of points in the curve. The successive images give an expression for Green's function. The cases when the image closes are naturally of interest. Then a point on the curve, which is one of its own images, will have other images on the curve. Such a curve may properly be called self-inverse.[1] A principal theorem on such curves is that if there be an isogonal mapping such that n points of one plane correspond to a single point of another plane, then to a line of the second plane will correspond a self-inverse curve of the first plane.

2. Professor Simpson began by quoting President Eliot who has said that "the main reason for the painfully slow progress of the human race is to be found in the inability of the great mass of people to establish correctly the premises of an argument." All thought starts with assumptions; they may be explicit or implicit, recognized or unrecognized. Out of implicit and unrecognized assumptions sophistry is bred. Failure to recognize assumptions as such has been the source of much bitter controversy, builders having sought to preserve the scaffoldings of thought as part of the permanent structure. Science is essentially pragmatic; the question of truth is the question of the truth of assumptions and its test is the test of experiment and experience.

Mathematics is perhaps more explicit in its assumptions than any other subject introduced into the curriculum at an early stage. It is better suited than any other to develop the sense of the significance of assumptions as well as the logical sense. The social significance of mathematics is to lie perhaps in its contribution to the building of a social consciousness which shall recognize that the stabilization of our institutions rests ultimately upon our ability to know and to test our assumptions, and upon a willingness to revise them without partisanship, or bitterness, or distress.

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[1] Self-inverse curves and surfaces have been termed by the French "anallagmatic"—EDITOR.

3. Professor Murnaghan's remarks were elementary and informal. They were directed to an exposition and criticism of several recent articles on relativity, showing in a simple manner just what mathematical problems the writers had set for themselves and to what extent satisfactory conclusions had been secured. Some work by Whitehead was examined with particular care.

4. Professor Tuckerman pointed out that the method for the solution of numerical algebraic equations published by Græffe in 1837 and developed and perfected by Encke in 1841, and later by Runge and other German mathematicians, is almost completely ignored in English and American text books. Illustrations using a computing machine and also a slide rule were given, both for real and complex roots. Attention was called to an erroneous statement, that "no reliable estimate of the error (of the roots) can be given." The computational errors of the method were discussed and a simple expression for the upper limit to the error in the modulus of any root was derived. For determination of the amplitude of the complex roots, Encke's method is preferable in the case of one or two pairs. When three or more pairs are present, a displacement of the zero is to be recommended. The method was extended by a simple process to equations with complex coefficients, and an illustration of the solution was given. The author believes that Græffe's method deserves more attention than it has received, and that in many cases it is to be preferred to the better known Horner's and Newton's methods, because of its generality, simplicity, and directness of attack.

5. Mr. Lambert's paper gave a brief account of the theory by which Sir Isaac Newton connected the force governing the motion of the moon with terrestrial gravity. When all necessary refinements are introduced (involving, for example, the ellipticity of the earth), the fundamental equation of the theory may be used to compute the parallax of the moon; the parallax so obtained may be called the "dynamic parallax." The dynamic parallax does not agree with the parallax determined by direct observation at Greenwich and Cape of Good Hope, unless a rather large ellipticity of the earth is assumed, namely, about 1/293 instead of 1/297, the value which is generally accepted as nearly correct. This discrepancy in the ellipticities is rather larger than can reasonably be attributed to observational error in the parallax or to uncertainties in the other assumed data of the fundamental equation. The explanation is probably to be looked for in the deflections of the plumb lines at Greenwich and the Cape, amounting sometimes to $30''$ or more. Deflections there of about $10''$ would reconcile the observed and dynamic values of the parallax for an ellipticity of 1/297. A computation of the effects of surface irregularities represented by the topography within radii of 2,500 miles of Greenwich and the Cape, with allowance for isostatic compensation, indicates deflections of the vertical in the right direction but by only about one fourth of the requisite amounts. The results, however, could be changed by changing the hypothetical distribution of isostatic compensation. A further partial explanation of the discrepancy might be found if we accept Helmert's determination of the figure of the earth as a triaxial ellipsoid. The detailed discussion will probably be published by the U. S. Coast and Geodetic Survey.

G. R. CLEMENTS, *Secretary-Treasurer.*

"STATISTICS" IN A MATHEMATICAL ENCYCLOPEDIC DICTIONARY.

By H. L. RIETZ, State University of Iowa.

In commenting on Professor G. A. Miller's article in this MONTHLY (*1918*, 383) giving meanings of *group* and *group theory,* Professor E. R. Hedrick, Chairman of the Association's Dictionary Committee, suggested (*1918*, 428) that further sample definitions be submitted. As it is not unlikely that there will be considerable difficulty in determining the extent to which terms from applied mathematics should be included in the proposed dictionary, the writer has been considering the question with special reference to the mathematics of statistics, probability, and insurance. As a result of such consideration, the following brief paper is submitted to explain the meanings of *statistics* and of the associated expressions,[1] *statistical data, statistical methods, theory of statistics, mathematical statistics, statistical probability,* and to suggest a list of terms and expressions from statistics, probability, and insurance that should probably be included in the dictionary. In the selection of the list of terms and expressions given below, the writer has been guided by his experience with seniors and first-year graduate students taking courses in statistics and actuarial theory, and has included only terms and expressions within the range of reading of such students.

Statistics (stá-tĭs'tiks), F. *statistique,* G. *Statistik,* I. *Statistica,* Sp. *estadistica.* The word statistics seems to be derived from the Latin *status,* used in the sense of a political state. Statistics is a comparatively new word. Its first occurrence in English thus far noted seems to be in J. F. von Bielfeld, *The Elements of Universal Erudition,* translated by W. Hooper, London, 1770. One of the chapters of this book is called *Statistics,* and the subject is defined as "The science that teaches us what is the political arrangement of all the modern states of the known world." The word occurs in 1787 with a somewhat changed meaning in the preface to E. A. W. Zimmerman, *A Political Survey of the Present State of Europe.* In this work it is stated that about forty years before the branch known as statistics was formed into a separate science in Germany. The German word *statistik* was used by Professor Achenwall of Göttingen in 1749, and the Latin, *statisticus,* was used at a somewhat earlier date. In *Meyers Konversationslexikon,* 6 ed., volume 18, under "Statistik", Schäzer (1735–1809), a pupil of Achenwall, defined: "Statistik ist stillstehende Geschichte; Geschichte ist fortlaufende Statistik."[2] In 1790 Sir John Sinclair stated in a letter to the Clergy of the Church of Scotland that "statistical inquiries" have been carried to a great extent in Germany, and adds that the expression "statistical inquiries" means "inquiries respecting the population, the political circumstances, the productions of a country, and other matters of state." *Statistics,* as thus used by German writers and by others in the eighteenth century, meant an exposition of the character of the state, and such expositions were usually verbal rather than numerical. With the growth of official numerical data, it was natural that numerical statements should begin to replace the verbal statements. Statistics thus gradually came to mean an exposition of the attributes of the state by numerical methods. Following this usage, the word next came to denote the figures used in such descriptions. Thus, the collections of numerical data were called statistics. This use of the word prevails at the present time but the data may refer to the state or to any other subject.

There is, however, an element in the meaning of the word statistics as used at the present time in the theory of statistics that is not necessarily involved in a collection of figures. Thus, a collection of 1000 numbers consisting

[1] These expressions should be listed alphabetically and references should be given to see their meanings under *statistics*.

[2] This citation is contributed by Professor A. J. Kempner.

of the number 5 written 1000 times would not constitute statistics. Numerical data known as statistics or *statistical data* have a certain element of variability. For example, statistics on the statures of men are variable from man to man. Statistics of social interest show great variability from individual to individual and from community to community. Statistics of meteorology show great variability from time to time and from place to place. The term statistics as used at present in the theory of statistics means numerical data that exhibit variability in individual items, where such variability is ascribed to a multiplicity of causes. See G. U. Yule, *Introduction to the Theory of Statistics*, London, 1922, pp. 1–5; H. Bruns, *Wahrscheinlichkeitsrechnung und Kollektivmasslehre*, Leipzig and Berlin, 1906, pp. 1–17; P. M. H. Laurent, *Statistique Mathématique*, Paris, 1908, pp. 1–16; and E. Blaschke, *Vorlesungen über mathematische Statistik*, Leipzig and Berlin, 1906, pp. 1–8.

The expression *statistical methods* means methods which are suitable for the description and characterization of statistical data. In the development of the meaning of statistical methods there have been influences in operation from three sources—the calculus of probability, the preparation of life and monetary tables under the name "political arithmetic," and the collection of data to be used in the machinery of government and business.

An exposition of the principles on which statistical methods are based is called the *theory of statistics*. A set of mathematical propositions that relate to statistical methods is often called *mathematical statistics*.

The relation of mathematical statistics to the theory of probability may be indicated by saying that the general problem of mathematical statistics in its ideal form is to determine a system of drawings to be carried out with urns of fixed composition, in such a way that the results of the set of drawings lead, with a high degree of probability, to a table of values identical with the statistical data (cf. E. Borel, *Éléments de la Théorie des Probabilités*, Paris, 1910, p. 167). Mathematical statistics is thus one branch of the theory of probability (cf. articles on *statistics* in German and French encyclopedias of mathematics). In fact, *a posteriori* or inductive probability is sometimes called *statistical probability*. (See E. Czuber, *Wahrscheinlichkeitsrechnung*, volume 2, Leipzig and Berlin, 1921, p. 6.) The concept of statistical probability is involved whenever the properties of an aggregate are predicted or inferred by observation of a sample taken from the aggregate. Many such inferences are drawn by persons unfamiliar with mathematical statistics, and there is practically no doubt that many conclusions thus obtained are invalid.

Mathematical statistics aims to establish criteria that give numerical values to the degrees of confidence to be placed in such inferences. In the development of these criteria statistical probability means the limiting value, as s becomes infinite, of m/s, where m is the frequency of happening of the event in s trials. The existence of the limit is assumed. The applications of mathematical statistics cover a wide range of scientific and social interests. These applications include the whole theory of insurance, and have an important place in biology, anthropology, psychology, economics, and even in the more exact sciences of chemistry and physics. To give a notion of the variety of applications, we may cite the following: *Biometrika*, "a journal for the statistical study of biological problems"; E. L. Thorndike, *Educational Psychology*, New York, 1913–1914; H. L. Rietz and H. H. Mitchell, "On the metabolism experiment as a statistical problem" (*Journal of Biological Chemistry*, volume 8, 1910, pp. 297–326); E. Rutherford and H. Geiger, "The probability variations in the distribution of α particles" (*Philosophical Magazine*, series 6, volume 26, 1910, pp. 698–707); J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, New York, 1902; F. Y. Edgeworth, "On the application of probabilities to the movement of gas-molecules" (*Philosophical Magazine*, series 6, volume 40, 1920, pp. 249–272). Lists of references on the methods and theory of statistics are given in G. U. Yule, *Introduction to the Theory of Statistics*, London, 1922, and E. Blaschke, *Vorlesungen über mathematische Statistik*, Leipzig and Berlin, 1906. A bibliography of applications prior to 1904 is given in C. B. Davenport, *Statistical Methods*, New York, 1904.

The following terms and expressions drawn from the theories of probability, statistics, and insurance are tentatively suggested for inclusion in the proposed dictionary. It seems that any term or expression that occurs infrequently, and that would require an appreciable amount of space to give its meaning in the dictionary, may well be treated by giving one or more references to its definitions

and use in the literature. In accord with this view, the mark "ref." is placed after each term that may, in the judgment of the writer, be treated in this way. Doubtless a considerable number of other terms may be treated by giving a very brief statement and references. In the preparation of the following list, the writer is indebted to Professor E. L. Dodd for valuable suggestions and additions.

Accumulation of discount, ref.
Actuarial theory
Adjustment of data, ref.
Advowson, ref.
Aggregate mortality table
Allocurtic, ref.
Amortization
Amortization of premium, ref.
Annual rent
Annuity,
- apportionable
- certain
- complete
- continuous
- deferred
- due
- forborne
- immediate
- intercepted
- joint life
- last survivorship
- life
- perpetual
- reversionary
- survivorship
- temporary

A posteriori probability
A priori probability
Array
Arithmetical mean
Association, theory of, ref.
Assurance
Automatic policy loan
Average
Average deviation
Bayes rule, ref.
Benefit of insurance, ref.
Bernoulli series, ref.
Bernoulli theorem, ref.
Bias in sampling
Biometry
Capitalized cost
Charlier Coefficient of Disturbancy
Claims, death
Class
Class frequency
Class interval
Class mark
Coefficient of association
Coefficient of contingency, ref.
Coefficient of variability
Commutation columns
Commutation symbols
Complement of life, ref.
Complete expectation of life
Compound reversionary addition, ref.
Contingency coefficient, ref.
Continuous instalment
Convertible term
Copyhold, ref.
Correlation,
- multiple, ref.
- normal
- partial
- rank
- spurious

Correlation coefficient
Correlation ratio
Cost of insurance
Cumulative graphs
Curtate expectation of life
Death rate, central
Death strain
Decile
Dependent events
Differences,
- central
- finite
- order of

Discount
Dispersion
Dividend (in life insurance),
- annual
- deferred

Endowment, pure
Endowment insurance
Equally likely
Equated time
Equation of life,
- of payments
- of value

Error
Errors, theory of
Expectation of life,
- complete
- curtate

Expiry
Extended insurance

Extrapolation
Figurate numbers
Fluctuations in sampling
Force of discount
Force of interest
Force of mortality
Frequency curve
Frequency distribution,
    binomial
    Gaussian
    normal
    skew
Frequency polygon
Frequency surface
Gain and loss exhibit
Gauss curve of error
Geometrical mean
Geometrical probability
Gompertz's law, ref.
Goodness of fit
Graduation of data
Group insurance
Hardy's formula, ref.
Harmonical mean
Heteroclitic, ref.
Heterograde series, ref.
Heteroscedastic, ref.
Heteroscedasticity, ref.
Histogram, ref.
Historigram, ref.
Homoclitic, ref.
Homogeneity in statistics
Homograde series, ref.
Homoscedastic, ref.
Homoscedasticity, ref.
Homotyposis, ref.
Incontestible
Independent events
Index of abmodality
Index number
Industrial insurance
Initial expense
Insurable interest
Insurance,
    assessment
    capital redemption
    casualty
    fraternal
    life
    property
Interest,
    accumulative rate of
    compound
    continuously convertible
    effective rate of
    frequency of conversion of
    instantaneous rate of
    nominal rate of
    rate of
    remunerative rate of
    simple
International actuarial notation, ref.
Isocurtic, ref.
Isotropic, ref.
Joint life annuity
Joint life insurance
Joint life probability
Lag
Lapse
Last survivorship annuity, ref.
Last survivorship insurance, ref.
Least squares, ref.
Legal reserve insurance
Level premiums
Lexis ratio, ref.
Lexis scheme, ref.
Life interest
Lifetime, most probable
Limited payment policy
Loading
Logarithmic paper, ref.
Lorenz graph
Lubbock's formula, ref.
Makeham's laws
Mathematical risk, ref.
Mathematics of statistics
Maturity
Mean contingency, ref.
Mean deviation
Mean error
Mean square contingency, ref.
Mean value
Median
Method of moments, ref.
Mode
Modulus as a measure of dispersion
Moments
Mortality rate,
    instantaneous
Mortality table,
    aggregate
    select
    ultimate
Moving average
Mutuality
Natural premium
Net premium
Net valuation
Nominal interest
Non-forfeiture
Normal frequency curve
Normal law of error
Normal probability curve
Ogive
Old line insurance
Ordinary life policy

GEORGE BRUCE HALSTED.

By FLORIAN CAJORI, University of California.

[For several years the MONTHLY has had no leading article concerning a deceased mathematician. An exception is now made in the case of one who in its earlier years assisted very materially in the financial support of the MONTHLY, and contributed extensively to its pages.

Dr. Halsted's articles in the MONTHLY were over fifty in number, and one of them, "Non-euclidean geometry, historical and expository," consisting chiefly of his translation of Saccheri's *Euclides Vindicatus*, extended as a serial through the first five volumes. Twenty of the articles were biographical sketches, usually accompanied by portraits, in the series which Professor Finkel established and carried on for a number of years. These sketches were as follows: Alasia (volume 9), Barbarin (15), Beltrami (9), Bolyai Farkas (3), Cayley (2), De Morgan (4), Frost (6), Hoüel (4), Klein (1), Lambert (2), Lie (6), Lobachevsky (2), F. Schmidt (8), D. M. Y. Sommerville (19), Sylvester (1 and 4), Tchebychev (2 and 5), Tucker (7), and Vasiliev (4). Most of Dr. Halsted's other articles dealt with topics of non-euclidean geometry or the foundations of geometry.

An appreciative sketch of Dr. Halsted, by Professor L. E. Dickson, appeared in the first volume of this MONTHLY, *1894*, 337–340; another by C. Alasia may be found in *Le Matematiche*, Città di Castello, volume 2; and others in B. F. Finkel's *Mathematical Solution Book*, fourth edition, 1902, and in *National Cyclopædia of American Biography*, volume 3, new edition. The first three are accompanied by portraits. Halsted was the first student in mathematics at the Johns Hopkins University and his insistence on a course in the theory of algebraic forms started the brilliant investigations that characterized Sylvester's work in Baltimore. This is related in Cajori's *Teaching and History of Mathematics in the United States*, 1890, pp. 264–266. We have already indicated the leading facts of Dr. Halsted's life (*1922*, 187); see also this issue of the MONTHLY, page 352. A fairly complete list of his writings may be derived from: Sommerville's *Bibliography of Non-Euclidean Geometry*, 1911; Poggendorff's *Biographisches-literarisches Handwörterbuch*, 1898 and 1904; and the Royal Society's *Catalogue of Scientific Papers*, volumes 10 and 15—EDITORS.]

On March 19, 1922, there passed away in New York City, at the age of sixty-nine years, a unique and picturesque figure among American mathematicians. One of the earliest pupils of J. J. Sylvester at the Johns Hopkins University, a student in Germany and an instructor at Princeton University, he was later for nineteen years professor of mathematics at the University of Texas. Leaving Texas in 1903 he was for nine years successively connected with several institutions, passing rapidly from one to another. His closing years were spent with a son at Greeley, Colorado.

Halsted's scientific activity may be said to have penetrated three fields: (1)

the translation, with commentaries, of noted foreign works, particularly of the great researches of Lobachevsky, Bolyai and Saccheri, and some of the popular writings of Poincaré; (2) studies in the logic of mathematics, particularly of geometry; (3) criticisms of the mathematical text-books of his day.

American mathematicians are indebted to Halsted for making the writings of the creators of non-Euclidean geometry accessible to them in the English language. His commentaries were always spicy and valuable, even though, as a historian, Halsted was not always able to maintain the attitude of an impartial judge. At his hands Gauss, for instance, received scant justice.

The most conspicuous of his efforts in logic was his *Rational Geometry*, the first edition of which, published in 1904, was based on David Hilbert's set of axioms of 1898. The book was widely and favorably reviewed, both in this country and abroad. Certain logical defects were pointed out, which caused Halsted to prepare a revised edition (1907) in which the words on the title-page of the first edition, "Based on Hilbert's Foundations," were dropped. This revised edition was well received and was translated into several foreign languages. Logical precision was the dominant motive in the preparation also of his earlier texts, chief of which are his *Elements of Geometry* (1885) and *Synthetic Projective Geometry* (1906).

Among critics Halsted ranked as the most outspoken American opponent of slip-shod methods of reasoning, such, for example, as occurred in the same-direction theory of parallel lines and in the unrestricted employment of hypothetical constructions. Perhaps rather over-confident of the correctness of his own views, his criticisms of mathematical writers, and sometimes also of university administrators, were at times so violent as to recoil upon himself. Nevertheless, his influence upon the teaching of mathematics in this country has been decidedly beneficial. Several of our most active mathematicians of the present time received inspiration while they were pupils of Halsted.

I myself met Halsted in person only two or three times at scientific meetings and then, each time, only for a few moments. But I was in occasional correspondence with him since 1888. When I first entered upon historical work, I received from him valuable suggestions and much-needed encouragement. About 1895 he read the geometrical part of the manuscript of my *History of Elementary Mathematics*. In a letter to me he made a most unsparing attack, called forth by one of my criticisms of Euclid's logic and a clerical error that I had made in copying a reference to Euclid. In my reply, I acknowledged the clerical error, and made a few changes and additions to my account of Euclid. I also endeavored to point out to him the slight unreasonableness of his attack. By return mail I received an answer, which indicated that the storm had passed and that Halsted had fully re-entered the atmosphere of serene cordiality. He did not indulge in flattery; his criticisms were honest and based on a firm grasp of fundamentals; his correspondence was stimulating.

Halsted had in his private library some rare books which he prized highly. He himself has published statements relating to his copy of Saccheri. His copy

of the first (1637) edition of Descartes' *Géométrie,* which, through the courtesy of Professor C. I. Palmer, the present writer examined two years ago, contained on the first leaf the signature of Letenneur and the following inscription:

"Offert à Monsieur J. J. Sylvester Souvenir affectioné de son dévoué Chasles. Feb. 1847."

Further on appears the entry:

"To Dr. Halsted, with the kind regards and all good wishes of J. J. Sylvester."
2d May 1893.

And at the end of the book is pasted a small sheet containing the following note written by Halsted himself:

"This book, *La Geometrie,* treasured in the family of Letenneur as gift of the author and by the great geometer and historian of geometry M. Chasles as the gem of his collection, was given by him to Sylvester as recompense for a great service and by Sylvester to his favorite pupil Halsted, by whom it was conveyed to Professor C. I. Palmer.

"The signature of Letenneur with his minute but characteristic rubric, that of Chasles after his glowing words and of Sylvester after his gracious lines
Make the book priceless.
G. B. H."

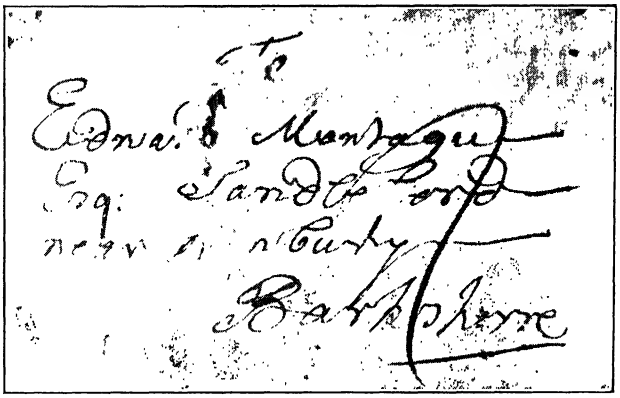
The spirit of Halsted has fled—gone to meet face to face the great masters he admired—Saccheri, Lobachevsky, Bolyai.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

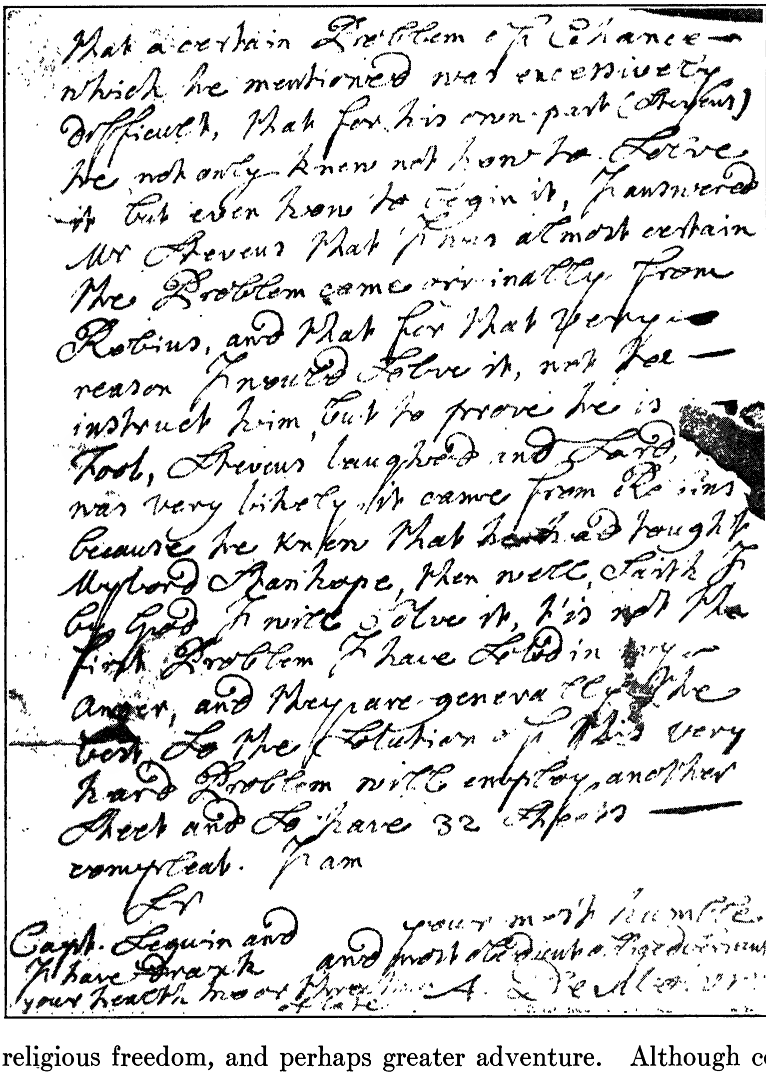
28. De Moivre Expresses Himself.

Of all the mathematicians who added to the reputation of England in the closing years of Newton's life, no one arouses a more sympathetic interest than Abraham De Moivre, author of the well-known *Doctrine of Chances,*[1] of the even



[1] London, 1718; second edition, London, 1738; third edition, posthumous, 1756.

more notable *Miscellanea Analytica*,[1] of a work on annuities, and one on series, and of various monographs on geometry and the Newtonian calculus. Born in France in 1667, he studied mathematics as a boy under Ozanam, one of the best teachers in Paris, and, at the age of eighteen, went to London in order to find


that a certain Problem of Chance
which he mentioned was excessively
difficult, that for his own part (Sir)
he not only knew not how to solve
it but even how to begin it, I answered
Mr Stevens that I was almost certain
the Problem came originally from
Robins, and that for that very
reason I would solve it, not to
instruct him, but to prove he is a
Fool, Stevens laughed and said it
was very likely it came from Robins
because he knew that he had taught
Mylord Stanhope, then well, saith I
by God I will solve it, 'tis not the
first Problem I have solved in my
anger, and they are generally the
best. The solution of this very
hard Problem will employ another
sheet and a square 32 ...
compleat. I am
Sir
your most humble
and most obedient servant
A. De Moivre
Capt. Lequin and
I have drank
your health more than once.

greater religious freedom, and perhaps greater adventure. Although compelled by narrow circumstances to forego the ideal life of a student, he became a member of the mathematical group of which Newton was the leader, and was recognized as a man of genuine ability. He made his living by teaching mathematics and by solving such problems as were brought to him by the curiously inclined, but such an existence is always a precarious one, and De Moivre's later years were

[1] London, 1730.

those of a rather lonely old man whose intimate friends had long since departed this world. He frequented Slaughter's famous coffee house in St. Martin's Lane, and that he imbibed something besides the drink from which the place received its name is not to be wondered, considering the period in which he lived and the circumstances which surrounded him.

Among my autographs is the last page of what was originally a four-page letter written by him to Edward Montague, a member of one of the most prominent of the noble families of England.[1] The letter is here reproduced in facsimile, being more legible than is usually the case with those that have come down to us from that period.

In this letter De Moivre speaks of a Mr. Stevens, probably Henry Stuart Stevens.[2] He also refers to a certain problem arising in the then popular game called "Hazard." This problem does not appear in the first edition of De Moivre's celebrated *Doctrine of Chances*, but it is No. XLVI of the edition of 1738. Todhunter,[3] in his *History of the Theory of Probabilities*, quotes De Moivre as saying:

"After I had solved the foregoing I spoke of my solution to Mr. Henry Stuart Stevens, but without communicating to him the manner of it. As he is a gentleman who, besides other uncommon qualifications, has a peculiar sagacity in reducing intricate questions to simple ones, he brought me, a few days after, his investigation of the conclusion set down in my third corollary; and as I had occasion to cite him before in another work, so I here renew with pleasure the expression of esteem which I have for his extraordinary talents."

Now by reference to the letter itself and to the above quotation and others from De Moivre, it is possible to reconstruct the probable facts of the case. This particular problem XLVI is "to find at Hazard the gain of 'the box' for any number of games divisible by 3," and is first given in the edition of 1738. In his *Miscellanea Analytica* of 1730, De Moivre refers to it, however, saying: "Septem aut octo abhinc annis D. Stevens Int. Templ. Socius, vir ingenuus, singulari sagacitate præditus id sibi propositum habens ut problema superius allatum solveret, hac ratione solutionem facile assecutus est, quam mihi his verbis exhibuit."

Now seven or eight years before 1730 would be about 1722–1723. But in the 1738 edition of the *Doctrine of Chances* De Moivre says that he solved the problem about twelve years before, which would be about 1726 (provided the revision was made the year the book was printed, which was probably not the case.) After De Moivre solved the problem, perhaps about 1724, he probably showed the solution to Stevens, who gave him a better one. We then have the date of the letter about 1723 or 1724.

[1] He was born in 1697 and was admitted as a Fellow of the Royal Society in 1746, having been elected the preceding year. He died in 1761. His signature may be found in facsimile in a book of the Royal Society, published in London in 1912.

[2] Born in 1707; died in 1782. He was elected to the Royal Society in 1740 and was admitted as a fellow the following year.

[3] Isaac Todhunter, born in 1820; died in 1884. He was successful as a writer of textbooks and his two historical contributions rank very high.

It should be stated, however, that it is by no means certain that the problem referred to in the letter is No. XLVI of the second edition of the *Doctrine of Chances*. It may be some other problem to which we have no certain way of attaching Stevens's name.

Coming now to the letter itself, it seems that the problem was given to De Moivre by Stevens about 1724; that De Moivre said that the problem very likely came originally from one Robins, probably Benjamin Robins,[1] to which opinion Stevens agreed; and that De Moivre had a rather poor opinion of Stevens at that time. The Lord Stanhope mentioned as having been tutored by Robins was doubtless Philip Dormer Stanhope, fourth Earl of Chesterfield, the well-known author of Chesterfield's letters, whose education was chiefly received through private instructors.

The reference in the letter to having worked out some of his best solutions "in my anger," might better, no doubt, have read "in my cups." The Captain Seguin, referred to in the post scriptum, is not mentioned in the standard works on biography. The "32 Sheets compleat" may refer to the revision of the *Doctrine of Chances*, already under way, or possibly to the *Miscellanea Analytica*. Books were not written in a hurry in those days.

The story, so often told, that De Moivre decided to sleep fifteen minutes longer each night, until finally he never awoke, is probably based upon some pleasantry of his in connection with his disease of somnolence which finally ended a life that had become quite unendurable, even with the help of such libations as were then poured out so freely.[2]

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

The discussions which follow give cause for special satisfaction in that two of them are in some sense replies to previous papers, while the third closes with a question inviting further investigation. It would seem, from such indications, that this department is at times answering its purpose as a department of discussion.

The letter of Brigadier General Bixby, though it arises out of Professor Candy's article on mechanical solution of equations, is concerned more specifically with graphical solution, and emphasizes in a suitable way the great and increasing importance of this problem. In particular, the writer argues, from his experience, the advantages of Lill's graphical method, and recalls his own attempt in 1879 to make this method known to English reading students, by means of a pamphlet.

[1] Born at Bath in 1707; died at Fort St. David, Madras, in 1751; author of *Mathematical Tracts*, printed posthumously in 1761.

[2] I am indebted to Mr. Jekuthial Ginsburg for assistance in tracking down some of the facts in connection with this interesting letter.

On account of the great simplicity and utility of the method, and of its almost universal omission from texts in English, some readers of the MONTHLY will perhaps be glad to find a portion of General Bixby's pamphlet quoted at length.

The note by Professor Underwood completes the author's previous work (*1921*, 374–376), and bears some relation to his reply to Question **36** (*1922*, 255).

Professor Mathews proves directly a theorem regarding concyclic sets of points on an equilateral hyperbola which is the inverse of a theorem already known on a bicircular quartic.[1] If, in the notation of the paper, the points $\{B\}$ and $\{C\}$ are kept fixed, the sixteen circles AB_iC_jD through a point A define sixteen transformations of A into D. The question raised at the end of the paper is then how to determine the eight points $\{B\}$ and $\{C\}$ so that the transformations generate a finite group.

I. Graphical Solution of Numerical Equations.[2]

By W. H. Bixby, Washington, D. C.

I was much interested in the article of Professor A. L. Candy on mechanical solution of equations of the nth degree in a single variable.[3] Theoretically, the method is excellent; but practically there will be much difficulty in using it. It will be difficult to determine or secure the exactitude of the right-angles concerned, and the necessary length of the main bar may be awkward. The most perfect and practical mechanism for such equations is the multiple weighing machine, which has several graduated bars and sliding weights, the final weight indicating a root at each place where the machine balances. It would be costly, but no more so than the modern refined planimeters.

From my own personal experience in such matters, the method of Mr. Lill, Austrian engineer, developed by him about 1867 and exhibited by him at the Vienna World Exposition a little later, is the best graphical method yet developed, and far easier, quicker, and more exact, than any other graphical or mechanical method. I read of this about 1878, and published it in 1879 by a privately printed pamphlet.[4] At that date I had not seen Lill's 1867 printed article.[5] A few months ago I found that Luigi Cremona had also described Lill's method and made it public to English readers in 1888.[6] My pamphlet failed to attract much attention. A few engineers showed some interest in it; but appar-

[1] R. Lachlan, "On a theorem relating to bicircular quartics and twisted cubics," *Proceedings of the London Mathematical Society*, vol. 21, 1890, pp. 274–280; J. L. Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1915, p. 217.

[2] Extract from a letter to the editors.

[3] A. L. Candy, "A Mechanism for the Solution of an Equation of the nth Degree," this MONTHLY, *1920*, 195–199.

[4] W. H. Bixby, *Graphical Method for Finding the Real Roots of Numerical Equations of Any Degree if Containing but One Variable*, West Point, 1879.

[5] M. E. Lill, "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," *Nouvelles Annales de Mathématiques*, series 2, vol. 6, 1867, p. 359.

[6] L. Cremona, *Graphical Statics*, translated by T. H. Beare, Oxford, 1890, pp. 70–76. [More recent accounts of the method are found in C. Runge, *Graphical Methods* (Columbia University Lectures, 1909–1910), New York, 1912, pp. 11–12, and *Praxis der Gleichungen*, Berlin and Leipzig, 1921, pp. 101–110. See also this MONTHLY, *1911*, 159–162—EDITOR.]

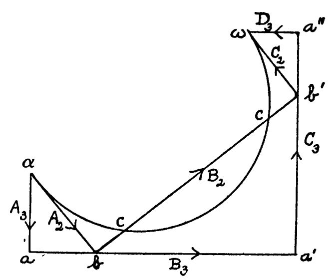
ently there were only a very few who at that time had any occasion for solving such equations in their daily practice.

The following extract from the author's 1879 pamphlet will explain the method under discussion.[1] The proof may be easily supplied by the reader. The figure appended is for the case $n = 3$.

"Suppose any numerical equation

$$A_n x^n + B_n x^{n-1} + C_n x^{n-2} + \cdots + T_n x + U_n = 0,$$

in which A_n is any positive number, either whole or fractional.



"Commence, on a blank sheet of paper, at any assumed point α, and using any convenient scale, lay off in a downward direction a distance αa equal to A_n; through a, draw a perpendicular, and lay off upon it with the same scale as before, the value of B_n (laying off this distance to the right if B_n is positive, to the left, if B_n is negative); through the end of B_n, draw a perpendicular to B_n, upon which lay off the value of C_n, upward if positive, downward if negative; and so on. . . . [For each new line the positive direction turns through a right angle counter-clockwise.] Letter the end of the last line ω. We will then have a rectangular contour (that is, a broken line all of whose angles are right angles) of $n + 1$ sides commencing at α and ending at ω.

"Now starting again at α, draw at random any straight line cutting B_n in some point as b; through b draw a perpendicular to αb cutting C_n in some point as b', and so on; the result will be a new rectangular contour of n sides. If the nth side passes through the point ω, then $ab/\alpha a$, taken with its sign changed, is a root of the given equation. There will be as many such contours of n sides, and therefore as many points b, as there are real roots to the given equation.

"These rectangular contours can be readily determined by the aid of a ruler and right-angled triangle.

"Suppose only one root is found by the above method, giving us one new rectangular contour, $\alpha bb' \cdots \omega$, of n sides; call its first side A_{n-1}, its second B_{n-1}, and so on; this new rectangular contour, $\alpha bb' \cdots \omega = A_{n-1}, B_{n-1}$, etc., is the contour which represents the equation obtained by reducing the given equation to the $n - 1$ degree, by dividing out the root, $x = ab/\alpha a$, already found; treat this new contour of n sides like the preceding, obtaining a new rectangular contour of $n - 1$ sides whose first vertex is at some point c upon the line bb'; then $bc/\alpha b$, taken with its sign changed, will be another root of the given equation; and so on.

"The graphic solution of equations, by the above method of trial, will be an exceedingly simple process if the reader will proceed as follows:

"Procure a sheet of paper ruled in squares, each square being about a tenth of an inch on its side (such paper is easy to obtain, as it is nowadays in common use among surveyors and draughtsmen); procure also a sheet of tracing paper or cloth. Lay the tracing sheet upon the ruled paper, and, by the aid of the divisions upon the ruled paper, lay off on the tracing sheet the lines $ABCD$, etc.; thrust a needle, or pin, through the beginning of the A line and also through any convenient intersection upon the ruled paper; then, turning the tracing sheet about the needle as a pivot, the reader can readily follow by eye the ruled lines of the lower sheet, and can easily detect the auxiliary rectangular contours without having to pencil even a single unnecessary line.

"This graphical solution is especially applicable to cases where the desired roots lie between $\pm \frac{1}{2}$ and ± 5; if the desired roots lie beyond these limits, the given equation may advantageously be transformed into another whose roots will lie between the above limits."

In the case of a quadratic equation (represented by the inner contour) the

[1] For the extension of the method to complex roots and complex equations, see the article "d'après M. Lill" "Résolution graphique des équations algébriques qui ont des racines imaginaires," *Nouvelles Annales de Mathématiques*, series 2, vol. 7, 1868, p. 363—EDITOR.

two solutions, if real, may be found by means of a circle on $\alpha\omega$ as diameter, as indicated in the figure.[1]

II. Supplementary Note on the Irrationality of Certain Trigonometric Functions.

By R. S. Underwood, Alabama Polytechnic Institute.

Theorem: *When an angle is rationally expressible in degrees and not a multiple of 30° or 45°, its trigonometric functions are irrational.*

The tangent of any rationally expressible angle except one of the form treated below may be proved irrational by a method so closely analogous to that used in my former paper that the details are omitted here. There remain angles of the form $(m/n)^\circ$, where m/n is an irreducible fraction and where $m = k45$.

In the expansion

$$\tan n\theta = \frac{n \tan\theta - \frac{n(n-1)(n-2)}{3!}\tan^3\theta + \cdots \pm \tan^n\theta (\text{or} \pm n \tan^{n-1}\theta, \text{ when } n \text{ is even})}{1 - \frac{n(n-1)}{2!}\tan^2\theta + \cdots \pm n\tan^{n-1}\theta \ (\text{or } \tan^n\theta, \text{ when } n \text{ is even})}$$

the substitution $\tan n\theta = \pm 1$ gives an nth degree equation whose only possible rational roots (± 1) are excluded by the conditions of the problem. When $\tan n\theta = 0$ or $\pm\infty$, it is evident that the roots obtained by equating the numerator and denominator respectively to zero are either integers or the reciprocals of integers. Furthermore, if $\tan\theta$ is rational, $\tan 2\theta$, when it is finite, is rational. But if $\tan\theta = k$ or $1/k$, k being an integer, $\tan 2\theta = \pm 2k/(1 - k^2)$, which is neither an integer nor the reciprocal of an integer, since the denominator lacks the factor k. Hence $\tan\theta$ is irrational.

The above theorem is interesting in that it brings out in a striking way for pedagogical purposes the great preponderance of irrational over rational numbers. Divide a right angle into n equal parts. Then there are only three distinct angles in all this infinite family which possess rational trigonometric functions.

Incidentally I have shown that the algebraic equations of the form (n odd)

$$x^{n-1} - \frac{n(n-1)}{2!}x^{n-3} + \cdots \pm n = 0,$$

and

$$nx^{n-1} - \frac{n(n-1)(n-2)}{3!}x^{n-3} + \cdots \pm 1 = 0,$$

have no rational roots.

[1] General Bixby would be glad to hear from any engineer, physicist or mathematician who is frequently required to solve higher equations, what method he uses, how much time is required, and what accuracy is deemed necessary.

III. Concyclic Points on an Equilateral Hyperbola and on its Inverses.

By R. M. Mathews, Wesleyan University.

The equation of an equilateral hyperbola may be written

$$xy - x - y = 0, \tag{1}$$

where the origin is a vertex and the axes of coördinates are parallel to the asymptotes. Writing $y = mx$, we can obtain the parametric equations

$$x = \frac{1 + m}{m}, \qquad y = 1 + m. \tag{2}$$

There is one to one correspondence between all finite values of m (except zero) and all finite points on the curve.

These expressions (2) substituted in the equation of an arbitrary circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \tag{3}$$

lead to the quartic equation

$$m^4 + 2(1 + f)m^3 + (2 + 2g + 2f + c)m^2 + 2(1 + g)m + 1 = 0. \tag{4}$$

The roots of this equation are the parameters of the points of intersection of the circle with the hyperbola. It is easy to show that the necessary and sufficient condition that four points of an equilateral hyperbola (2) be concyclic is that their parameters be such that

$$m_1m_2m_3m_4 = 1.$$

Take three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ of parameters k_i, l_i, m_i, respectively ($i = 1, 2, 3, 4$). The circle through A_i, B_i, C_i will cut the hyperbola in a fourth point D_i (parameter n_i) such that

$$k_il_im_in_i = 1 \qquad (i = 1, 2, 3, 4).$$

From these equations, with the help of the facts that the sets of A's, B's and C's are severally concyclic, we obtain

$$n_1n_2n_3n_4 = 1.$$

Thus we have proved the theorem:

Given three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ ($i = 1, 2, 3, 4$) *on an equilateral hyperbola; let the circle through* $A_iB_iC_i$ *cut the curve in a fourth point* D_i; *then the four points* D_i *are concyclic.*

As specializations of this proposition we have two known theorems, the first due to Cazamian.[1]

Two circles meet an equilateral hyperbola in $\{A_i\}$ *and* $\{C_i\}$ ($i = 1, 2, 3, 4$),

[1] "Sur l'hyperbole équilatère et sur ses inverses," *Nouvelles Annales de Mathématiques*, series 3, vol. 13, 1894, pp. 265–280.

respectively; draw the circle through A_i and tangent to the curve at C_i; the four circles so obtained meet the conic again in four concyclic points.

The four circles which osculate an equilateral hyperbola in four concyclic points cut the curve again in four concyclic points.

When an equilateral hyperbola is inverted about a point not on it, the transform is a bicircular quartic with a singular point at the center of inversion and the tangents there are at right angles (for they are parallel to the asymptotes of the conic). Thus the quartic is *orthotomic.* In particular, when the point of inversion is at the center of the hyperbola, the transform is a lemniscate. If the hyperbola be inverted about a point on it, the transform is a strophoid, oblique, in general, but a right strophoid, or Booth's logocyclica, when the center of inversion is a vertex of the conic. Every orthotomic bicircular quartic and every strophoid may be obtained by such an inversion.

As an inversion carries circles into circles, in general, the theorems just stated hold true when "equilateral hyperbola" is replaced by "orthotomic bicircular quartic" or by "strophoid," and the circles are conditioned as not passing through the node.

When one of the circles of the hyperbola passes through the center of inversion, special theorems result after the inversion. For example, in the main theorem for a strophoid let A_1 be at ∞, and then:

Let A_2, A_3, A_4 be three collinear points of a strophoid while $\{B_i\}$ and $\{C_i\}$ ($i = 1, 2, 3, 4$) are two sets of concyclic points. Let the line B_1C_1 cut the curve in D_1, while the circle on $A_iB_iC_i$ ($i =$ 2, 3, 4) meets the cubic in D_i. Then the four points $\{D_i\}$ are concyclic.

In the case of one of the quartics the initial points may be arranged as: (1) one set of four collinear points with two sets of concyclic; (2) two sets of collinear points with one concyclic; (3) three sets of collinear points. None of the lines or circles of the quartic may pass through the singular point.

It is evident that the main theorem carries on to curves of still higher degree obtained by successive inversions.

On returning to the original theorem, we observe that there is no order assigned for the choice of a point triple to determine an (ABC) circle. When the three sets of concyclic points are assigned, it is evident that there are 64 circles (ABC) possible; each is determined by a triple of points, one from each set. Accordingly, there are 64 D-points which are concyclic in groups of four. To find how many of the possible ${}_{64}C_4$ are admissible here we observe: first, that each D-set is determined by a certain four of (ABC) circles; second, that in this set must be one circle for each of the A's; third, under this line of A's must be written some permutation of the four B's and some permutation of the four C's. Thus there are $\lfloor 4 \cdot \lfloor 4 = 576$ such sets and so 576 D-circles.

There are now $12 + 64 = 76$ points concyclic by fours in at least $576 + 64 + 3 = 643$ ways. It is easy to show that if a set of D-points be used with the B's and C's, the original A-set will be reached only once. It remains, then, to determine the conditions under which the configuration is closed as it stands, or will be closed after the addition of other points.

RECENT PUBLICATIONS.

REVIEWS.

Geschichte der Mathematik in systematischer Darstellung mit besonderer Berücksichtigung der Fachwörter. By J. TROPFKE. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger. Erster Band,[1] 1921, *Rechnen,* 8 + 177 pages; zweiter Band, 1921, *Allgemeine Arithmetik,* 4 + 221 pages; dritter Band, *Proportionen, Gleichungen,* 1922, 4 + 151 pages. Price \$.70 + \$2.00 + \$1.44.

Tropfke's *Geschichte* appeared first in 1902, as a two-volume work. It was well received as an interesting and worthy attempt at a systematic history of the fundamental ideas of arithmetic, algebra, geometry, and trigonometry.

The outstanding defect of the first edition was the reliance placed upon the authority of Cantor. This standpoint has been entirely changed in the present edition and the author has made every effort to found his conclusions upon source material. The preface indicates that the author has had the great advantage of the friendly criticism of Gustav Eneström, of Stockholm, and of H. Wieleitner, of Augsburg; no abler critics for work on the history of elementary mathematics could be found anywhere. Equally fortunate has Dr. Tropfke been in securing the aid of Julius Ruska, of Heidelberg, for points on Arabic mathematics and terminology.

The radical nature of the revision is only partly indicated by the fact that the first volume, *Rechnen* and *Algebra,* of the first edition required 332 pages, as opposed to a total of 549 pages now for these three sections, containing the corresponding material. More significant is the fact that the first edition of these sections included a grand total of slightly over 1200 footnotes, as opposed to over 2600 in the revised edition. Frequently, even yet, Tropfke cites Cantor when the original is easily available, as in respect to Egyptian duplication (note 251, volume 1) and in other places. The statement, page 52, volume 1, that the works of Jordanus had the widest circulation because of the cultivation of science largely by church schools is not all in accordance with historical fact. The works of Sacrobosco and Alexander de Villa Dei had the widest circulation, evidenced by the great number of manuscripts in European libraries, and these works also had the most enduring influence on subsequent works before the invention of printing; this conclusion has been definitely established in a thesis at Michigan, *A comparative study of the early treatises introducing into Europe the Hindu art of reckoning* (Concord, N. H., 1916) by Professor Suzan R. Benedict, of Smith College.

I note on page 13, volume 1, that the reference to the use of "Zeyfferzale" is incorrectly transcribed from the 1514 edition of Koebel's *Rechenbüchlein.* However, this is evidently only a slip as other citations checked are found accurately given.

The topics treated in the first part, of five parts planned, embrace numbers in general, including names and symbols, measures of time, of angles, and decimal

[1] Already reviewed in this MONTHLY, *1922,* 16–17.

measures, computation with integers, denominate numbers, properties of numbers, tables, fractions, and applied arithmetic. Under computation each fundamental operation is discussed with illustrations.

A notable omission is that form of division in which the remainders are written but not the partial products, commonly called the "Austrian method" and usually associated with the additive method of subtraction. Tropfke indicates that the complete "Austrian" process appeared first about the middle of the nineteenth century, but omission of partial products is found in Clavius (1603 ed. consulted), Manelli of 1659, Dechales of 1690, Malcolm's *A New System of Arithmetick* of 1730 and Sadler's *Complete System* of 1773; this method is designated as "a danda" or as Italian.

The second part of this work treats algebra and logarithms under similar detailed divisions. This method of treatment makes the work of great value to the teacher who wishes to find the historical development of any given topic of elementary arithmetic or algebra. It is highly to be recommended for this type of use.

The treatment of the algebraic symbols is given in a satisfactory manner; further light would have been obtained by consulting an article by Dr. Suzan Benedict on "The Development of Algebraic Symbolism," written at Teachers College, using the Plimpton collection and the private collection of D. E. Smith[1]. The statement (page 16) that the abbreviation used by Regiomontanus for *minus* is to be read "ig" is not at all accurate; the abbreviation is entirely regular for *minus*, the symbol like a 9 being standard for "us" and the "m" and "n" indicated by a bar above the "i."

So far as the computation with zero is concerned Tropfke leaves the impression that the early writers on algorism neglected this. However, both of the treatises published by Boncompagni (Trattati I and II), and those of Sacrobosco and Alexander de Villa Dei expressly mention this computation with zero.

As a late illustration of a mathematician who did not believe negative roots to be possible Tropfke cites Harriot's *Artis analyticae praxis*, 1631. A whole series of later illustrations is to be found in Maseres, *Tracts on Numerical Equations*.

Repeatedly in the treatment both of arithmetic and of algebra (*e.g.*, volume 1, pages 17–18; volume 2, pages 8–12; volume 3, pages 28, 36, 50, 112) Tropfke is compelled to treat the Hindu mathematics. Unfortunately Tropfke has taken the suspicions of Mr. G. R. Kaye concerning the non-Hindu origin of many mathematical ideas for fact. Historical things must be treated historically. Definite and concrete evidence has been found of numerous contributions to arithmetical and algebraical ideas by Hindu and Arabic scholars; any revision in favor of Greek origin must be based on historical evidence, not suppositions. Until the Greek documents are found no historian has any right to postulate the existence of Greek documents, and deny credit to Hindus, to Arabs, and to Babylonians solely on the basis of some hypothetical assumptions. Square root,

[1] *Teachers College, Columbia University, Circular, Department of Mathematics*, 1906–1907; *School Science and Mathematics*, vol. 9, 1909, pp. 375–384.

numerical symbols, negative numbers, the treatment of zero, fractions, algebraic equations, large numbers, and other related ideas are all topics which lead us historically to India; the whole is self-supporting and bears internal evidence of Hindu origin of more weight than any single document; but even the documentary evidence is available in mass.

Under *Allgemeine Arithmetik,* in the second volume, Tropfke treats in the first subdivision the history of our algebraic symbols and the introduction of general coefficients. The word algebra is discussed historically in a separate chapter. In a section devoted to the development of the concept of algebraic number Tropfke treats unity, zero, infinity, fractions, irrationals, negative numbers, and complex numbers. In the section devoted to the algebraic operations, the fundamental operations and also powers and roots are historically discussed. The second volume closes with a discussion of logarithms.

The historical development of topics connected with the theory of equations and of proportion is ably given in the third volume. The most notable omission seems to the writer of this review to be any mention in the discussion of quadratic equations of the algebra of Abu Kamil and of its influence upon the work of Leonardo of Pisa and Al-Karkhi [See my article in this MONTHLY (*1914,* 37–48), "The Algebra of Abu Kamil"; also in *Bibliotheca Mathematica,* volume 12, 1912, pp. 40–55]. To say that Al Karkhi followed only Greek models is nonsense; his work on quadratics is directly taken from Abu Kamil as I have shown. The date of Diophantos, 360 A.D., is too late by at least a century. The reference (page 42) to Greek normal forms of equations (*e.g.,* $x^2 = ax$, etc.) should be to Arabic forms.

The valuable list of problems and illustrations from classical authors has been increased, and an interesting appendix added on the Arabic treatment of the cubic equation.

Fortunately Tropfke has avoided, in general, the offensive national emphasis which so frequently mars the monumental work of Cantor. We see traces of Cantor's influence still in the references to Bürgi under decimal fractions in the first volume and under logarithms in the second volume.

This new edition is to be highly commended. Unfortunately no similar work is to be had in English, although even a translation of this work would be desirable. The scientific activity in Germany under the present undoubtedly distressing economic conditions arouses our admiration.

The prices in marks of books published in Germany vary almost from day to day; the publishers have affixed the prices indicated above. Notwithstanding the resentment which Americans may feel in the fact that these and other publishers in Germany make American prices higher than European it must in fairness be admitted that these prices are far less than recent English prices on volumes of this size. It is to be hoped that definite encouragement of this scholarly publication will be forthcoming from America, even though some of the orders go through German dealers to take advantage of the lower price in marks in Germany.

LOUIS C. KARPINSKI.

ARTICLES IN CURRENT PERIODICALS.

ALCALDE (Austin, Texas), volume 10, September, 1922: "George Bruce Halsted" by H. Y. Benedict, 1357–1359 ["Halsted was not a member of the first 1883–84 faculty of eight professors, but came the second year to succeed as Professor of Mathematics, William Le Roy Broun, who had gone back to Alabama to be president of the Alabama Polytechnic College. Halsted was for some time the first and only young 'professor,' all his colleagues being 'elder statesmen' who had already won their reputations in other institutions. The young, erratic, and somewhat bizarre Halsted presented quite a contrast to the older and more conventional Gould, Dabney, *et al.* During the 18 years that he remained in the University he was the most conspicuous and spectacular member of the faculty, at least so far as the students were concerned. There were noted men in the faculty and the students knew it, but they talked more of Halsted than of the others. This was because he was always saying or doing something peculiar or interesting. To this day the old-timers remember how he used to drive rapidly all about town a Shetland pony hitched to a wagon built of four toy wheels and a long plank, himself sitting astride of the plank which also carried three five-gallon coal-oil cans full of mash for his pigs. Let any student of the early years get to telling stories and you'll soon hear him tell, not without some exaggeration, anecdotes of Halsted—how he tried to feed his family exclusively on prickly-pear salad, how he built his house on stilts to avoid malaria; how, driving too rapidly, he thanked the policeman who was about to arrest him, for stopping his runaway horse.

"Halsted was a graduate of Princeton, where there is a Halsted Observatory donated by a member of his family, and at Johns Hopkins was a pupil of the great but eccentric Sylvester and a member of the small band there in 1876 that is usually regarded as the first real graduate school in the United States. For scientific research, for discovery of new natural laws, for philosophical speculation, for plucking some new fact out of the vast unknown, Halsted ever afterwards had an unabated zeal, an enthusiasm that communicated itself to others even while they smiled at his. 'Creation discovery,' 'the on go and the uprush of civilization,' 'nothing can take the place of contact with the living spirit of research,' were phrases ever on his tongue. The unknown fascinated him, conventions bored him, and he yearned to make discoveries.

"Although an unsystematic teacher who did not force his students to cover much ground, he was very interesting and at times elucidated complexities with startling clarity. His public lectures were extremely popular, and on interesting topics. In debate he was acute, and amusing. . . .

"Some of Halsted's phrases still cling after the passage of many years. 'The successful American,' he once wrote, 'is highly migratory; I have been all over the world.' At an ancient meeting of the Texas Academy of Science, at which Alexander Macfarlane, Professor of Physics, but known locally as 'Johnnie,' was arguing that the square root of minus one was a rotor or 90 degree turner, Halsted jumped to his feet and cried, 'It is no more 90 degrees than it is the left ear of a white elephant.'

"Much more could be written. Was he lucky in the pupils that he had, or did he inspire them to great things? That he had great pupils is certain: Len Dickson, M. B. Porter, R. L. Moore, G. W. Pierce, R. A. Thompson, Florence Lewis, and others. From any other institution comparable with the University of Texas, there has come in our generation no such group. . . .

"To 'George Bruce,' as we used to call him, it is scarcely fitting to say 'Rest in Peace.' In the other world, with William James and Elliot Cowes, with his beloved Bolyai and Lobatchevsky and Sylvester, he will find his happiness, not in rest, but in gazing curiously at the universe as it appears from the other side. Loving fine distinctions, splitting and resplitting the boundary line dividing convergent and divergent series, may not Halsted's spirit even now be seeking the Aum of the Buddhists, the Alpha and Omega of the Christians, the Alif of the Mohammedans?

'A hair divides the False from True,
Yes; and a single Alif were the clue
Could you but find it—to the Treasure House
And peradventure the Master too.'"]

ANNALS OF MATHEMATICS, volume 23, September, 1921: "On matrices whose elements are integers" by O. Veblen and P. Franklin, 1–15; "An algorism for differential invariant theory" by O. E. Glenn, 16–28; "The general theory of cyclic-harmonic curves" by R. E. Moritz, 29–39; "More theorems on the complete quadrilateral" by J. W. Clawson, 40–44; "A theorem on cross ratios in the geometry of inversion" by J. L. Walsh, 45–51; "The condition for an isothermal

family on a surface" by J. K. Whittemore, 52–55; "The reversion of class number relations and the total representation of integers as sums of squares or triangular numbers" by E. T. Bell, 56–67; "Note on the term maximal subgroup" by G. A. Miller, 68–69; "Reducible cubic forms expressible rationally as determinants" by L. E. Dickson, 70–74; "Note on the Picard method of successive approximations" by D. Jackson, 75–77; "A fundamental system of covariants of the ternary cubic form" by L. E. Dickson, 78–82; "The modular theory of polyadic numbers" by A. A. Bennett, 83–90; "Some algebraic analogies in matric theory" by A. A. Bennett, 91–96; "Generalized conjugate matrices" by P. Franklin, 97–100.

Bulletin of the American Mathematical Society, volume 28, June, 1922: "The February meeting of the American Mathematical Society" by R. G. D. Richardson, 233–244; "A property of continuity" by D. C. Gillespie, 245–250; "Kinematics in a complex plane and some geometric applications" by A. Emch, 251–257; "Note on some results concerning Fermat's last theorem" by H. S. Vandiver, 258–260; "Convex distribution of the zeros of Sturm-Liouville functions" by E. Hille, 261–265; "Books on Fourier series" by C. N. Moore, 266–270 [Review of E. W. Hobson, *Theory of Functions of a Real Variable and the Theory of Fourier's Series* (vol. 1, Cambridge, 1921) and of H. S. Carslaw, *Introduction to the Theory of Fourier's Series and Integrals* (2d edition, London, 1921)]; Review by R. D. Carmichael of *Oeuvres de G. H. Halphen* (vol. 3, Paris, 1921), by A. R. Crathorne of J. Lipka, *Graphical and Mechanical Computation* (New York, 1918), by A. Dresden of K. Knopp, *Funktionentheorie* (2 Parts, Berlin and Leipzig, 1918–20), and by F. Cajori of L. Carnot, *Réflexions sur la Métaphysique du Calcul Infinitésimal* (vols. 1 and 2 Paris, 1921), 271–273; Notes, 274–277; New publications, 278–280—July, 1922: "The April meeting of the San Francisco Section" by B. A. Bernstein, 281–285; "The Easter meeting of the Society" by R. G. D. Richardson and A. Dresden, 285–302; "A report on the scientific work of the Chicago Section, 1897–1922" by A. Dresden, 303–307; "Eliakim Hastings Moore Fund" by A. Dresden, 307–309; "Note on the division of a plane by a point set" by E. W. Chittenden, 310–312; "Note on steady fluid motion" by S. D. Zeldin, 313–315; "Two books on analysis" by A. Dresden, 315–317 [Review of G. Vivanti, *Lezioni di Analisi Infinitesimale* (2d edition, Torino, 1920) and of E. Pascal, *Lezioni di Calcolo Infinitesimale* (Milano, Part I, 4th edition, 1919; Part II, 4th edition, 1918, Part III, 2d edition, 1918)]; Review by E. B. Stouffer of V. Kommerell and K. Kommerell, *Allgemeine Theorie der Raumkurven und Flächen* (vols. I and II, 3d edition, Berlin and Leipzig, 1921), by G. E. Wahlin of L. J. Mordell, *Three Lectures on Fermat's Last Theorem* (Cambridge, 1921), by H. Bateman of A. A. Robb, *The Absolute Relations of Time and Space* (Cambridge, 1921), by E. P. Adams of L. Rougier, *Philosophy and the New Physics* (Philadelphia, 1921), and by C. N. Reynolds, Jr., of K. M. Kohler, *Das Exzentrizitätsprinzip als Korrelat zur Relativitätstheorie* (Vienna, 1921), 317–319; Notes, 320–325; New publications, 325–328.

Bulletin des Sciences Mathématiques, volume 46, June, 1922: "L'œuvre mathématique de Georges Humbert, quelques mots sur Camille Jordan" by H. Lebesgue, 220–223 [Extrait de la leçon inaugurale de mathématiques donnée au Collège de France le 6 janvier 1922. "Lorsqu'en 1912, Georges Humbert fut nommé professeur de mathématiques au Collège de France, ses travaux nombreux, variés et importants, qui lui avaient valu d'être élu membre de l'Académie des Sciences en 1901, avaient attiré sur lui, depuis longtemps, l'attention des mathématiciens du monde entier. Après 1912, malgré une longue maladie, son activité scientifique ne s'est pas ralentie; dans les derniers mois de sa vie, il rédigeait encore un Mémoire qui ne fut publié qu'après sa mort, dans le premier fascicule du *Journal de Mathématiques* de 1921. Georges Humbert a publié plus de 140 Notes et Mémoires, souvent étendus. . . . Dans toute œuvre véritable se révèle une continuité de pensée qui permet de grouper tout naturellement les divers Mémoires autour de quelques idées directrices, de quelques préoccupations dominantes. Dans cette esquisse rapide, je me bornerai au groupement qui s'impose tout d'abord: recherches antérieures à 1898, relatives à l'étude des fonctions algébriques; recherches postérieures à 1898, relatives à la théorie et à l'utilisation des fonctions abéliennes singulières. . . . Ici, dans cette chaire où j'ai le redoutable honneur de lui succéder, Humbert fut un professeur particulièrement apprécié, dont les leçons étaient impatiemment attendues d'un auditoire toujours fidèle. A mon avis, rien ne montre mieux les rares qualités d'Humbert que ses succès au Collège de France, car la tâche d'un professeur y est si difficile que, pour ma part, j'ai toujours été tenté de la déclarer impossible. On n'a pas le droit de n'y faire qu'un enseignement classique, fût-il excellent; il y faut un enseignement toujours en progrès, toujours renouvelé, toujours original; par le fond, si possible, tout au moins quant au groupement des matières et à la compréhension du sujet. . . . Comment Humbert a-t-il réussi à se rapprocher assez de cet idéal pour contenter toujours ses auditeurs? Le règle de conduite

qu'il a adoptée est à la fois habile et modeste; elle a été, de plus, fort utile aux progrès des études mathématiques en France. Au lieu de compter, comme il en aurait eu le droit, sur l'originalité de son esprit et la rapidité de sa compréhension, il a préféré s'appuyer sur la solidité et la précision de ses connaissances pour tout ce qui touche au domaine algébrique. Dans ce domaine, il a su trouver des sujets de Cours précis et variés, et cela d'autant plus facilement qu'il a continué à travailler exclusivement dans ce demaine et qu'il était constamment au courant de tout ce qui se publiait le concernant. En faisant ce choix, Humbert ne risquait guère de voir son enseignement faire double emploi avec quelque autre; les sujets étudiés par Humbert sont, en effet, presque tous en marge des questions soulevées par les principaux courants de la pensée mathématique pendant les trente dernières années. . . . Puisque ma tâche consiste, aujourd'hui, à parler de la chaire de Mathématiques du Collège de France et de ceux qui l'ont occupée le plus brillamment, j'ai la joie de pouvoir rendre hommage a l'illustre doyen des mathématiciens français, à M. Camille Jordan . . . . Voici toute une série de problèmes dont la liaison avec certains de ceux étudiés par Humbert n'est pas trop lointaine; quelle différence cependant entre les méthodes de ces deux éminents mathématiciens. Humbert utilise partout et toujours la fonction analytique, algébrique même, qu'il réussit ingénieusement à faire intervenir dans bien des questions, même dans des questions arithmétiques; à cet égard, il s'apparente à Poincaré. M. Jordan, suivant la voie ouverte par Galois, traite de questions, relatives par leur énoncé même au calcul algébrique, sans le secours de l'appareil analytique, à l'aide de raisonnements presque synthétiques. Dans ces raisonnements on procède toujours, si l'on veut dire, à une analyse; mais l'instrument qu'on y emploie est sans cesse variable, on le construit, on le modifie à chaque instant, et c'est en ce sens que les raisonnements sont synthétiques. . . . Les excursions faites par M. Jordan hors des domaines de recherches traditionnels avaient d'ailleurs fait réfléchir bien des mathématiciens; ceux-ci ont donné à la jeune école des encouragements précieux qui ont largement compensé les quelques reproches qu'elle a dû subir. Reproches bien injustes d'ailleurs, car, loin de mépriser le calcul, nous sommes persuadés que nos procédés synthétiques actuels céderont quelque jour la place à des procédés meilleurs, parce que plus analytiques; nous ne prétendons qu'à être les précurseurs et les annonciateurs peut-être d'un Viète ou d'un Newton qui introduira un symbolisme nouveau ou des notions nouvelles, peut-être d'un Descartes ou d'un Cauchy qui utilisera un symbolisme déjà connu et qui l'élargira pour des fins nouvelles, et nous soupirons après le nouvel algorithme qui, entre les mains d'hommes habiles comme l'était Humbert, donnera facilement et elégamment plus que ce que nous n'obtenons que péniblement et lourdement."]

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City. [*1918*, 187; *1921*, 387.]

The following meetings were held in 1921–1922:

March 3, 1921: Reception to freshmen.

March 7: Business meeting.

March 21: "Graphical solution of problems" by Isabel Graves '23.

April 4: "Problems from Jones's *Mathematical Wrinkles*" by Monica Gilloran '21.

April 18: "Geometric forms in art and nature" (illustrated with lantern slides) by Professor Lao G. Simons.

May 19: Business meeting.

May 20: Election of officers, as follows: President, Sarah Karnis '22; vice-president, Edna Kramer '22; treasurer, Rose Charlon '22; publicity manager, Henrietta Olidort '23; secretary, Adele Matzke '22.

September 26: "Women mathematicians" by Adele Matzke '22; "Sonya Kovalevski" by Edna Kramer '22.

October 10–November 7: Mathematicians distinguished in other fields: "Omar Khayyam,

Cowley showed photostats of an Italian mathematical manuscript of the Middle Ages, containing three of these problems.

May 18: Indoor picnic. The following officers were elected for 1922-1923: President, Harvia Wilson '23; vice-president, Erneste Goodman '23; secretary-treasurer, Mary Hall '24; members of executive committee, Professor Cowley, Jeannette Kinne '23.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2986. Proposed by C. F. GUMMER, Queen's University.

A triangle is inscribed in a circle. The arcs into which it divides the circumference are bisected at points forming the vertices of a second triangle. A third triangle is derived in the same way from the second, and so on. Prove that each set of alternate triangles approaches a limiting position.

2987. Proposed by PHILIP FITCH, North Denver High School, Colorado.

A flexible chain of length l and uniform weight is fastened at one end to the ridge of a roof with pitch p and slant height L. If the eaves of the roof are at a height h from the ground and the coefficient of friction between the chain and the roof is μ, how long will it take, after releasing the chain, for the highest end to reach the ground?

2988. Proposed by PHILIP FRANKLIN, Harvard University.

Prove, geometrically, that if in an ellipse the tangent at P cuts the directrices in Z, Z' and the remaining tangents from Z and Z' to the ellipse meet at T, PT is normal to the ellipse at P. (An analytic proof is given in the *Journal of the Indian Mathematical Society*, vol. 13, 1921, p. 234.)

2989. Proposed by L. M. HOSKINS, Stanford University.

How should the following questions be answered, assuming that the place referred to is in latitude 34° 8′?

A building twelve feet high has been erected 49 inches south of our lot line. We desire to erect a wall on our line six inches in thickness. (*a*) How high can we build the wall and have it wholly within the shadow cast by the building? (*b*) How high can we build the wall and have it within the shadow cast by the building during the winter months?

2990. Proposed by R. M. MATHEWS, Wesleyan University.

If a circle be bitangent to a conic, its center is on one of the axes of the curve.

2991. Proposed by E. J. OGLESBY, New York University.

Sum the infinite series,

$$1+\frac{3x^2}{2!}+\frac{4x^4}{4!}+\frac{6x^6}{6!}+\cdots,$$

where the numerators of the coefficients form a series of numbers whose third differences are all equal to 2.

SOLUTIONS.

2894 [1921, 184]. Proposed by PHILIP FRANKLIN, Harvard University, AND E. L. POST, Columbia University.

Given the following set of assumptions concerning a set S and certain undefined sub-classes of S, called m-classes:

I. If A and B are distinct elements of S, there is at least one m-class containing both A and B.

II. If A and B are distinct elements of S, there is not more than one m-class containing both A and B.

Def. Two m-classes with no elements in common are called *conjugates*.

III. For every m-class there is at least one *conjugate m-class*.

IV. For every m-class there is not more than one conjugate m-class.

V. There exists at least one m-class.

VI. Every m-class contains at least one element of S.

VII. Every m-class contains not more than a finite number of elements.

Develop some of the propositions of the "mathematical science" (cf. Veblen and Young, *Projective Geometry*, vol. I, pp. 1 f.) based on them and in particular develop a sufficient number of theorems to prove that the set of assumptions is categorical and give a concrete representation of the set S which satisfies them. Also prove that the assumptions are independent.

SOLUTION BY THE PROPOSERS WITH IMPROVEMENTS SUGGESTED BY W. E. CLELAND, R. HARTSHORNE, and G. E. RAYNOR, Princeton University.

Examples to prove the independence of assumptions I to VII.

I. The elements $A\ B\ C\ D$ and the sets AB and CD.

II. The combinations of six letters taken three at a time.

III. The elements $A\ B$ and the set AB.

IV. The elements $A\ B\ C, X\ Y\ Z$ and the sets: $ABZ, ACY, BCX, AX, BY, CZ, XY, YZ, XZ$.

V. S is the null class, and there are no m-classes.

VI. S is the null class, and there are two null-m-classes.

VII. S contains an infinite number of elements, and there are an infinite number of m-classes formed as follows: Start with two m-classes $A_1\,A_2$ and $A_3\,A_4$. Form all the classes required by I. Form new classes $B_1\,B_2$, $B_3\,B_4$, which are to be conjugate to these last. Satisfy IV by inserting in the classes common elements, $C_1, C_2, \cdots$, never using the same element in more than one pair of classes. Form all the classes with these new elements required by I, and repeat the process.

THEOREM 1. *Every m-class contains at least two elements.*

For suppose one m-class contained a single element A. By III there would exist an m-class not containing A, and by VI it would contain an element B. By I there would be a class AB and by III, a conjugate class containing neither A nor B. Thus we would have two m-classes conjugate to the class with a single element A, contradicting IV.

THEOREM 2. *S contains at least four elements.*

By V there exists at least one m-class, which by theorem 1 contains at least two elements. By III there exists a conjugate m-class which also contains at least two elements.

THEOREM 3. *S contains at least six m-classes.*

The previous proof shows the existence of two conjugate m-classes, each containing at least two elements. Each of the elements of one of these m-classes, with one of the elements of the second of these classes, by I determines an m-class, and these m-classes, at least four in number, are all distinct, since if two coincided, by II it would coincide with one of our original classes, and these would not be conjugate.

THEOREM 4. *No m-class contains more than two elements.*

Let A and C be any two non-conjugate m-classes. By VII we may suppose that the number of elements in A, say p, is equal to or greater than the number of elements in C. Let the elements of A be $A_1, A_2, \cdots, A_p$, A_1 the element common to A and C.

Let $B_1, B_2, \cdots$, be the elements of B, the m-class conjugate to A, B_1 the element common to B and C. C then contains A_1 and B_1, and if there are any other elements in C we may call them $C_1, C_2, \cdots, C_k$, so that $k \leqq p - 2$.

Now if B_x is any one of the B's except B_1, the m-classes $B_xA_2, \cdots, B_xA_p$ will be distinct and no one of them can contain A_1 or B_1. Otherwise one of them would be the class A and contain B_x, or the class B and contain one of the A's, which is impossible, since A and B are conjugate

classes. One of them may be the conjugate of C, but each of the others, if $p > 2$, must contain one of the elements of C by IV, and that must be one of the C's and not A_1 or B_1. On the other hand, no two of them can contain the same C for they already have B_x in common. Therefore, there must be at least $p - 2$ of the C's, and, as k is not greater than $p - 2$, it follows that $k = p - 2$ and that C contains the same number of elements as A.

It follows also that one of these classes is the conjugate of C; that is, that the conjugate of C contains B_x, which is any one of the B's except B_1. This means that there can be only one B besides B_1, and that the class B can contain only two elements.

Now C was any class non-conjugate to A. Therefore, every class except B contains p elements while B contains only two. But if we start with some other class in place of A, we can prove that its conjugate contains two elements and that B contains p. This is possible only if $p = 2$ and every class contains only two elements.

THEOREM 5. *The elements of S and the m-classes are isomorphic with the elements A, B, C, D and the sets AB, AC, AD, BC, BD, CD.*

The four elements of theorem 2 may be labelled A, B, C, D and the six m-classes of theorem 3 the six of this proposition. By theorem 4 there are no more elements in any of these m-classes, and by II no more m-classes containing only these elements. Let X be an additional element. By I there is a class AX, and by theorem 4, it contains no more elements. We then have classes AX and AB both conjugate to CD, and since this violates IV it proves that there are no elements X. The six sets above evidently satisfy I to VII.

NOTE. In connection with the above solution Professor Veblen's comments on the origin and discussion of the problem will be of interest to the readers of the MONTHLY.

Professor VEBLEN says, "The problem originated in my course in Projective Geometry which I began, as usual, by a discussion of the abstract point of view in mathematics. In order to emphasize the point that our logical processes should be independent of any particular set of mental images or, indeed, of any knowledge of what the propositions are about, I proposed that certain members of the class should make up a set of postulates which would give the properties of some set of objects chosen but not divulged by them. The other students were then challenged to make logical deductions from the postulates and thereby deduce enough theorems to learn what the postulate makers had in mind.

"As a result of this suggestion, Mr. Post and Mr. Franklin brought forward a set of postulates which is essentially the one offered in the enclosed problem and a solution was found by several of the other students in precisely the manner indicated. One of the students indeed went further and pointed out that one of the postulates in the original set was redundant.

"The exercise was a success in showing how mathematical deductions can be made without knowledge as to what one is reasoning about. It also brought out vividly the problem of the significance of the logical processes as a method of discovery. While there is a sense in which it is true that you cannot get anything out of a set of postulates except what has been put in them, you can at least find out what was put in them. This is what the solver of this problem has to do in a simple case. In the more complicated case of ordinary geometry the student is apt to think that he understands what is in the axioms, but every time that he witnesses the derivation of a new theorem it turns out that there was something in them that he had not seen before.

"By a slight modification this problem can also be used to propose another problem which it seems to me may turn out to be an important one. Let us replace Assumption VII by the assumption that no m-class contains more than 6 elements. The mathematical science based on the assumptions could then be built up without ever counting beyond, let us say, 24. The question arises, how much of logic is needed to develop so limited a mathematical science? Would it be possible to single out a subset of the postulates of logic which would suffice for the purpose? If so, what processes of logic can be omitted? What sort of a logic results if the omitted processes are replaced by others? If it should turn out that the logic required for a satisfying theory of this finite system (or any other partciular system as, for example, a particular finite projective space) stops short of that required for larger systems, we would be in the presence of a criterion for the classification of logical processes which might help toward deciding the question as to what logical processes are legitimate in dealing with various types of infinite sets."

2899 [1921, 228]. Proposed by NORMAN ANNING, University of Michigan.

A, B, C, and P are any four coplanar points. P describes a sextant about A when the line AP turns about A through $+60°$. Show that P moves in a closed curve when it describes sextants in succession *either* about $A, B, A, B, A, \cdots$ *or* about $A, B, C, A, B, C, \cdots$.

I. Solution by C. F. Gummer, Queen's University.

These are particular cases of a more general theorem. Consider n coplanar points $A_1, A_2, \cdots, A_n$; and let a point in the plane, starting at P, revolve successively about $A_1, A_2, \cdots, A_n, A_1, \cdots$ through angles $\phi_1, \phi_2, \cdots, \phi_n, \phi_1, \cdots$, until all the points $A_1, \cdots, A_n$ have been used m times each. The resultant displacement from P after the first set of rotations about $A_1, A_2, \cdots, A_n$ is the sum of a number of vectors, namely PA_1, A_1A_n and those obtained by turning A_nA_{n-1} through ϕ_n, $A_{n-1}A_{n-2}$ through $\phi_{n-1} + \phi_n$, $\cdots$ A_2A_1 through $\phi_2 + \phi_3 + \cdots \phi_n$ and finally A_1P through $\phi_1 + \phi_2 + \cdots \phi_n$. If Φ denotes the sum of the n angles, the vector sum after m sets of rotations contains two vectors of length PA_1 inclined at an angle $\pi + m\Phi$, and it contains m vectors of length A_iA_{i-1} (including the case of A_1A_n) inclined successively at the angle Φ so that they may be regarded as a series of equal chords placed end to end in a circle of suitable size. This is true for each value of i from 2 to n and for A_1A_n.

Three cases occur:

(1) $\Phi/(2\pi)$ *is rational but not integral.* A value of m other than 1 may then be found so that $m\Phi$ is a multiple of 2π. It follows that after m sets of rotations the two vectors derived from PA_1 are equal and opposite, and the vectors derived from A_iA_{i-1} are the sides of a closed regular polygon (possibly interlacing); therefore the resultant displacement vanishes, and the point has moved in a closed curve. The problems proposed belong to this case.

(2) $\Phi/(2\pi)$ *is irrational.* No group of vectors can be made to have a zero sum; and it will be found that the path cannot be closed (except for special positions of P). The path however lies in a finite region, since each group of vectors has a sum not greater than the diameter of its corresponding circle; and it may be shown that the path returns to positions indefinitely near to the initial point.

(3) $\Phi/(2\pi)$ *is an integer.* On taking $m = 1$, A_1P and PA_1 again destroy one another; but the remaining vectors, being now in groups of one, have not generally a zero sum. The resultant transformation is a translation of the entire plane, which by repetition (unless it happens to vanish) carries every point to infinity.

II. Solution by A. A. Bennett, University of Texas.

The problem will be treated in the more general case as follows: Given m and n two positive integers each greater than unity. Let $A_1, A_2, \cdots A_m, B_1$ be any $m + 1$ coplanar points. Let P describe a curve starting from B_1, made up of circular arcs, each of which is a one-(mn)th part of a circumference but with various radii as follows: With center A_1 describe one-(mn)th part of a circumference, positively from B_1, terminating at B_2. With center, A_2, describe one-(mn)th part of a circumference positively from B_2, terminating at B_3. Continue cyclically taking as the $(m + 1)$st center A_1, $(m + 2)$nd center A_2, etc. Show that B_{mn+1} coincides with B_1.

Let us use vector methods and denote the positive turn of one-(mn)th part of a circumference by the operator T. Then $T^{mn} = 1$. We shall then have the following relations:[1]

$$\begin{aligned} B_2 - A_1 &= T(B_1 - A_1), \\ B_3 - A_2 &= T(B_2 - A_2), \\ &\cdots\cdots \\ B_{k+1} - A_k &= T(B_k - A_k), \\ &\cdots\cdots \end{aligned}$$

Collecting terms,

$$B_{m+1} = T^mB_1 + (1 - T)(T^{m-1}A_1 + T^{m-2}A_2 + \cdots + TA_{m-1} + A_m)$$

and finally,

$$B_{mn+1} = T^{mn}B_1 + (1 + T^m + T^{2m} + \cdots + T^{(n-1)m})(1 - T)(T^{m-1}A_1 + T^{m-2}A_2 + \cdots + TA_{m-1} + A_m).$$

Since $T^m \neq 1$, and $(T^m - 1)(1 + T^m + T^{2m} + \cdots + T^{(n-1)m}) = T^{mn} - 1 = 0$, it follows that

[1] In this notation we may regard the difference of two points as a vector and the sum of a point and a vector as a point after the manner of Grassmann (see E. W. Hyde, *The Directional Calculus*, Boston, 1890, p. 2), but when we apply the distributive law to T, writing $T(B - A)$, for example, as $TB - TA$, we must understand A and B to represent vectors drawn from some arbitrary point O; that is, we may say that TB and TA stand for $T(B - O)$ and $T(A - O)$.

the operator $(1 + T^m + T^{2m} + \cdots + T^{(n-1)m})$ is a null-operator, so that

$$B_{mn+1} = T^{mn}B_1 = B_1$$

as desired.

It is of interest to carry out the work for an infinite number of points.

Let $x(t)$, $y(t)$ be the coördinates of a closed curve as the parameter t ranges from zero to 2π and be periodic with period 2π. Let $X(t)$ and $Y(t)$ be the coördinates of a derived curve obtained by a limiting process from the above discussion. Let us describe an infinitesimal circular arc with $x + \Delta t\cdot x'$, and $y + \Delta t\cdot y'$ as coördinates of the center, starting from the point with coördinates $X(t)$ and $Y(t)$, and described positively with an angle equal to the circumference divided by $n/(\Delta t/2\pi)$. The terminal point of the arc will be denoted by $(X + \Delta t\cdot X', Y + \Delta t\cdot Y')$. Since the arc is circular, we shall have

$$[(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')]^2 + [(X + \Delta t\cdot X') - (x + \Delta t\cdot x')]^2 = [Y - (y + \Delta t\cdot y')]^2 + [X - (x + \Delta t\cdot x')]^2. \tag{1}$$

The slope of the initial position of the radius is $[Y - (y + \Delta t\cdot y')]/[X - (x + \Delta t\cdot x')]$, and of the terminal position is $[(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')]/[(X + \Delta t\cdot X') - (x + \Delta t\cdot x')]$. The tangent of the angle of rotation is to be equal to $\tan(\Delta t/n)$; thus,

$$\frac{\dfrac{(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')}{(X + \Delta t\cdot X') - (x + \Delta t\cdot x')} - \dfrac{Y - (y + \Delta t\cdot y')}{X - (x + \Delta t\cdot x')}}{1 + \dfrac{(Y + \Delta t\cdot Y') - (y + \Delta t\cdot y')}{(X + \Delta t\cdot X') - (x + \Delta t\cdot x')}\cdot\dfrac{Y - (y + \Delta t\cdot y')}{X - (x + \Delta t\cdot x')}} = \tan\left(\frac{\Delta t}{n}\right). \tag{2}$$

Simplifying and dropping higher powers of Δt, we have from (2)

$$\frac{\dfrac{Y-y}{X-x}\left[1 + \Delta t\left(\dfrac{Y'-y'}{Y-y} - \dfrac{X'-x'}{X-x}\right)\right] - \dfrac{Y-y}{X-x}\left[1 - \Delta t\left(\dfrac{y'}{Y-y} - \dfrac{x'}{X-x}\right)\right]}{1 + \left(\dfrac{Y-y}{X-x}\right)^2} = \frac{\Delta t}{n}$$

or

$$Y'(X - x) - X'(Y - y) = [(Y - y)^2 + (X - x)^2]/n. \tag{3}$$

From (1), we have similarly,

$$Y'(Y - y) + X'(X - x) = 0; \tag{4}$$

whence,

$$-nX' = Y - y, \qquad nY' = X - x. \tag{5}$$

Eliminating between these we have the following pair of equations, to determine $X(t)$ and $Y(t)$, in terms of $x(t)$ and $y(t)$:

$$n^2X'' + X = x + ny', \qquad n^2Y'' + Y = y - nx'. \tag{6}$$

These are to be taken subject to the initial condition that when $t = 0$, $X(0)$ and $Y(0)$ have assigned values, and, from (5)

$$-nX'(0) = Y(0) - y(0), \qquad ny'(0) = X(0) - x(0). \tag{7}$$

The solutions are therefore determined.

Since $x(t)$ and $y(t)$ are periodic of period 2π, and since the solution of the homogeneous equations, $n^2X'' + X = 0$, $n^2Y'' + Y = 0$, are periodic of period $2n\pi$, it follows for this example that X and Y have each the period $2n\pi$. Thus for this "infinite" case also the set of derived points closes after the original set is described cyclically n times. It is to be noted that for each new choice of a variable t, the given curve whose parametric equations are $x = x(t)$, $y = y(t)$, is regarded as the limit of a new finite set of points.

NOTE BY THE EDITORS.—Professor Gummer's solution differs from the first part of Professor Bennett's only in that the rotations are different, making his solution more general.

These solutions may also be expressed in terms of complex quantities without any introduction of the notion of vectors or any operator T.

Also solved by T. M. BLAKSLEE and F. L. WILMER. Professor Blakslee sent in four different solutions, one of which was the generalized solution.

2893 [1921, 184]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Find the locus of the mid-point of the segment determined by two given skew lines in a variable plane turning about a fixed axis, not coplanar with either of the given lines.

SOLUTION BY J. K. WHITTEMORE, Yale University.

Let the axis of the variable plane be the z axis, and suppose the x and y axes chosen parallel one to each of the given skew lines. This coördinate system is generally oblique. The equations of the skew lines are (1) $y = b_1$, $z = c_1$; (2) $x = a_2$, $z = c_2$; the variable plane, $y = \lambda x$, intersects the two lines in points of coördinates, (1) $x = b_1/\lambda$, $y = b_1$, $z = c_1$; (2) $x = a_2$, $y = \lambda a_2$, $z = c_2$. The coördinates of the mid-point of the segment, a point of the required locus, are

$$x = \frac{a_2}{2} + \frac{b_1}{2\lambda}, \qquad y = \frac{b_1}{2} + \frac{\lambda a_2}{2}, \qquad z = \frac{c_1 + c_2}{2}.$$

Eliminating the parameter λ, the equations of the required locus are

$$\left(x - \frac{a_2}{2}\right)\left(y - \frac{b_1}{2}\right) = \frac{a_2 b_1}{4}, \qquad z = \frac{c_1 + c_2}{2}.$$

The locus is a hyperbola lying in a plane parallel to both given skew lines and half way between them; its asymptotes are parallel to the given skew lines; its center is the point of intersection of the diagonals of the parallelopiped, three of whose edges lie on the axis of the variable plane and the given skew lines; it intersects the axis of the variable plane.

Also solved by WILLIAM HOOVER.

2900 [1921, 277]. Proposed by I. A. BARNETT, University of Saskatchewan.

AB is the diameter of a circle and Q_0 any point on the circumference; $Q_1, Q_2, Q_3, \cdots$, are the points of bisection of the arcs $AQ_0, AQ_1, AQ_2, \cdots$; to prove that the product of the chords of the circle $BQ_1, BQ_2, BQ_3, \cdots, BQ_n$ is equal to $OA^n \cdot (AQ_0/AQ_n)$, O being the center of the circle.

SOLUTION BY A. M. HARDING, University of Arkansas.

Arc AQ_1 = arc Q_1Q_0. Hence, $\angle OBQ_1 = \angle AQ_0Q_1 = \angle Q_1AQ_0$. Hence, the isosceles triangles, OBQ_1 and Q_1Q_0A are similar. Then,

$$\frac{BQ_1}{AQ_0} = \frac{OB}{AQ_1}; \qquad \text{that is,} \qquad BQ_1 = OA \cdot \frac{AQ_0}{AQ_1}.$$

In a similar manner, it may be shown that

$$BQ_2 = OA \cdot \frac{AQ_1}{AQ_2},$$
$$BQ_3 = OA \cdot \frac{AQ_2}{AQ_3},$$
$$\vdots$$
$$BQ_n = OA \cdot \frac{AQ_{n-1}}{AQ_n}.$$

Multiplying these equations gives

$$BQ_1 \cdot BQ_2 \cdot BQ_3 \cdot \cdots BQ_n = OA^n \cdot \frac{AQ_0}{AQ_n}.$$

Also solved by T. M. BLAKSLEE, ARTHUR PELLETIER, and A. V. RICHARDSON.

2945 [1922, 29]. Proposed by T. M. BLAKSLEE, Ames, Iowa.

A point P in the plane of the triangle ABC rotates in a given direction around the vertices taken in either cyclical order, in each case through an angle equal to the corresponding angle of the triangle. That is, for example, AP rotates around A through an angle equal to the angle A of the triangle; then BP around B through an angle equal to the angle B, and so on. Prove

that P coincides with its original position at the end of six of these rotations. (See problem 2899, *1921*, 228.)

This is completely solved in Professor Gummer's solution of 2899 given above.

Also solved by J. B. REYNOLDS and the PROPOSER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

R. M. DEMING, instructor of mathematics at Case School of Applied Science, has been appointed professor and head of the mathematics department at Upper Iowa University, Fayette, Iowa.

Mrs. W. C. GRAUSTEIN née CURTIS (*1921*, 334), formerly assistant professor of mathematics at Wellesley College (*1920*, 382), is again to become a member of the department there for 1923–1924.

GIUSEPPE LAIS, vice-director of the Vatican observatory since 1889, died December 26, 1921. He was born at Rome, April 15, 1845. Most of his meteorological, astronomical, and astrographical publications are to be found in *Nuovi Lincei, Atti*, Rome, from 1875 on. A biographical sketch and portrait are given in *Archivio di Storia della Scienza*, May, 1922.

ORAZIO TEDONE, born at Ruvo di Puglia, Italy, May 10, 1870, was killed in a railway accident April 14, 1922. He received his doctorate from the University of Pisa in 1892 and was a correspondent of the Academy of the Lincei. In 1899 he was appointed professor of higher analysis in the University of Genoa; in 1903 his title was changed to that of professor of rational mechanics. To volume IV–4 of the *Encyklopädie der mathematischen Wissenschaften* he contributed (1906): "Allgemeine Theoreme der mathematischen Elastizitätslehre (Integrationstheorie)," pages 55–124; and "Spezielle Ausführungen zur Statik elastischer Körper," in collaboration with A. Timpe, pages 125–214.

ALFONSO DEL RE, professor of descriptive geometry and design in the University of Naples since 1899, died September, 1921. He was born at Calitri, Italy, October 9, 1859. In 1886 he received his doctorate at the University of Naples and was there an assistant in projective geometry, 1886–1889. Among his books are the following: *Lezioni di algebra della logica ad uso degli studenti delle facoltà di matematica e di filosofia e lettere* (Naples, 1907); *Geometria proiettiva ed analitica, Lezioni* (22 + 394 pages, Modena, 1900; 984 pages, Rome, 1900). He was a collaborating editor of *Giornale di Matematiche*, 1910–1913, and in this periodical, as well as in *Rendiconti del Circolo Matematico di Palermo*, and publications of the Modena, Naples and Rome academies, many of his papers are to be found.

DÉSIRÉ ERNEST LEBON, honorary professor at Lycée Charlemagne, Paris, died February 12, 1922. He was born at Audigny, France, August 25, 1846. He was professor of mathematics, or of descriptive geometry, in various lycées

from 1873 to 1898, the year of his appointment as professor of descriptive geometry at Lycée Charlemagne. He is the author of a score of books and many articles. Half of his books deal with descriptive geometry, and other topics in French secondary schools. In America he is probably best known by his *Histoire abrégée de l'Astronomie* (Paris, 1899), by his 7 volumes in *Savants du Jour* series (Poincaré, Picard, Darboux, Appell, Lippmann, Gautier, and Haller, Paris, 1909–1913) and by the first fascicule of his last work, *Table de Caractéristiques de base 30030* ··· (Paris, 1920), reviewed by D. N. Lehmer in an interesting manner for *Bulletin of the American Mathematical Society*, 1921, volume 27, pp. 225–227. He was chief editor, during the 8 years of its existence, of *Bulletin Scientifique*, Paris, 1886–1894, a periodical devoted almost wholly to mathematics.

ALFRED BRAY KEMPE, born in London, England, July 6, 1849, died April 21, 1922. He graduated as twenty-second wrangler at Cambridge in 1872, became a barrister at the Inner Temple in 1873 and bencher in 1909. He was president of the London Mathematical Society 1893–1894. He was elected a fellow of the Royal Society in 1887 (vice-president, 1902), and created a knight in 1921. He published a number of mathematical papers 1872–1895 and his lecture on linkages[1] was reprinted in book form with the title, *How to Draw a Straight Line* (1877). In a paper of 1876 he proved that a linkwork can be found to describe any algebraic curve.[2] Sir Alfred's paper published in the *American Journal of Mathematics*, 1879, was accepted for eleven years as containing a proof (Heawood discovered the flaw) of the famous map-coloring problem[3] mentioned by Möbius in lectures of 1840. In the same *Journal*, for 1913, G. D. Birkhoff made a notable contribution in the discussion of the problem.

HEINRICH SUTER, professor of mathematics in the Gymnasium of Zürich since 1886, died March 17, 1922. He was born at Hedingen, near Zürich, January 4, 1848. After studying at the Universities of Berlin and Zürich he received his doctorate from the latter in 1872. His dissertation was the first edition of the first part of his *Geschichte der mathematischen Wissenschaften*, namely, *Von den æltesten Zeiten bis Ende des XVI. Jahrhunderts*, the second edition of which was published at Zürich in 1873. The first edition of the second part, *Von Anfange des XVII. bis gegen das Ende des XVIII. Jahrhunderts*, appeared in 1875. A Hungarian translation of the first part, by Sandor, was published at Budapest in 1874; and a Russian translation by Manujlow, at Kischinew in 1876. His *Die Mathematik auf den Universitäten des Mittelalters* appeared in a Programm of

[1] For a bibliography of this subject, 1796–1882, see *Bulletin des Sciences Mathématiques et Astronomiques*, vol. 18, 1883, pp. 145–160. Among the articles in this MONTHLY dealing with linkages are the following: "On kinematic geometry—a new inversor" by J. J. Quinn, *1905*, 105–106; "The solution of an equation by a frame" by T. M. Blakslee, *1911*, 159–162; "A cardiograph" by C. M. Hebbert, *1915*, 12–13; "Linkages" by D. H. Leavens, *1915*, 330–334.

[2] The mechanical production of rectilinear motion, aside from its practical importance, has exercised a peculiar fascination on the minds of geometers, and the allied theory of three-bar curves is still being investigated. For an interesting popular account of the theory of linkages see a paper by F. V. Morley in *The Scientific Monthly*, vol. 9, 1919, p. 366. Mr. Morley has an analytic treatment of the problem in *Proceedings of the London Mathematical Society*, June 28, 1922.

[3] See *Encyklopädie der mathematischen Wissenschaften*, III-1-1, 1907, pp. 177–178.

the Kantonschule, Zürich, 1887, and in "Festschrift der Kantonschule zur 39. Versammlung Deutscher Philologen und Schulmänner" published at Zürich in the same year. His most important publications are probably: *Mathematiker-Verzeichnis im Fihrist des Ibn Abî Jakûb an-Nadîm* (Leipzig, 1892), and *Die Mathematiker und Astronomen der Araber und ihre Werke* (Leipzig, 1900–1902, 8 + 278 + 8 + 337 pages) which appeared in volumes 6, 10 and 14 of *Abhandlungen zur Geschichte der Mathematik.* His most recent large work was: *Al-Kwarizmi, Mohammed Ibn Musa. Die astronomischen Tafeln in der Bearbeitung des Maslama Ibn Ahmed Al-Madjzetti und der lateinischen Uebersetzung des Athelhard von Bath auf Grund der Vorarbeiten von A. Björnbo und R. Besthorn. Herausgegeben und kommentiert von H. Suter* (Copenhagen, 1914; 284 pages). He was also an extensive contributor to *Bibliotheca Mathematica* since 1889, and to *Zeitschrift für Mathematik und Physik* since 1884. A tiny portrait of Suter is given on page 17 of Eneström's Bibliotheca Mathematica, General Register, 1887–1896, 1897.

It has been the custom at the United States Military Academy (West Point) and the United States Naval Academy (Annapolis) to have the major part of the instruction committed to officers of the military and naval services, respectively. The study of military tactics has been left to army officers and naval manœuvers as a subject for instruction has never been in the hands of civilians. As a matter of fact, instruction has frequently been delegated to recent graduates of the respective academies, to young men of excellent habits and high ideals but with no pedagogical experience and with neither taste nor opportunity for scholarly research. For many years past the Naval Academy has made an exception of the courses in mathematics which constitute an important share of the midshipman's curriculum, and there has been a large corps of expert civilian instructors in mathematics at Annapolis. Recently a move has been made which threatens to result in the wholesale dismissal of civilians from the teaching staff of the Academy, the future instruction in calculus, mechanics, probability, and other technical mathematical topics to fall to such naval officers as may be spared for this work. It is to be hoped that the future defenders of our shores may not be robbed of the best scientific training available as an outcome of politics or prejudice, whatever decision is made.

Professor ARCHIBALD HENDERSON, of the University of North Carolina, gave a paper on "How the Einstein theory of relativity was verified" before the physics section of the North Carolina Academy of Science at Chapel Hill on May 5 and 6. He was elected president of the Academy for the coming year.

Attention is called to the fact that the date of the Ohio Section meeting should be March 30–31, instead of March 29–30, as announced in the preliminary notice.

Published February 26, 1923.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the...

....Fund, and to be used

for { Endowment—the income only of which may be expended.
Special Projects—for which both principal and income may be expended.

Witness: Signature...

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Seventh Summer Meeting of the Association, University of Rochester, September 6–7, 1922
Seventh Annual Meeting, Harvard University, December 28, 29, 1922

The following are dates of Section meetings of the Association in 1922 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29
IOWA, Des Moines, November 3; Cornell College, Mount Vernon, April 27-28, 1923
KANSAS, Topeka, January 21; January 20, 1923
KENTUCKY, Georgetown College, April 8; University of Kentucky, April, 1923
MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Washington, December 9; Baltimore, May, 1923
MINNESOTA, St. Paul, June 4, 1921; St. Paul, May 27
MISSOURI, Kansas City Junior College, November 18; University of Missouri, Columbia, November 30–December 1, 1923
OHIO, Columbus, Apr. 14–15; Mar. 30–31, 1923
ROCKY MOUNTAIN, Greeley, Colo., April 14–15; University of Colorado, April, 1923
SOUTHEASTERN, Atlanta, Ga., April 29; Agnes Scott College, Decatur, Ga., March 10, 1923
TEXAS, Dallas, November 25, 1921; Houston, December 1–2

AMERICAN MATHEMATICAL MONTHLY

SUPPLEMENT, MAY-JUNE, 1923.

INDEX TO VOLUME XXIX, 1922.

By MILDRED E. CARLEN, Brown University.

Misprinted names in the text are corrected, and missing initials supplied, in this index. The authors of certain anonymous contributions are also indicated.

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QUESTIONS AND DISCUSSIONS—QUESTIONS.

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PROBLEMS—SOLUTIONS.

Numbers in black-face type refer to problems, those in light-face to pages.

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PERSONAL MENTION—NECROLOGY.

ADDENDA AND CORRIGENDA.

P. 27, l. 27, *for* "Crowly" *read* "Crowley."
P. 29, problem **2949** corrected on p. 420.
P. 42, l. 24, *for* "H" *read* "S."
P. 58, l. 3 from bottom, *for* "k'" *read* "k."
P. 78, l. 22, *for* "Harshberger" *read* "Harshbarger."
P. 92, ll. 3–4 from bottom, *for* "R. V. RICHARDSON" *read* "A. V. RICHARDSON."
P. 94, l. 24, *for* "SPENCELY" *read* "SPENCELEY."
P. 94, l. 2 from bottom, *for* "BRAZLER" *read* "BREZLER."
P. 129, l. 21 from bottom, *for* "J. W. M." *read* "J. H. M."
P. 135, l. 6 from bottom, *for* "had" *read* "have."
P. 173, l. 18, *for* "**Toscane**" *read* "**Toscana.**"
P. 177, l. 10, *for* "Housen" *read* "Emily E. Howsen."
P. 228, l. 20, *for* "length s" *read* "lengths."
P. 232, l. 21, *for* "Reichenan" *read* "Reichenau."
P. 233, l. 12, *for* "NÖTHER" *read* "NOETHER."
P. 266, l. 2, *for* "*Kongelige*" *read* "*Koneglige.*"
P. 274, l. 17, *for* "$\sqrt{(2a^2 - r^2) - r}$" *read* "$\sqrt{(2a^2 - r^2)} - r$."
P. 275, l. 16, *for* "l_2" *read* "l_3."
P. 279, l. 19, *for* "van Rhijh" *read* "VAN RHIJN."
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P. 407, l. 12 from bottom, *for* "roots"." *read* "roots."
P. 407, l. 11 from bottom, *for* "$- 9r^2$.)" *read* "$- 9r^2$.)"."
P. 417, l. 4, *for* "Leypolt" *read* "Leypoldt."
P. 452, l. 15, *delete* "the last."
P. 498, l. 34, after Moore, E. H., *delete* "276."
P. 498, l. 35, after Moore, R. L., *add* "276."
P. 499, l. 19, after Schottenfels, Ida M., *for* "107" *read* "100."
P. 499, l. 20, after Schreiber, E. W., *for* "100, 100" *read* "100, 107."

Published July 31, 1923.

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The REGISTER OF OFFICERS AND MEMBERS of the Association issued as Number 6, June–July, was sent only to those specially subscribing for it. The paging of the MONTHLY for August continues from that for May, so that the 1922 volume will be complete aside from the REGISTER. The INDEX to this volume will be published later as a supplement and will be sent out with one of the early numbers of the 1923 volume.

RATES OF EXCHANGE.

By HUBERT E. BRAY, The Rice Institute, Houston, Texas.

The general subject of rates of exchange is, of course, a complicated one. In this paper an elementary discussion of the subject is presented, following Cournot,[1] under the special hypothesis that there is no flow of gold between the various markets involved. Thus the rates of exchange are determined by a system of simultaneous linear homogeneous equations in n unknowns, the ratios of which are the rates of exchange between the n markets. The matrix of this system is of a special type which possesses interesting properties.

1. Preliminary Example. For the sake of clearness consider first a special case in which only three financial markets are involved, *e.g.* New York, London, Paris. Suppose that these form a closed system, *i.e.*, that they have dealings only with each other. We will suppose also that transactions are carried on by exchanges of credit, *i.e.*, that there is no flow of gold from one market to another.

Let the markets in question be indicated by subscripts 1, 2, 3, respectively, and suppose that

New York possesses m_{12} pounds of credit at London,
New York possesses m_{13} francs of credit at Paris,
London possesses m_{21} dollars of credit at New York,
etc.

Suppose also that

One dollar at New York is equal in value to C_{12} pounds at London,
One dollar at New York is equal in value to C_{13} francs at Paris,
etc.

The quantities C_{12}, C_{13}, C_{23}, etc., are called *rates of exchange*. Since there is no flow of gold between markets, we must have:

$$\begin{aligned} m_{21} + m_{31} &= m_{12}C_{21} + m_{13}C_{31}, \\ m_{12} + m_{32} &= m_{21}C_{12} + m_{23}C_{32}, \\ m_{13} + m_{23} &= m_{31}C_{13} + m_{32}C_{23}. \end{aligned} \qquad (A)$$

The first equation states that the number of dollars of credit which London and Paris together hold at New York is equal to the combined value in dollars at New York of the number of pounds of credit at London and the number of francs of credit at Paris held by New York. The other equations have a similar meaning.

Since a dollar at New York is worth C_{12} pounds of credit at London, and a

[1] Augustin Cournot, *The Mathematical Principles of the Theory of Wealth*, chapter III, New York, 1897. The equations, which are due to Cournot, were discussed in a course given by Professor Evans at the Rice Institute.

pound at London is worth C_{23} francs at Paris, it follows that

$$C_{12}C_{23} = C_{13}.$$

Together with equations (A) the rates of exchange therefore satisfy the equations

$$C_{ij}C_{jk} = C_{ik}, \qquad C_{ii} = 1. \qquad i, j, k = 1, 2, 3. \quad (B)$$

To solve equations (A) and (B) we multiply the first of (A) by C_{11} ($= 1$), the second by C_{21}, the third by C_{31}, thus converting the sums involved into dollars. We get

$$\begin{aligned} (m_{21} + m_{31})C_{11} - m_{12}C_{21} - m_{13}C_{31} &= 0, \\ - m_{21}C_{11} + (m_{12} + m_{32})C_{21} - m_{23}C_{31} &= 0, \\ - m_{31}C_{11} - m_{32}C_{21} + (m_{13} + m_{23})C_{31} &= 0. \end{aligned} \qquad (C)$$

Since the sum of the left-hand members is equal to zero, these equations are linearly dependent. On solving the last two equations we get

$$C_{21} = \frac{\begin{vmatrix} m_{21} & -m_{23} \\ m_{31} & m_{13} + m_{23} \end{vmatrix}}{\begin{vmatrix} m_{12} + m_{32} & -m_{23} \\ -m_{32} & m_{13} + m_{23} \end{vmatrix}} = \frac{\begin{vmatrix} m_{21} + m_{31} & -m_{13} \\ -m_{31} & m_{13} + m_{23} \end{vmatrix}}{\begin{vmatrix} m_{12} + m_{32} & -m_{23} \\ -m_{32} & m_{13} + m_{23} \end{vmatrix}}.$$

Similarly

$$C_{31} = \frac{\begin{vmatrix} m_{21} + m_{31} & -m_{12} \\ -m_{21} & m_{12} + m_{32} \end{vmatrix}}{\begin{vmatrix} m_{12} + m_{32} & -m_{23} \\ -m_{32} & m_{13} + m_{23} \end{vmatrix}}.$$

In general, provided none of the two-rowed determinants involved is equal to zero, the solution of equations (C), satisfying (A) and (B), is

$$C_{ij} = \frac{D_i}{D_j}, \qquad i, j = 1, 2, 3,$$

where D_i is the determinant obtained by striking out the ith row and column of the matrix

$$(\mathbf{m}) = \begin{Vmatrix} m_{21} + m_{31} & -m_{12} & -m_{13} \\ -m_{21} & m_{12} + m_{32} & -m_{23} \\ -m_{31} & -m_{32} & m_{13} + m_{23} \end{Vmatrix}.$$

Since the m's are essentially non-negative, the C_{ij}'s are all positive, for

$$\begin{aligned} D_1 &= m_{12}m_{13} + m_{12}m_{23} + m_{13}m_{32}, \\ D_2 &= m_{23}m_{21} + m_{23}m_{31} + m_{21}m_{13}, \\ D_3 &= m_{31}m_{32} + m_{31}m_{12} + m_{32}m_{21}. \end{aligned}$$

What happens if the rank of ($\mathbf{m}$), which in general is two, is reduced to unity?

If we require that none of the quantities C_{ij} be equal to zero, we find that one of the following three cases must subsist:

$$
\begin{array}{ll}
(a) & m_{12} = m_{13} = m_{21} = m_{31} = 0, \\
(b) & m_{23} = m_{21} = m_{32} = m_{12} = 0, \\
(c) & m_{31} = m_{32} = m_{13} = m_{23} = 0.
\end{array}
$$

In case (a) the three equations (C) amount to the single equation:

$$m_{32}C_{21} - m_{23}C_{31} = 0,$$

i.e.,

$$C_{23} = \frac{C_{21}}{C_{31}} = \frac{m_{23}}{m_{32}}.$$

If m_{23} and m_{32} are both different from zero, the rates of exchange between London and Paris are uniquely determined. These two markets have no dealings with New York. Their rates of exchange with New York are indeterminate. The situation here disclosed, which in the case of only three markets is perhaps trivial, is a special example of a more interesting fact in the case of n markets. In fact, it will be proved in general that:

If the rank of the matrix $(\mathbf{m})$ *is* r, *and none of the rates of exchange is zero, then the* n *markets are divided up into* $n - r$ *separate groups, each group consisting of a certain number of markets which have dealings with each other but not with any market in any other group. The rate of exchange between any two markets of the same group is uniquely determined. Between two markets of different groups the rate of exchange is indeterminate.*

Let us consider the general case.

2. Let there be n markets, $M_1, M_2, \cdots, M_n$, and suppose that

$$M_i \text{ holds } m_{ij} \text{ } j\text{-units of credit at } M_j,$$

and that

$$\text{one } i\text{-unit at } M_i \text{ is equal in value to } C_{ij} \text{ } j\text{-units at } M_j.$$

The quantity C_{ij} is called the *rate of exchange at* M_i *on* M_j. As before, we suppose that business between the markets is in equilibrium, *i.e.*, there is no flow of gold between them. The following equations are therefore true:

$$
\begin{array}{l}
m_{21}+m_{31}+\cdots+m_{n1}= m_{12}C_{21}+m_{13}C_{31}+\cdots+m_{1n}C_{n1}, \\
m_{12} +m_{32}+\cdots+m_{n2}=m_{21}C_{12} +m_{23}C_{32}+\cdots+m_{2n}C_{n2}, \\
m_{13}+m_{23} +\cdots+m_{n3}=m_{31}C_{13}+m_{32}C_{23} +\cdots+m_{3n}C_{n3}, \qquad (1) \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
m_{1n}+m_{2n}+\cdots+m_{n-1,n} =m_{n1}C_{1n}+m_{n2}C_{2n}+\cdots+m_{n,n-1}C_{n-1,n}.
\end{array}
$$

$$C_{ij}C_{jk} = C_{ik}, \qquad i, j, k = 1, 2, \cdots, n. \qquad (2)$$

It should be noted that equations (2) are not independent; also that they contain the equations $C_{ii}^2 = C_{ii}$, which will be interpreted as $C_{ii} = 1$.

If we multiply the first equation of (1) by C_{1p}, the second by C_{2p}, etc., thus

converting the amounts of money into currency units of M_p, and if we make use of the relations $C_{ij}C_{jp} = C_{ip}$, we obtain the set of n homogeneous equations:

$$\begin{array}{l} \sum_i m_{i1}C_{1p} - m_{12}C_{2p} - m_{13}C_{3p} - \cdots - m_{1n}C_{np} = 0, \\ - m_{21}C_{1p} + \sum_i m_{i2}C_{2p} - m_{23}C_{3p} - \cdots - m_{2n}C_{np} = 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ - m_{n1}C_{1p} - m_{n2}C_{2p} - m_{n3}C_{3p} - \cdots + \sum_i m_{in}C_{np} = 0. \end{array} \tag{3}$$

Here p is one of the numbers $1, 2, \cdots, n$ and $m_{ii} = 0$ is inserted to simplify the notation. Our object is to solve equations (1) in conjunction with (2) and to discuss the effect of the rank of the matrix of (3) upon the solutions.

3. We consider matrices of two kinds, namely:

$$(\mathbf{a}) = \begin{Vmatrix} \sum_i a_{i1} & - a_{12} & - a_{13} & \cdots & - a_{1n} \\ - a_{21} & \sum_i a_{i2} & - a_{23} & \cdots & - a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ - a_{n1} & - a_{n2} & - a_{n3} & \cdots & \sum_i a_{in} \end{Vmatrix},$$

$$(\mathbf{b}) = \begin{Vmatrix} \sum_i a_{i1} + b_1 & - a_{12} & \cdots & - a_{1n} \\ - a_{21} & \sum_i a_{i2} + b_2 & \cdots & - a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ - a_{n1} & - a_{n2} & \cdots & \sum_i a_{in} + b_n \end{Vmatrix},$$

in which all of the numbers a_{ij}, b_i are non-negative and $a_{ii} = 0$. The matrix (**a**) is a special case of the matrix (**b**). In general we shall refer to a determinant (matrix) in which each element in the principal diagonal is either greater than or equal to the sum of the absolute values of the other elements in its column, the latter being non-positive numbers, as a *determinant* (*matrix*) of *type* (**b**). The special case in which each principal element is *equal* to the sum of the absolute values of the other elements in its column will be called a *determinant* (*matrix*) of *type* (**a**). The determinant (**a**) is equal to zero.

THEOREM 1. *The value of the determinant* (**b**) *is not negative.*

For the proof by induction, first note that the theorem is evident when (**b**) is of order 1 or 2. If we expand (**b**) as a polynomial in the b_i's, we find that the constant term is precisely the determinant (**a**); the coefficient of b_k is the determinant found by omitting the kth row and column of (**a**)—this is a determinant of type (**b**) of order $n - 1$; similarly the coefficient of $b_k b_l$ is a determinant of type (**b**) of order $n - 2$; in fact the coefficient in every term of the polynomial is a determinant of type (**b**) of order less than n, except the coefficient of $b_1 b_2 \cdots b_n$, which is unity, and the constant term which is the determinant (**a**) and is equal to zero. Thus if the theorem is true when the order is less than n, it is true when the order is equal to n.

THEOREM 2. *If the determinant* (**b**) *is zero, it contains a principal minor of type* (**a**).

If all of the b's are zero, (**b**) itself is of type (**a**). If not all of the b's are zero, the coefficient of any product of non-vanishing b's must be zero, since there are no negative terms in the expansion of (**b**) as a polynomial in the b's. In particular, the product of all the b's must be zero and therefore at least one b is zero. Since the coefficient of any product of b's is a determinant of type (**b**) of lower order than the given determinant, the theorem follows by induction.

If a determinant, of type (**b**) or (**a**), of order n contains no principal minor of type (**a**) of order less than n, it is said to be a *simple* determinant of type (**b**) or (**a**). Two simple minors of type (**a**) are quite separate.

If (**b**) is equal to zero, we can now arrange its rows and columns in a specially convenient order. Let $\alpha_1, \alpha_2, \cdots \alpha_s$ be the simple principal minors of type (**a**) which it contains and let their orders be $k_1, k_2, \cdots k_s$. By shifting rows and columns similarly α_1 can be made to lie in the first k_1 rows and columns, α_2 in the next k_2 rows and columns and so forth. The elements of the last l rows and columns, $l = n - (k_1 + k_2 + \cdots + k_s)$, will form a non-vanishing determinant β of type (**b**). When the rows and columns of (**b**) are arranged in this manner —or when the variables and equations (3) are correspondingly arranged—the arrangement is said to be *compact.*

The following theorems are now easily proved:

THEOREM 3. *If a certain principal minor α of order k, is equal to zero, then every minor of order k taken from the same k columns which contain α is equal to zero.*[1]

THEOREM 4. *Every $(n - 1)$-rowed principal minor of a simple determinant of type* (**a**) *is different from zero.*

COROLLARY. *The rank of a simple determinant of type* (**a**) *is $n - 1$.*

THEOREM 5. *The rank of* (**b**) *is $n - s$, s being the number of simple principal minors of type* (**a**) *which* (**b**) *contains.*

THEOREM 6. *The cofactor of every element of the jth column of the determinant* (**a**) *is equal to D_j, the determinant obtained by striking out the jth row and column.*

4. We can now apply the results of **3** to the solution of equations (1) and (2). The following notation is used:

$$(\mathbf{m}) = \left\| \begin{array}{cccc} \sum_{i=1}^{n} m_{i1} & -m_{12} & \cdots & -m_{1n} \\ -m_{21} & \sum_{i=1}^{n} m_{i2} & \cdots & -m_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ -m_{n1} & -m_{n2} & \cdots & \sum_{i=1}^{n} m_{in} \end{array} \right\| .$$

This matrix is supposed to be compactly arranged. Its determinant is equal

[1] If a principle minor of order h of (b) is itself of type (a), then all the b's in that minor are zero and all the elements outside of that minor in its h columns are zero, for we have conditions of the form

$$\sum_{i=1}^{n} a_{ij} + b_j = \sum_{i=1}^{h} a_{ij}, \qquad \text{whence} \qquad \sum_{i=h+1}^{n} a_{ij} + b_j = 0,$$

and since there are no negative terms in this polynomial each term must be zero separately.

to zero; $s =$ the number of simple principal minors, $\alpha_1, \alpha_2, \cdots \alpha_s$, of type (**a**). By theorem 5 the rank of (**m**) is $n - s$; $k_1, k_2, \cdots k_s$ are the respective orders of $\alpha_1, \alpha_2, \cdots \alpha_s$; $l =$ the order of the minor determinant β, of type (**b**), lying in the last l rows and columns; $\beta \neq 0$; $k = k_1 + k_2 + \cdots + k_s$; $n = k + l$.

There are three cases to consider:

Case I. (**m**) is of rank $n - 1$, $s = 1$, $k = n$, $l = 0$. (**m**) is simple.

Case II. (**m**) is of rank $n - s$, $s > 1$, $k = n$, $l = 0$. (**m**) contains s simple principal minors of type (**a**) which contain all of the non-zero elements.

Case III. (**m**) is of rank $n - s$, $s \geqq 1$, $k < n$, $l > 0$. There is a principal minor determinant β, lying in the last l rows and columns, of type (**b**) and positive.

Case I. Recalling that $C_{pp} = 1$ ($p = 1, 2, \cdots, n$), we see that equations (3) have one and only one solution, since the rank of (**m**) is $n - 1$. This solution is:

$$C_{ip} = \frac{D_i}{D_p} > 0,$$

where D_i is the cofactor of every element of the ith column, by theorem 6. This formula gives the only solution of (1) and (2).

Case II. In this case equations (1) consist of s separate sets, each of which is of the kind considered in Case I. In fact, by using equations (2) and multiplying the k_1 equations of the first set in order by the numbers

$$C_{1p_1}, C_{2p_1}, \cdots, C_{k_1p_1},$$

where p_1 is some one of the numbers $1, 2, \cdots, k_1$, we obtain a set of equations of the form (3) of the type discussed in Case I, the unknowns being

$$C_{1p_1}, C_{2p_1}, \cdots, C_{k_1p_1}.$$

Similarly the second set of k_2 equations is reduced to the form (3), the unknowns being

$$C_{k_1+1, p_2}, C_{k_1+2, p_2}, \cdots, C_{k_1+k_2, p_2},$$

where p_2 is one of the numbers $k_1 + 1$, $k_1 + 2$, $\cdots$, $k_1 + k_2$. The remaining sets are treated in the same way. The s sets of equations, having matrices of ranks $k_1 - 1$, $k_2 - 1$, $\cdots$, $k_s - 1$, can be solved uniquely for the quantities C_{ij} when i, j correspond to two markets of the same set. But it is evident that the C_{ij}'s are indeterminate when i, j correspond to markets of different sets.

Case III. In this case it is impossible to obtain a solution which contains no zeros. For, consider the last l equations of (1). If we multiply them in order by the quantities

$$C_{k+1,p}, C_{k+2,p}, \cdots, C_{n,p},$$

we obtain a set of homogeneous equations in these quantities. The determinant

β of the set is different from zero. It follows, therefore, that they are satisfied only by

$$C_{k+1,p} = C_{k+2,p} = \cdots = C_{n,p} = 0;$$

and this result holds when $p = 1, 2, \cdots, k$. If now we substitute these zeros in the first k equations (1), we obtain a set of k equations of the type considered in Case II. We see therefore that it is possible to find values of C_{ij} $[i = 1, 2, \cdots, n;$ $j = 1, 2, \cdots, k]$ which satisfy the first k equations of (1) together with those of equations (2) which do not involve the subscripts $k + 1, k + 2, \cdots, n$. The last l equations (1) can not be satisfied in conjunction with those of equations (2) which involve these subscripts.

From the point of view of economics this result means that there are s separate sets of solvent markets (corresponding to the s simple matrices of type (**a**)) such that no two markets of different sets have any dealings with each other. The rate of exchange between two such markets is indeterminate, but between two markets of the same set it is determined uniquely as in Case II. The l markets not included in these s sets are bankrupt; whereas the credit of each of them at any solvent market is zero, some of them have debit balances with at least one of the solvent markets.

A SIMPLE THEORY OF COMPETITION.

By GRIFFITH C. EVANS, The Rice Institute, Houston, Texas.

1. Postulates. Significant theories of economics may be discussed mathematically in very simple terms, and results may be stated, in precise fashion, which a non-mathematical theory could not be expected to deduce at all. Moreover the assumptions are simple enough to be verified approximately in actual societies, and yield results which may be easily calculated in definite cases. Such a rudimentary theory of competition is given below, in terms of linear and quadratic functions, and the elements of the calculus.

Let

$$q(u) = Au^2 + Bu + C \tag{1}$$

represent, as a quadratic function, the total cost of producing (that is, putting on the market) u units of a commodity in unit time. Let

$$y = ap + b \tag{2}$$

represent, as a linear function, the amount of goods, y, which, if the price is p, will be bought in the market in unit time.

Suppose there are two producers, each manufacturing, subject to the same cost function given by (1), amounts u_1 and u_2, respectively, in unit time, and each trying to make his profit a maximum. What will be the amounts u_1 and u_2 manufactured, and the price p at which the goods are sold? This is the problem of competition expressed in simplest terms.

If we denote the respective profits by π_1 and π_2, we shall have, in a situation where as much is sold as is produced, the equations

$$\pi_i = pu_i - q(u_i), \qquad i = 1, 2, \tag{3}$$
$$y = u_1 + u_2 = ap + b, \tag{4}$$

since pu_i is the selling value of the amount u_i. It remains however to add a determining postulate for the stationa y values of the quantities π_i.

1.1. As a possible determining postulate we may take that of Cournot:[1]

(*a*) *Each competitor assumes that the production of the other or others is independent of his, and tries to make his profit a maximum.*

The mathematical expression of this postulate is to regard u_1 and u_2 as independent variables, and write

$$\frac{\partial \pi_1}{\partial u_1} = 0, \qquad \frac{\partial \pi_2}{\partial u_2} = 0, \tag{5}$$

that is,

$$0 = p + u_i \frac{\partial p}{\partial u_i} - q'(u_i), \qquad i = 1, 2,$$

and since, by (4), $\partial p / \partial u_i = 1/a$,

$$0 = p + \frac{u_i}{a} - 2Au_i - B, \qquad i = 1, 2. \tag{5'}$$

If from these equations we eliminate u_1 and u_2 by adding them together and then making use of (4), we shall have a single equation which we may solve immediately for p and obtain

$$p = \frac{b - 2Aab - 2Ba}{- a(3 - 2Aa)}, \qquad u_1 = u_2 = \frac{b + Ba}{3 - 2Aa}. \tag{6}$$

1.2. It may be noticed that the postulate just used is not entirely equivalent to this other:

(*c*) *Each producer tries to determine the amount u_i of his production per unit time so as to make the total profit a maximum.*

The total profit is given by

$$\pi = (u_1 + u_2)p - q(u_1) - q(u_2),$$

and the condition which expresses this postulate is

$$0 = \frac{\partial \pi}{\partial u_1} = \frac{\partial \pi}{\partial u_2}, \tag{7}$$

or

$$\begin{aligned} 0 &= p + (u_1 + u_2)\frac{\partial p}{\partial u_i} - 2Au_i - B \\ &= p + \frac{ap + b}{a} - 2Au_i - B, \end{aligned}$$

[1] Augustin Cournot, *Recherches sur les principes mathématiques de la théorie des richesses*, Paris, 1838; translated by N. T. Bacon, *Researches into the Mathematical Principles of the Theory of Wealth*, London, 1897; see chapter VII.

so that, in the same way as in the previous case, the u_1 and u_2 may be eliminated by means of (4) and the value of p obtained:

$$p = \frac{2b - 2Aab - 2Ba}{-a(4 - 2Aa)}, \qquad u_1 = u_2 = \frac{b + Ba}{4 - 2Aa}. \tag{8}$$

This latter phenomenon we may define as *coöperation.*

1.3. An instructive standard of comparison is the monopoly price. For this, the natural postulate was given by Cournot:[1]

(*m*) *The producer determines his production per unit time so as to make his profit a maximum.*

In this case

$$\pi = pu - q(u),$$

and the condition is simply

$$\frac{d\pi}{du} = 0,$$

that is,

$$0 = p + \frac{u}{a} - 2Au - B.$$

Hence

$$p = \frac{b - 2Aab - Ba}{-a(2 - 2Aa)}, \qquad u = \frac{b + Ba}{2 - 2Aa}. \tag{9}$$

1.4. There is another situation which it is worth while to describe; and it is a second kind of competition. Each competitor may by slightly changing the price—say by underselling the other—try to obtain whatever portion of the trade he can handle with the maximum return or profit. That is, he regards the price as fixed—say, just less than the market price—and produces the amount which would give him the maximum profit at that price. For this "cut-throat" competition we have therefore the following postulate:

(*b*) *Each competitor regards the price as fixed and tries to make his profit a maximum.*

In this case we have the equation

$$\left(\frac{\partial \pi_1}{\partial u_1}\right)_{p \text{ constant}} = 0, \tag{10}$$

and, on account of the symmetry of the situation,

$$\left(\frac{\partial \pi_2}{\partial u_2}\right)_{p \text{ constant}} = 0$$

whence

$$0 = p - q'(u_1) = p - q'(u_2),$$
$$p = 2Au_1 + B = 2Au_2 + B.$$

[1] Cournot, *l. c.*, chapter V.

With reference to (4) then we have

$$p = \frac{Ab + B}{1 - Aa}, \qquad u_1 = u_2 = \frac{b + Ba}{2 - 2Aa}. \tag{11}$$

2. Phenomena with n producers. Let us denote the prices given by the postulates (a), (c), (m) and (b) respectively by letters p with a corresponding index. It is instructve to write down the formulæ for $p^{(a)}$, $p^{(c)}$ and $p^{(b)}$ when there are n producers instead of two. With reference to the postulates given in these cases, it is easily calculated that their values are the following:

$$p^{(a)} = \frac{b - 2Aab - nBa}{-a(n + 1 - 2Aa)} \quad \text{with} \quad u_i^{(a)} = \frac{b + Ba}{n + 1 - 2Aa}, \tag{12}$$

$$p^{(c)} = \frac{nb - 2Aab - nBa}{-a(2n - 2Aa)} \quad \text{with} \quad u_i^{(c)} = \frac{b + Ba}{2n - 2Aa}, \tag{13}$$

$$p^{(b)} = \frac{2Ab + nB}{n - 2Aa} \quad \text{with} \quad u_i^{(b)} = \frac{b + Ba}{n - 2Aa}, \tag{14}$$

and if we let n become large, we have the approximate formulæ

$$p^{(a)} = p^{(b)} = B \quad \text{with} \quad u_i^{(a)} = u_i^{(b)} = \frac{b + Ba}{n}, \tag{15}$$

$$p^{(c)} = \frac{1}{2}\left(B - \frac{b}{a}\right) \quad \text{with} \quad u_i^{(c)} = \frac{b + Ba}{2n}, \tag{16}$$

so that the total production in the case of coöperation would be about half of what it would be in either case of competition.

3. Nature of coefficients. In regard to the nature of the quantities which appear in all these formulæ, and which are the coefficients in the functions representing q and y, it may be pointed out that in the obviously typical case we have

$$a < 0, \qquad b > 0.$$

If, as has been done here, for $q(x)$ we regard the quadratic expression as an approximate representation of the cost function for all values of u, we may regard the A, B, C, determined statistically for this purpose, as all positive. In fact, C represents the cost of producing nothing per unit time, that is, the overhead expense, B represents the additional cost of producing the first unit, and the fact that A is positive represents the assumption that on the whole the cost curve is concave upward.

Since B is the cost of producing the first unit, and since both A and C are > 0, the average cost per unit of any quantity u, which is q/u, must be $\geqq B$. Hence no industry will be started unless the demand y is positive for $p = B$; and therefore we assume, with reference to (2), $b + Ba > 0$. In this connection, notice equations (11) to (16), and (21) and (23′) below.

These hypotheses would not be legitimate if the quadratic expression were regarded as an approximation to the cost curve over an interval somewhat short. Thus, if the cost curve, more accurately represented, happened to be concave

downward in its first part, and concave upward in the second part, we should be able for the latter portion, which is the interesting region for the variable, to assume a quadratic expression in which A would be positive but not necessarily B and C. These possibilities are interesting when we wish to get some idea of the nature of taxation and its incidence, a subject in which generally our interest is confined to a small portion of the cost curve. But for a rough approximation to the phenomena under investigation we may return to our original assumptions, with $A, B, C > 0$. It is justifiable also to assume all our producers working with the same cost curve, since we wish to obtain not an idea of the effect of a divergence of cost curves, but a rough notion of the phenomenon of competition itself.

There are one or two special cases which we may note in passing. If we put $n = 1$, the prices $p^{(a)}$, $p^{(c)}$ become identical with the $p^{(m)}$ for monopoly. This is not the case however with $p^{(b)}$ which reduces to

$$p^{(b)} = \frac{2Ab + B}{1 - 2Aa}, \qquad \text{with} \qquad u^{(b)} = \frac{b + Ba}{1 - 2Aa}\,. \tag{17}$$

In fact, since $-2Aa > 0$, the u for (17) is greater than the u for monopoly given by (9), and accordingly $p^{(b)} < p^{(m)}$.

Another situation to note is where the demand y is independent of the price, *i.e.*, where $a = 0$. In this case

$$p^{(m)} = p^{(c)} = p^{(a)} = \infty, \tag{18}$$

or in other words, the prices are pushed up beyond the region in which the hypothesis of demand independent of price remains valid, not only for monopoly and coöperation, but also for competition as described by Cournot. On the other hand, for $p^{(b)}$ we get the value

$$p^{(b)} = 2A\frac{b}{n} + B, \qquad \text{with} \qquad u_i^{(b)} = \frac{b}{n}\,. \tag{18$'$}$$

The two kinds of competition represent therefore quite distinct situations.

Some essential differences are brought out if we consider an industry where aside from overhead cost most of the cost is labor—a situation which we may characterize by writing $A = 0$. We obtain

$$\begin{aligned} p^{(a)} &= \frac{n}{n+1}B - \frac{b}{(n+1)a}, \\ p^{(c)} &= p^{(m)} = \frac{1}{2}B - \frac{b}{2a}, \\ p^{(b)} &= B. \end{aligned}$$

4. Relative values. There are simple inequalities that hold in general for the various kinds of prices. In fact, from (12), (13), (14) it follows obviously that

$$u_i^{(c)} < u_i^{(a)} < u_i^{(b)},$$

for any particular value of n, $n > 1$. Hence, since $y = \Sigma u_i = nu_i$,

$$y^{(c)} < y^{(a)} < y^{(b)},$$

and since $y = ap + b$,

$$p^{(c)} > p^{(a)} > p^{(b)}. \tag{19}$$

Moreover from (13),

$$y^{(c)} = nu_i^{(c)} = \frac{b + Ba}{2 - \frac{2Aa}{n}}.$$

But the quantity $-Aa$ being positive,

$$\frac{b + Ba}{2 - \frac{2Aa}{n}} > \frac{b + Ba}{2 - 2Aa},$$

that is,

$$y^{(c)} > y^{(m)},$$

and therefore,

$$p^{(c)} < p^{(m)}. \tag{19'}$$

Hence, finally, for any value of n, $n > 1$, the prices are arranged in descending order, as follows: monopoly, coöperation, competition (a), and competition (b). The total productions are arranged in the reverse order. It is possible, but hardly worth while for the present paper, to get simple formulæ for the direct measure of these differences.

5. Failure. If the price falls below a certain value, any producer will fail to make a profit; and if it falls still lower, since $C \neq 0$, the producer will lose at a definite rate. It remains to find this critical price.

The line $q = pu_i$ cuts the curve $q = Au_i^2 + Bu_i + C$ in two points, real, coincident or imaginary. In the first case only is it possible to obtain values of u_i for which $pu_i > q(u_i)$, *i.e.*, for which there shall be a profit. The two points are determined by the condition

$$0 = Au_i^2 + (B - p)u_i + C,$$

and will therefore be coincident when

$$(B - p)^2 = 4AC,$$

that is, when

$$p = B + 2\sqrt{AC},$$

the plus sign being taken with the radical, since the other value of p would correspond to a negative value of u. This is deduced immediately from the shape of the curve for q.

The condition $p^{(b)} \geqq p$ is equivalent to the condition

$$nB + 2Ab \geqq (n - 2Aa)(B + 2\sqrt{AC}),$$

from which a necessary condition for profit, under postulate (b), follows:

$$n < \sqrt{\frac{A}{C}}(b + Ba + 2a\sqrt{AC}). \tag{21}$$

This, it turns out, is under (*b*) also a sufficient condition, for the value of u_i is determined by the equation $p = q'(u_i)$, and the value of u_i which corresponds to p by this equation must by the law of the mean lie between the two intersections which the line $q = pu_i$ makes with the graph of q, when these are real. We have here then a definite finite limit on the number of competitors who may safely engage in a given production. Similar limitations apply to the cases governed by the other postulates (*a*) and (*c*); in all such limitations the importance of overhead expense is manifest, as in (21).

6. Offer. Equation (17) represents a sort of monopoly price. If there is a single producer, and if for each price he would produce the amount which would give him the maximum profit at that price, then the price in the market and the amount of goods produced and sold in unit time would be given by (17). In this case then (which hardly corresponds to any true monopoly) the producer regards himself as having a definite offer of the commodity for each price p, and the price in the market is obtained by observing the intersection of the curves of offer and demand. In a similar fashion, in the case (*b*) in general, each producer regards his individual offer as a function of the price, and the total offer is again defined. Offer is the amount of the commodity which would, if the price were p, be placed on the market in unit time.

It is most important to notice that in the general cases (*a*), (*c*), (*m*) offer in this sense has no relation to the problem, and the offer-demand diagrams lead to wrong results. In fact, in the problems (*m*), (*c*) and even (*a*) (unless n is so large that the quantity u_i/a in (5′) may be neglected, and (*a*) and (*b*) become thereby identical), the quantity u_i cannot be obtained as a function of p without introducing information (in $\partial p/\partial u_i$) derived from the demand curve. Thus offer as ordinarily interpreted loses its significance. On the other hand, any other definition of offer would be artificial, and the calculation of a new curve to fit into the offer-demand diagram, in order to give the proper intersection, would involve knowing in advance the solution of the problem.

6.1. One sees in this way that there may not be much sacredness in the "law of supply and demand," as an objection to the fixing of prices. The effect of price fixing is to make the situation entirely a type (*b*) problem, and thus to create an offer of goods in the technical sense, as we have just defined it, and the offer so created may be actually greater than the amount of production would be, which would give the maximum profit under less restricted conditions. It is in this way that the equations (17) become of practical interest, as portraying the case (*b*) when there is only one producer.

As an instance, suppose, let us say, that under war-time conditions the quantities q and y are given by functions (1) and (2), with certain values for the coefficients, and it is required to find what price $p_0 < p^{(m)}$ may be imposed on a monopoly product, without changing the amount produced. If the price is fixed at p_0, the relation of u to p_0 is fixed by postulate (*b*), so that u is given by the equation

$$p_0 = 2Au + B, \qquad \text{whence} \qquad u = \frac{p_0 - B}{2A}. \tag{22}$$

From this equation and (9) we have

$$\frac{p_0 - B}{2A} = \frac{b + Ba}{2 - 2Aa},$$

so that, finally,

$$p_0 = \frac{B + Ab}{1 - Aa}. \tag{23}$$

If the price were not fixed, it would become $p^{(m)}$, given by (9), and the relation between the two prices is therefore the following:

$$p^{(m)} - p_0 = \frac{b + Ba}{-a(2 - 2Aa)}, \tag{23'}$$

in which, in fact, the right-hand member denotes a positive quantity.

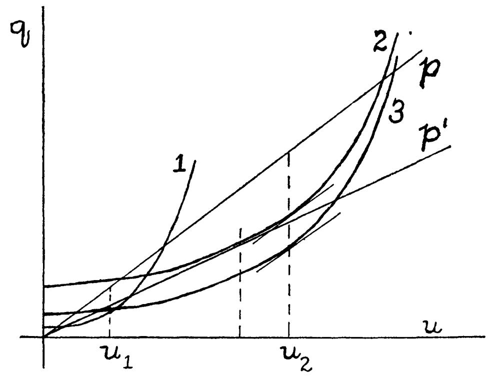
6.2. Before completing this subject, let us consider very briefly a related one. Is it possible to tax an industry in such a way that prices will not be changed? We notice in fact that a change in C has no effect on the price, so that either a tax in the shape of a fine of so many dollars a year, or a bounty of so many dollars a year will leave the price and the amount produced unchanged. In general, if we denote by $\bar{\pi}(u)$ the profit under the modified conditions, it is sufficient for the price not to change that we have

$$\bar{\pi}_i(u_i) = \alpha\pi_i(u_i) + \beta_i,$$

where α and β_i are constants, for in this case $\bar{\pi}_i'(u_i)$ and $\pi_i'(u_i)$ vanish for the same value of u_i. In other words, any procedure which takes away a definite proportion of the profit, or a definite amount, per unit time, or both, will not modify the price. We must guard against saying that an income tax will not modify prices, for an income tax affects a considerable proportion of incomes, and will thus presumably affect the a and b of the demand function for practically all commodities. An excess profit tax, however, is not open to this qualification, and satisfies the above conditions.

One may fix the price against a monopoly by (23), and then fine it by excess profit taxes, charter taxes, confiscation of dividends, and so on, however unjust the process may be, without diminishing the rate of production in the slightest degree. And thus, in connection with (23), it may be that the price p_0 will cause a loss rather than a profit, and this can only be met by passing dividends, or even going into a receiver's hands; yet the production will not diminish. On the other hand, if the price is fixed at less than p_0, the production will diminish even if there is still a profit, and even if the cost curve is modified by passing dividends and failing to pay interest on bonds. For these are modifications of the C in the expression for $q(u)$ and have no effect on the maxima and minima involved. In exceptional times it may be quite desirable by fixing prices at restrictive values to decrease the production in certain industries, say luxuries, and increase competition in basic industries and new necessities, with the object of keeping the price system as a whole fairly stable.

7. Modifications in the cost curve. Rather than consider the total variety of cost curves among producers, let us consider a type of modification which is characteristic in our problem. When prices are in a certain condition, under postulate (b), with say n competitors producing at a profit, some of the producers may wish to make a larger profit by increasing their overhead expenses, decreasing their other expenses, and producing larger amounts. In the diagram, the curves (1) and (2) represent an individual cost curve in these two states, the amount which would be produced in each case being that which would make the distance between the line $q = pu$ (which represents the selling value) and the cost curve a maximum. This is the position in which the tangent to the cost curve is parallel to the price line $q = pu$. The transition contemplated is from (1) to (2).



In (b), as we have seen, an offer curve is defined, so that offer is given as a function of price. The transition from (1) to (2), however, changes this function, and as is evident from the remark just made, will increase the offer at the price p. Since $a < 0$, this will decrease the price, and if the price goes below a certain amount p', indicated on the diagram by the line $q = p'u$, there will no longer be any profit in the expanded business. The change from (1) to (2) is unfortunately usually not a reversible process. And the usual modification of the cost curve which is now enforced is from (2) to (3), by omitting to pay dividends, and by such other changes in the value of C as ultimately provoke the bondholders to demand a receivership. Thus even apart from any phenomenon connected with the rate of interest,[1] bankruptcy may be regarded as a normal event in a system of competition.

8. General points of view. In a general system of economics we cannot for a proper discussion restrict ourselves to functions of a single variable. Mathematically this is not so essential a modification as it has been sometimes regarded. An extension of the problem which goes deeper is what is obtained when we remember that what a producer is interested in is not to make his momentary profit a maximum, but his total profit over a period of time of considerable extent, with reference to cost functions which are themselves changing as a whole with respect to time, such as in the instance discussed in the previous section. The mathematical discipline which enables us to find functions which make a maximum or a minimum quantities which depend upon them throughout periods of time is the calculus of functionals, or in special cases the calculus of variations. But the quantity which we want to make a maximum over a period of time

[1] Irving Fisher, *The Purchasing Power of Money*, New York, 1913; see chapter IV.

need not be the total profit; it may be the total production, or whatever other quantity we wish to take as a desirable characteristic of the social system we discuss. The author regrets that at the present time he can refer only to his lecture courses for a further treatment of this point of view. Nevertheless it seems the most fruitful way that a really theoretical economics may be developed.

ON KELLOGG'S DIOPHANTINE PROBLEM.[1]

By D. R. CURTISS, Northwestern University.

1. Kellogg's Problem. Two Applications. In a recent number of the MONTHLY,[2] Professor Kellogg has presented a very interesting discussion of the Diophantine equation

$$\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}=1, \tag{1}$$

in which he gives reasons for believing that the maximum value of any of the unknowns that can occur in a solution in positive integers is u_n, where

$$u_1=1, \qquad u_{k+1}=u_k(u_k+1). \tag{2}$$

Thus the successive u's are 1, 2, 6, 42, 1806, $\cdots$. I propose here to give a proof of the correctness of this statement, a proof in which we use sequences of inequalities, each containing one less x than the preceding. The method may be of some interest in itself, and of some value in similar problems. That the proof is hardly so simple as the statement of the problem will cause no surprise, at least to one familiar with Diophantine analysis.

Before we take up this proof it may add some interest to note two problems, one geometrical, the other arithmetical, whose solution depends on finding particular sets of integers satisfying (1). The first is that of laying non-overlapping sets of floor-tiles, each tile being a regular polygon whose sides are of unit length, so as to cover the plane, or a portion of the plane, just once or multiply; the polygons of a set will not, in general, all have the same number of sides. For example, suppose we are to fit n such tiles against each other so that all shall have a common vertex, and the piece of surface formed by them shall wind k times about this vertex, the last tile being in such a position as to fit without overlapping against the first tile if it were in the first layer instead of the last. In other words, the tiles are to generate without overlapping a piece of a Riemann surface in which the sheets form one cycle about a branch point. If the n tiles

[1] Read before the American Mathematical Society, December, 1921.

[2] *1921*, 300–303. See references there to Carmichael's *Diophantine Analysis* and to his review of Dickson's *History of the Theory of Numbers*, vol. 2, in this MONTHLY. This subject is very close to that of Sylvester's paper, "On a point in the theory of vulgar fractions," *American Journal of Mathematics*, vol. 3, 1880, pp. 332–335 and 388–389, which, Sylvester says, was suggested by the account in Cantor's *Geschichte der Mathematik* of the ancient Egyptian treatment of fractions by resolution into a sum of fractions each having unity as its numerator.

are regular polygons of $m_1, m_2, \cdots, m_n$ sides, respectively, so that their interior angles are

$$\pi - \frac{2\pi}{m_1}, \quad \pi - \frac{2\pi}{m_2}, \quad \cdots, \quad \pi - \frac{2\pi}{m_n},$$

the sum of these angles must be $2k\pi$. From this we deduce the relation

$$\frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_n} = \frac{n}{2} - k.$$

If n is even we divide through by $(n/2) - k$ and are thus led to seek solutions of (1) in which the x's have a common factor $(n/2) - k$. If n is odd we may add $\frac{1}{2}$ to both sides of the above equation and reduce to form (1) by dividing through by $((n+1)/2) - k$. Another reduction to (1) will suggest itself if we require all the m's to be even. We are thus led to consider solutions of (1) under certain restrictions.

The other problem is that of finding perfect numbers, a positive integer being defined as perfect if it is equal to the sum of all its different divisors less than itself, including unity. Thus 6 is a perfect number since $6 = 1 + 2 + 3$. For even perfect numbers we have a formula ascribed to Euclid, but it is not known whether an odd perfect number exists.[1] Let a_n be a perfect number, so that

$$a_n = 1 + a_1 + a_2 + \cdots + a_{n-1},$$

where $1, a_1, a_2, \cdots, a_{n-1}$ are all the divisors of a_n, not including a_n itself. The quotient of a_n by any a is an a, so, if we divide the above equation by a_n and rearrange, we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n} = 1.$$

Since the largest a is a_n itself, the correctness of Kellogg's statement for (1) has this consequence:

A perfect number with n divisors less than itself (unity included) cannot be greater than the number u_n defined by (2).

This result may be of some use in discussing the existence of an odd perfect number. We may lower this upper bound for an odd perfect number by trying to find the maximum x in a solution of (1) where all the x's are required to be odd and unequal.

Both these applications suggest the discussion of solutions of (1) under restrictions, and especially the determination of the maximum x under such added conditions. Problems of this type present difficulties, and I shall not attempt their discussion; perhaps some reader will be interested to try his fortune in this field.

2. A Theorem which Includes Kellogg's. Let us return now to the problem we have undertaken to solve. Where no ambiguity is thereby introduced we

[1] Compare this MONTHLY, *1921*, 140–141.

shall find it convenient to use the symbol $f_r(x)$ defined by

$$\frac{1}{f_r(x)} = 1 - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_r}. \tag{3}$$

With this notation, (1) can be put in the form

$$x_n = f_{n-1}(x).$$

The result which we are to prove, and which includes Kellogg's theorem, we will state as follows:

THEOREM I. *The maximum finite value of $f_{n-1}(x)$ for all positive integral values of $x_1, x_2, \cdots, x_{n-1}$ is u_n as defined by (2). There is but one set of x's which gives this maximum value, namely that in which*

$$x_k = u_k + 1, \qquad k = 1, 2, \cdots, n-1.$$

Note that we are not here restricting ourselves (as in Kellogg's formulation) to values of the x's that make $f_{n-1}(x)$ integral; we find, however, that the maximum of this expression under the given conditions is an integer.

The reader will find in Kellogg's paper a proof that the set $x_k = u_k + 1$ actually gives to $f_{n-1}(x)$ the value u_n.

3. **Necessary Conditions for a Maximum. Reduced Sets.** As a first step in our proof, a glance at (3) shows that if all but one of the x's are fixed, then the best choice for the remaining x, *i.e.*, the value that makes $f_{n-1}(x)$ as large as possible, but finite, is the least value of that x for which $1/f_{n-1}(x)$ is positive. Thus we should take x_{n-1} as the least integer such that

$$1 - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_{n-1}} > 0,$$

i.e., such that

$$x_{n-1} > f_{n-2}(x).$$

If we use the symbol $E(a)$ to denote the greatest integer which does not exceed a, this gives us as our best choice for x_{n-1},

$$x_{n-1} = E(f_{n-2}(x)) + 1. \tag{4}$$

In particular, *a necessary condition for a maximum of $f_{n-1}(x)$ is that the x's be a set in which the member or one of the members having the largest value, taken as x_{n-1}, verifies (4).*

In a set which has the above property we cannot decrease any one member, leaving the others unaltered, without making $1/f_{n-1}(x)$ negative or zero; let us call such a set a *compact set.* When a set is compact, x_{n-1} being the largest member, we have

$$f_{n-1}(x) \geqq x_{n-1}(x_{n-1} - 1),$$

for this is equivalent to

$$\frac{1}{f_{n-1}(x)} \leqq \frac{1}{x_{n-1} - 1} - \frac{1}{x_{n-1}}, \qquad \text{or} \qquad 1 - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_{n-1} - 1} \leqq 0,$$

which follows from the definition of a compact set.

We will use the term *reduced set* for a compact set which has but one largest member. Let us consider two ways of transforming any given set of integers $x_1, x_2, \cdots, x_r$ for which $1/f_r(x)$ is positive into a reduced set $x_1', x_2', \cdots, x_r'$. The first step, naturally, is to take the largest x (or one of them if there are equal largest x's) and decrease it to the smallest value which will make $1/f_r(x)$ positive, while the other x's are left unaltered; if a largest member of the resulting set can be decreased we proceed as in the first instance, repeating the process until we obtain a compact set $X_1, X_2, \cdots, X_{r-1}, X_r$, which we will suppose arranged in order of magnitude. In case $X_{r-1} = X_r$, two methods of procedure are suggested by formulas indicated by Kellogg. In the first we replace the set X by the set Y in which

$$Y_1 = 2, \quad Y_2 = 2X_1, \quad Y_3 = 2X_2, \quad \cdots, \quad Y_{r-1} = 2X_{r-2}, \quad Y_r = X_r. \tag{5}$$

It is easily verified that

$$f_r(Y) = 2f_r(X).$$

We now make the Y set compact by the process already described. Then, if the resulting set has equal largest members, apply (5) so as to obtain a set Z. If these operations are continued long enough we must finally obtain a reduced set. In fact, a set x whose first k members are $2, 4, 8, \cdots, 2^k$ cannot contain any other member as small as 2^k without making $1/(f_r(x))$ zero or negative. But the set Y contains one member 2, the set Z one member 2 and one 4, and so on, and if we have not at some previous stage obtained a reduced set we shall have transformed our set into the set $2, 4, 8, \cdots, 2^r$, and by making this compact we have the reduced set $2, 4, 8, \cdots, 2^{r-1}, 2^{r-1}+1$. Note that *every step has increased* $f_r(x)$.

The other method of reduction proceeds from the identities[1]

$$\frac{2}{x} \equiv \frac{1}{\frac{x}{2}+1} + \frac{1}{\frac{x}{2}\left(\frac{x}{2}+1\right)}, \qquad \frac{2}{x} \equiv \frac{1}{\frac{x+1}{2}} + \frac{1}{x\left(\frac{x+1}{2}\right)} \tag{6}$$

If a compact set X has two largest members equal, so that $X_{r-1} = X_r$, we replace the set X by the set

$$X_1, X_2, \cdots, X_{r-2}, \frac{X_r}{2}+1, \frac{X_r}{2}\left(\frac{X_r}{2}+1\right), \tag{7}$$

when X_r is even, and by

$$X_1, X_2, \cdots, X_{r-2}, \frac{X_r+1}{2}, X_r\left(\frac{X_r+1}{2}\right), \tag{8}$$

when X_r is odd; from (6) we see that the new set gives to $f_r(x)$ the same value as does the set X. We then make the new set compact as before; and this compact set will also be a reduced set. For by (7) or (8) we have replaced two equal numbers X_r by two integers, one less than X_r and the other greater. If we could reduce the latter to be less than or equal to one of the other X's, we should have a set in which no number is greater than the corresponding X and at least one

[1] The first of these is given in Kellogg's paper.

number is less, which is impossible, since the set X was compact by hypothesis.

As an illustration, consider the set 3, 4, 5, 6. The first method transforms this into 3, 4, 5, 5 (compact), and then into 2, 6, 8, 5, which, rearranged, is 2, 5, 6, 8 (reduced). By the second method we obtain successively 3, 4, 5, 5, then 3, 3, 4, 15, and finally 3, 3, 4, 13 (reduced).

The set (7) or (8) gives to $f_r(x)$ the *same* value as the set X, so that we cannot say for the second method, as we did for the first, that at every step we increase $f_r(x)$. Whether a set (7) or (8), derived from a compact set X for which $X_{r-1} = X_r$, may ever be compact I have not been able to determine.[1] Until that point is settled we cannot say that in every case the reduced set obtained by the second method gives to $f_r(x)$ a greater value than the original set. This is, however, true of the first method, so that we may strengthen the condition for a maximum given at the beginning of this section as follows:

THEOREM II. *A necessary condition that a set x maximize $f_{n-1}(x)$ is that it be a reduced set; i.e.; we can so choose our notation that we have*

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-2} < x_{n-1}, \tag{9}$$
$$x_{n-1} = E(f_{n-2}(x)) + 1. \tag{4}$$

We assume here and in what follows that $n > 2$, the case $n = 2$ being easily disposed of.

4. The Function $\phi_{n-2}(x)$. Necessary Conditions for Maximizing $\phi_{n-2}(x)$. Having restricted ourselves to reduced sets x, we can now assign an upper bound for $f_{n-1}(x)$. In fact, this last function is expressible as a fraction whose numerator is the product $x_1x_2 \cdots x_{n-1}$, and whose denominator is a positive integer. We thus have

$$f_{n-1}(x) \leqq x_1x_2 \cdots x_{n-1} \tag{10}$$

where x_{n-1} is given by (4), and all the x's satisfy (9).

Since

$$E(f_{n-2}(x)) + 1 \leqq f_{n-2}(x) + 1,$$

we can replace (9) and (10) by the inequalities

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-2} \leqq f_{n-2}(x), \tag{11}$$
$$f_{n-1}(x) \leqq x_1x_2 \cdots x_{n-2}[f_{n-2}(x) + 1]. \tag{12}$$

We will denote the second member of (12) by $\phi_{n-2}(x)$.

The next step is to investigate the maximizing of $\phi_{n-2}(x)$ for sets $x_1, x_2, \cdots, x_{n-2}$ subject to (11). We shall prove the following result:

THEOREM III. *A necessary condition that a set $x_1, x_2, \cdots, x_{n-2}$ subject to (11) maximize $\phi_{n-2}(x)$ is that it be a reduced set.*

To prove this, let us start with any set verifying (11) which makes $1/f_{n-2}(x)$

[1] While the sets (7) and (8) are not usually compact, they are so in some cases. For example, 7, 8, 10, 10, 10, 11, 11, 12, 12, 12, is compact, and the set (7) derived from it, 7, 8, 10, 10, 10, 11, 11, 12, 7, 42, is also compact. In a similar manner, 9, 9, 10, 11, 11, 11, 11, 12, 13, 13, 13, is compact as is also the corresponding set (8), namely, 9, 9, 10, 11, 11, 11, 11, 12, 13, 7, 91. For this second example $f_{11}(x)$ has the remarkably large value of 25,740—EDITOR.

positive and transform this set into a reduced set by the second method. We will show that at each step we have a set verifying (11) and giving to $\phi_{n-2}(x)$ a value greater than it had before. The former of these statements is easily disposed of, for our second method employs two kinds of processes. The first process decreases one element at a time, which increases $f_{n-2}(x)$ and leaves (11) true when the symbols are suitably arranged. The second substitutes (7) or (8) for a compact set X with two largest members equal. It is, then, only the effect of this latter substitution that we need examine; we give the details for X_{n-2} even. The last member of (7) is the largest, and $f_{n-2}(x)$ is the same for the set (7) as for the set X, so that we have to prove

$$\frac{X_{n-2}}{2}\left(\frac{X_{n-2}}{2}+1\right) \leqq f_{n-2}(x).$$

But since X is compact $f_{n-2}(X) \geqq X_{n-2}(X_{n-2}-1)$, and since X_{n-2} must be greater than 2, we have $X_{n-2}(X_{n-2}-1) > (X_{n-2}/2)\ ((X_{n-2}/2)+1)$. Therefore the new set satisfies (11). We have a similar result when X_{n-2} is odd, depending on the fact that here $X_{n-2} > 3$.

Having shown that each transformed set for our second method satisfies (11), we must now prove that each step increases $\phi_{n-2}(x)$. These steps consisted either in decreasing one member at a time, or in passing from a compact set X with two equal largest members to a set (7) or (8). To prove that a step of the former sort increases the value of $\phi_{n-2}(x)$, suppose we have a set x satisfying (11), which we transform into a set x' by changing only the member x_{n-2}; this latter we replace by $x'_{n-2} = x_{n-2} - 1$. We suppose $f_{n-2}(x')$ positive. Since

$$\frac{1}{f_{n-2}(x)} - \frac{1}{f_{n-2}(x')} = \frac{1}{x_{n-2}(x_{n-2}-1)},$$

we have

$$f_{n-2}(x') = \frac{x_{n-2}(x_{n-2}-1)f_{n-2}(x)}{x_{n-2}(x_{n-2}-1) - f_{n-2}(x)},$$

and $\phi_{n-2}(x') - \phi_{n-2}(x)$ is equal to $x_1x_2 \cdots x_{n-3}$ multiplied by

$$\begin{aligned}&(x_{n-2}-1)[f_{n-2}(x')+1] - x_{n-2}[f_{n-2}(x)+1]\\ &= \frac{x_{n-2}(x_{n-2}-1)^2f_{n-2}(x)}{x_{n-2}(x_{n-2}-1)-f_{n-2}(x)} - x_{n-2}f_{n-2}(x) - 1\\ &= \frac{x_{n-2}[f_{n-2}(x)]^2 - (x_{n-2}^2 - x_{n-2} - 1)f_{n-2}(x) - x_{n-2}(x_{n-2}-1)}{x_{n-2}(x_{n-2}-1) - f_{n-2}(x)}.\end{aligned}$$

The denominator of this fraction is positive, being the denominator of the expression for $f_{n-2}(x')$. The numerator is a quadratic in $f_{n-2}(x)$ with both roots less than x_{n-2}. For if we substitute x_{n-2} for $f_{n-2}(x)$ it reduces to $2x_{n-2}$, which is positive. Therefore the numerator of the fraction is positive and

$$\phi_{n-2}(x') > \phi_{n-2}(x).$$

It remains only to show that the substitution of a set (7) or (8) for a compact

set X in which $X_{n-3} = X_{n-2}$ gives to $\phi_{n-2}(x)$ a value greater than $\phi_{n-2}(X)$. We carry this through for the case where X_{n-2} is even; the reader will easily see how to treat the other case. The substitution to be carried out leaves $f_{n-2}(x)$ unaltered, and in fact the only change in $\phi_{n-2}(x)$ is to substitute for the two equal factors X_{n-2} in $\phi_{n-2}(X)$ the pair $X_{n-2}/2 + 1$, $(X_{n-2}/2)\,(X_{n-2}/2 + 1)$. But the product of this pair is greater than X_{n-2}^2, for $X_{n-2} > 2$, so that

$$\left(\frac{X_{n-2}}{2} - 1\right)^2 > 0, \qquad \left(\frac{X_{n-2}}{2} + 1\right)^2 > 2X_{n-2}, \qquad \frac{X_{n-2}}{2}\left(\frac{X_{n-2}}{2} + 1\right)^2 > X_{n-2}^2.$$

Thus $\phi_{n-2}(x)$ has been increased by this substitution.

Since each step in our second method, which must terminate with a reduced set, increases $\phi_{n-2}(x)$, while all the sets used satisfy (11), we have proved Theorem III.

5. Proof of Theorem I. If we now take $x_1, x_2, \cdots, x_{n-2}$ as a reduced set, so that

$$x_{n-2} = E(f_{n-3}(x)) + 1 \leqq f_{n-3}(x) + 1,$$

we obtain by comparison with (10), (11), and (12) the inequalities

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-3} \leqq f_{n-3}(x), \tag{13}$$

$$f_{n-1}(x) \leqq \phi_{n-2}(x) \leqq x_1x_2\cdots x_{n-2}(x_1x_2\cdots x_{n-2} + 1) \leqq \phi_{n-3}(x)[\phi_{n-3}(x) + 1]. \tag{14}$$

To maximize the right member of (14) is to maximize $\phi_{n-3}(x)$, and we are thus led to repetitions of section 4. Our final inequalities are

$$x_1 \leqq f_1(x), \qquad f_{n-1}(x) \leqq U_{n-2}, \tag{15}$$

where U_{n-2} is defined by the recurrence relations

$$U_1 = \phi_1(x),\ U_{k+1} = U_k(U_k + 1), \qquad k = 1, 2, \cdots, n-3.$$

But (15) gives us $x_1 \leqq 1/(1 - (1/x_1))$, $x_1 = 2$, since $x_1 < 2$ is impossible. With this value for x_1 we have

$$U_1 = \phi_1(x) = x_1\left(\frac{1}{1 - \dfrac{1}{x_1}} + 1\right) = 6 = u_3, \qquad \text{and} \qquad U_{n-2} = u_n,$$

to use the notation of (2). Thus we have

$$f_{n-1}(x) \leqq u_n.$$

But the value u_n is actually attained by giving to the x's the values $u_k + 1$, so that u_n is the maximum of $f_{n-1}(x)$.

One way of stating the necessary conditions at each reduction of our problem was to require that each of the sets,

$$x_1;\quad x_1, x_2;\quad \cdots;\quad x_1, x_2, \cdots, x_{n-1},$$

be reduced. The only x's that satisfy these conditions are those of the set $x_k = u_k + 1$. With this remark our proof of Theorem I is complete.

AUTHOR'S NOTE—Since the above was put into type, the author has received reprints of an article by Tanzô Takenouchi, entitled "On an indeterminate equation," which has just appeared in the *Proceedings of the Physico-Mathematical Society of Japan* (third series, volume 3, pp. 78–92). In this paper the number 1 on the right side of equation (1) is replaced by a fraction b/a in which $a \geqq b$, and in case a is of the form $(m+1)b-1$, it is shown that the maximum x in all solution sets $(n>1)$ is A_n, as given by the recurrence formula

$$A_1 = m, \qquad A_2 = a(A_1+1), \qquad \cdots, \qquad A_{k+1} = A_k(A_k+1).$$

When $b = m = 1$, this is Kellogg's theorem (see p. 91). The results of the present paper are thus anticipated in a general way, though Theorem I is not proved by Takenouchi. The methods employed are sufficiently different to make the present paper of interest, and its greater simplicity and brevity may recommend it. The methods we have here used apply with very little change to Takenouchi's problem. We cannot, in general, use the first method of reduction for a compact set with equal largest members, so that the necessity of some of our conditions does not follow, but we easily obtain Takenouchi's result. We can, in fact, go further than he has, and obtain the following analogue of Theorem I, where

$$\frac{1}{f_r(x)} = \frac{b}{a} - \frac{1}{x_1} - \frac{1}{x_2} - \cdots - \frac{1}{x_r},$$

and $b \leqq a$:

The maximum finite value of $f_{n-1}(x)$, $n>1$, for all positive integral values of x_1, x_2, $\cdots$ x_{n-1} is not greater than B_n, where

$$B_1 = E\left(\frac{a}{b}\right), \; B_2 = a(B_1+1), \qquad B_3 = B_2\left(\frac{1}{\frac{b}{a} - \frac{a}{B_2}} + 1\right),$$

$$B_{k+1} = B_k(B_k+1), \; k > 2.$$

If $a = (m+1)b-1$, we have $B_n = A_n$, so that this upper bound is actually reached.

In Takenouchi's example

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{5}{11},$$

which does not come under the case where he has given a general solution but for which he observed that the maximum x is at least 220, the upper bound noted above is $305\frac{1}{4}$.

A MODEL FOR THE PEANO SURFACE.

By ARNOLD EMCH, University of Illinois.

1. Introduction. In his *Calcolo differentiale e principi di calcolo integrale*, published in 1884, Peano gave the first rigorous treatment of the theory of maxima and minima of functions of several variables. The development of this theory may also be found on pp. 181–188, in notes 133–136, and on p. 332 of the German translation (1899). The note contains a discussion of the now famous function, representing a certain quartic surface which may properly be called the Peano surface.[1] By this example Peano demonstrated the falsity of some of the hitherto accepted criteria for maxima and minima of functions of several variables. One of these false criteria will be mentioned in the constructive discussion of the Peano surface. The investigations of Peano induced others to penetrate still more deeply into the theory of maxima and minima[2] and related domains of the calculus of variations. Some very reputable mathematicians and authors, like Serret,[3] Bertrand,[4] Todhunter,[5] and others, have fallen into this pit of faulty criteria.

The first one in America to call attention to these defects was Professor J. Pierpont in an interesting article on maxima and minima of functions of several variables.[6]

In most cases the errors are fundamentally due to the theorem, which in general is not true, that in Taylor's development of a function of several variables, the ratio of the remainder, following a certain term of the expansion, to this term, approaches zero as a limit when the increments of the variables approach zero as a limit. For cases in which this limit is zero, see Peano (German translation, theorem III, p. 173).

It is the purpose of this paper to discuss the geometrical properties of the Peano surface and to make them intuitively apparent by the aid of a model constructed by the writer. Peano's important criticisms can be comprehended very readily by means of this geometric visualization.

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[1] Georg Scheffers, *Lehrbuch der darstellenden Geometrie*, vol. 2, 1920, pp. 261–263.

[2] L. Scheffer, "Theorie der Maxima und Minima einer Funktion von zwei Variablen," *Mathematische Annalen*, vol. 35, 1890, pp. 541–576. Scheffer arrived independently at some of the conclusions of Peano, but uses Peano's function as an example.

V. v. Dantscher, "Zur Theorie der Maxima und Minima einer Funktion von zwei Veränderlichen," *ibid.*, vol. 42, 1893, pp. 89–131.

O. Stolz, *Grundzüge der Differential- und Integralrechnung*, vol. 1, Leipzig, 1893, pp. 213–228. Also *Sitzungsberichte der math.-natur. Klasse der Akademie des Wissenschaften*, Vienna, vol. 99, p. 499.

[3] J. A. Serret, *Cours de calcul différentiel et intégral*, vol. 1, 3d ed., Paris, 1886, p. 216. Harnack corrects the error to which Peano refers in the table of corrections of the first volume of Harnack's German translation (1897) of Serret's calculus.

[4] Bertrand, *Calcul différentiel*, Paris, 1864, where the same mistake occurs on p. 504.

[5] I. Todhunter, *A Treatise on the Differential Calculus*, London, 1875, pp. 226–236. G. Battaglini's Italian translation (fifth edition, 1913) of Todhunter's calculus, vol. 1, pp. 255–261, presents the problems under discussion apparently without cognizance of Peano's criteria. The treatment is therefore also unsatisfactory.

[6] J. Pierpont, "Maxima and minima of functions of several variables," *Bulletin of the American Mathematical Society*, 2d series, vol. 4, 1898, pp. 535–539.

2. **Peano's Criterion.**[1] Before taking up the discussion of the Peano surface, it is perhaps well to state the criterion upon which, according to Peano, the existence of maxima and minima depends.

We recall a few preliminary statements concerning definite forms.

A rational homogeneous function of several variables of degree n is called a *form* of the nth degree. In Taylor's expansion for

$$f(x_0 + h, y_0 + k, \cdots)$$

the successive terms are forms of degree 0, 1, 2, $\cdots$ in $h\ k, \cdots$

A form is called *definite,* if it vanishes only when all variables vanish simultaneously. It is *indefinite* if it may assume negative and positive values. Thus, a form of odd degree, which does not vanish identically, is always indefinite. A form may be *neither definite nor indefinite.* This happens for a form which has always the same sign when not zero, but which vanishes for a set $(h, k, \cdots)$ in which *not all* variables vanish. This is what happens in Peano's example.

The criterion for maxima and minima of $f(x, y, \cdots)$ is as follows: *If for $x = x_0$, $y = y_0$, $\cdots$ all partial derivatives of less than the nth order vanish, and if in Taylor's expansion for $z = f(x_0 + h, y_0 + k, \cdots)$, the term which is a homogeneous function of $h, k, \cdots$ of the nth degree is an indefinite form, then z is neither a maximum nor a minimum at $(x_0, y_0, \cdots)$. If however this form is definite and positive, then z is a minimum; if the form is definite and negative, then z is a maximum.*

The Peano criterion never gives a wrong result, but it must be understood that the cases where the forms involved are neither definite nor indefinite require individual treatment, like the surface examined hereafter. On the other hand, the other methods to which reference is made actually give wrong results in these cases.

3. **The Peano Surface.** Serret, *l.c.*, p. 219, sets up the criterion: "*the maximum or the minimum takes place when for the values of h and k for which d^2f and d^3f (third and fourth terms) vanish, d^4f (fifth term) has constantly the sign $-$, or the sign $+$.*" To show that this is erroneous, Peano considers the function

$$z = f(x, y) = (y^2 - 2px)(y^2 - 2qx), \tag{1}$$

in which $p > q > 0$. Putting $x_0 = 0$, $y_0 = 0$ in Taylor's expansion for $f(x_0 + h, y_0 + k)$, there results

$$f(h, k) = 4pqh^2 - 2(p + q)hk^2 + k^4.$$

The second term, d^1f, which is a form of the first degree, here vanishes identically. The system of the terms of the second degree remains different from zero for all values of h and k, except for the values of the sub-set $(0, k)$. Hence the third term d^2f is neither definite nor indefinite. For the set $(0, k)$ the terms of the second and third degree, d^2f and d^3f, vanish and the fifth term is positive. Hence

[1] See *l.c.*, German translation, and the theory on pp. 170–177, preceding maxima and minima. The whole theory of Peano is so simple, and at the same time rigorous, that it should be generally adopted by writers of textbooks.

according to Serret's criterion z would have a minimum, which is false. To show this, let (x, y) approach $(x_0 = 0, y_0 = 0)$ through the function $y^2 = 2lx$. Then

$$z = f(x, \sqrt{2lx}) = 4(l - p)(l - q)x^2,$$

which is negative when $q < l < p$, positive when $l > p$ or $l < q$. Hence there is neither a maximum nor a minimum at $(0, 0)$. Applying Peano's criterion, we note, as above, that the form of the first degree, d^1f, vanishes identically while d^2f remains positive for $h \neq 0$ and vanishes for the subset $(0, k)$. It is neither definite nor indefinite in h and k. The criterion, therefore, does not settle the question in this case. It is by means of the function $y^2 = 2lx$, as shown above (or $x^2 + y^2 = \epsilon^2$, ϵ as small as we please), that the behavior of the function may be investigated.

Geometrically the situation is as follows: The function $z = f(x, y)$ given by (1) is represented by a quartic. For the sake of convenience of construction and modelling we assume it in the form

$$z = -(y^2 - 2px)(y^2 - 2qx), \tag{2}$$

or

$$z = 2(p + q)xy^2 - 4pqx^2 - y^4. \tag{3}$$

Introducing the variable t to make (3) homogeneous ($t = 0$ being the equation of the plane at infinity), we have

$$zt^3 = 2(p + q)xy^2t - 4pqx^2t^2 - y^4. \tag{4}$$

From this is seen that the intersection of the xz-plane with the plane at infinity is a tacnodal line on the quartic. Consequently a plane $y = k$ through this line cuts the quartic in a residual curve which is a parabola, namely,

$$z = 2(p + q)k^2x - 4pqx^2 - k^4.$$

The y-axis has a four-fold contact with the surface. The infinite point of the z-axis is a triple point of the quartic and has the plane at infinity as a three-fold plane. The plane at infinity is a uniplane at the infinite point of the x-axis.

The parabolic cylinder $y^2 = 2lx$ passes through the tacnodal line of the quartic and, consequently, intersects the quartic in a quartic residual curve, whose projection upon the xz-plane has the equation

$$z = -4(l - p)(l - q)x^2,$$

representing a parabola (to be counted twice). On this quartic curve, z is either always negative or always positive (0 for $x = 0$), according to the values of l, as has been found above.

A plane $y = \lambda x$ through the z-axis intersects the surface in a quartic, whose projection upon the xz-plane has the equation

$$z = 2(p + q)\lambda^2x^3 - 4pqx^2 - \lambda^4x^4.$$

Since

$$\frac{dz}{dx} = 6(p+q)\lambda^2x^2 - 8pqx - 4\lambda^4x^3,$$

and

$$\frac{d^2z}{dx^2} = 12(p+q)\lambda^2x - 8pq - 12\lambda^4x^2,$$

it follows that at (0, 0, 0) for all values of λ the first derivative vanishes and the second derivative has the constant value[1] $-8pq$. Hence:

Every plane through the z-axis, without exception, cuts the surface in a quartic with a maximum at the point (0, 0, 0).

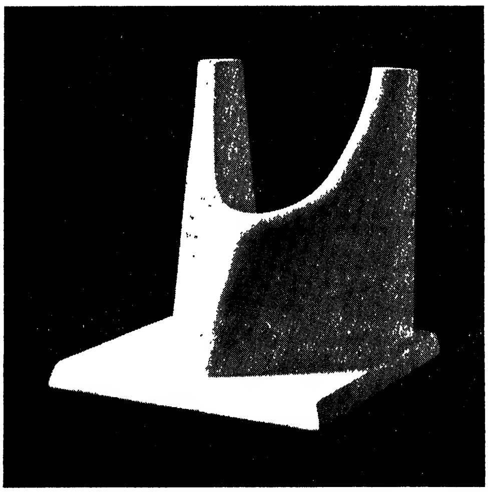

FIG. 1.

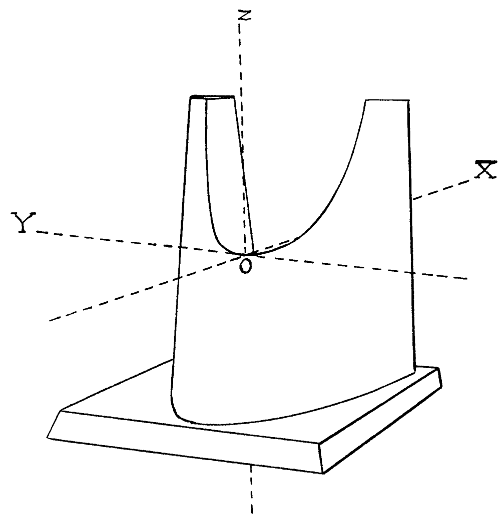

FIG. 2.

The non-critical, purely intuitive mind would ordinarily proclaim such a point a maximum (summit) of the surface. A model of the Peano surface shown in Fig. 1 is a *demonstratio ad oculos* that such is not the case. Fig. 2 shows the location of the surface with respect to the coördinate system. The scale on the z-axis has been chosen as one tenth of that of the x- and y-axes in order to avoid extremely steep slopes. The model represents the true form of the quartic surface defined by the equation

$$z = -\tfrac{1}{10}(y^2 - 5x)(y^2 - x),$$

so that

$$p = 2.5, \qquad q = 0.5.$$

This Peano surface[2] has been constructed on a system of horizontal contour-lines cut out by the pencil of parallel planes $z = \lambda$.

[1] It should be observed, however, that the section for $\lambda = \infty$, that is, the section formed by the yz-plane, cuts the quartic in a curve with a flat point, which projects upon the xz-plane in a segment of a straight line, so that dz^2/dx^2 assumes the indeterminate form $\infty \cdot 0 - 8pq - \infty \cdot 0$. This difficulty disappears for this value of λ if we project the curve of intersection upon the yz-plane. Then for $\lambda = \infty$ the first three derivatives of z with respect to y all vanish, while the fourth derivative is negative. Under these circumstances the flat point may still be defined as a maximum.

[2] The mould for this plaster model has been preserved, so that casts might be made for parties wishing to possess such a model in their collection.

THE FIRST ATTEMPT AT A TABLE OF INTEGRALS.

By NOAH R. BRYAN, University of Maine

There is apparent an ever-growing interest in the calculus in the fields of applied mathematics and of secondary education. In the field of applied mathematics, we find that an author of an elementary book on wireless telegraphy has considered it necessary to include a chapter on the calculus;[1] we find that chemists no longer look askance at the introduction of formal mathematics into their subject but are beginning to welcome the use of more calculus in their science;[2] and, since today the aëroplane is built according to the knowledge obtained by the careful methods of the laboratory, it is only natural to find its construction described in the language of the calculus.[3] In the field of secondary education, a study of the status of mathematics in the curricula in schools abroad reveals the fact that in ten of these countries some calculus is taught before the end of the year which corresponds to our twelfth school year or the last year in high school.[4] Also, not only has it been recommended that an elective course in the simpler parts of the calculus should be offered in our first-class high schools,[5] but such a course is already being given along with the course in more extended algebra in a number of our high schools.

To the ultra-conservative professor of mathematics, this interest in the calculus on the part of outsiders may seem like trespassing upon sacred ground. And, although the lover of pure mathematics is open-minded, he will ask whether a study of the calculus can be of much benefit unless the subject is approached from the scientific point of view. Therefore, it may be well to recall that our usual method of approach to the subject through limits is a relatively recent one so far as rigor is concerned. This method depends upon the thorough work on limits and infinitesimals by Carnot (1797) and the obtaining of the fractional form of the derivative by the method of limits as given by Cauchy (*c.* 1821). Yet, as we well know, wonders had been accomplished with the calculus long before this scientific method had become traditional.

It may, therefore, be of interest to consider a noteworthy attempt at classification of simple integral forms by Johann Bernoulli, the first great expositor of the calculus. Although his work did not appear in book form, his manuscripts written in 1691–1692 form the first textbook in integral calculus.[6]

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[1] S. J. Willis, *A Short Course in Elementary Mathematics and Their Application to Wireless Telegraphy*, London, 1917.

[2] J. W. Mellor, *Higher Mathematics for Students of Chemistry and Physics*, New York, 1919.

[3] *Flight*, September 2, 1920 (Official organ of the royal aero group of the United Kingdom); *Aeronautics*, September 2, 1920, London; *The Aeronautic Journal*, September, 1920; *Aircraft Engineering*, January, 1920, London.

[4] J. C. Brown, *Curricula in Mathematics*, United States Bureau of Education Bulletin, No. 45, 1914.

[5] *Elective Courses in Mathematics*, The National Committee on Mathematical Requirements, March, 1921.

[6] Johann Bernoulli lived from 1667 to 1748. His *Lectiones mathamaticae de methodo integralium* was published in the third volume of *Opera Omnia*, Lausannæ & Genevæ, 1742, pp. 385–558, but the differential part of his manuscript was not published. It is stated that this was because the ground had been fully covered by the Marquis de l'Hospital (1661–1704) in his *Analyse des*

His first lecture concerns the nature and the calculation of the integral. His method is that of the inverse of differentiation. He says that it is known that dx is the differential of x, and xdx the differential of $\frac{1}{2}xx$ or $\frac{1}{2}xx \pm$ a constant, and so on. Then

$$\begin{aligned} adx &\text{ is the differential of } \quad ax \pm \text{ a constant,}\\ axdx &\text{ is the differential of } \tfrac{1}{2}axx \pm \text{ a constant,}\\ axxdx &\text{ is the differential of } \tfrac{1}{3}ax^3 \pm \text{ a constant,}\\ ax^3dx &\text{ is the differential of } \tfrac{1}{4}ax^4 \pm \text{ a constant,}\end{aligned}$$

and so on.[1]

From this he deduces the general formula that ax^pdx is the differential of $ax^{p+1}/(p+1)$.

In his second lecture, Bernoulli considers the plane surface as divided into an infinite number of parts, each of which may be regarded as the differential of the surface. If we have the integral of this differential, that is, the sum of these parts, then will we also know the desired area. Those infinitely little parts of the plane surface can be thought of as obtained in different ways according to the most convenient subdivision of the figure. For example, plane surfaces would be directly divided, as is the custom, by means of parallel lines, as in Fig. 1,

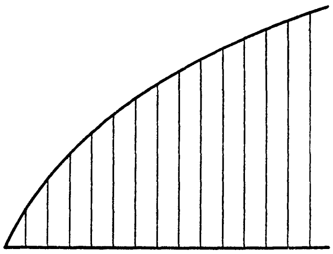


FIG. 1.

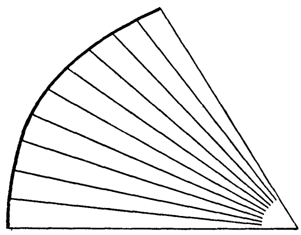


FIG. 2.

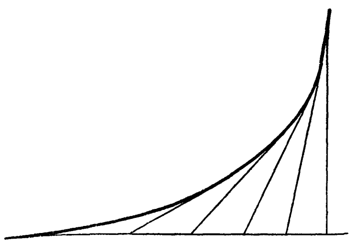


FIG. 3.

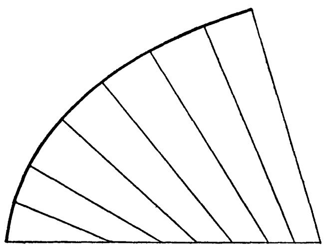


FIG. 4.

or by means of an infinite number of straight lines which are concurrent, as in Fig. 2, or by means of an infinite number of tangents, as in Fig. 3, or by means of an infinite number of normals to the curve, as in Fig. 4.

infiniment petits, Paris, 1696. There has been some controversy as to whether the latter book may not have really been based upon the manuscript of Bernoulli more fully than would be inferred from the general acknowledgment in the Preface. The manuscript of this part of Bernoulli's work written in Latin, has recently been found in the library of the University of Basel and is reproduced in full, with an Introduction by Paul Schaftheitlin, in *Verhandlungen der Naturforschenden Gesellschaft in Basel*, vol. 34, 1922, pp. 1–32, with four plates of figures. See also Paul Schaftheitlin, "Johann Bernoulli's Differentialrechnung," *ibid.*, vol. 32, 1921, pp. 230–235—EDITOR.

[1] *Opera Omnia*, vol. 3, pp. 387–388.

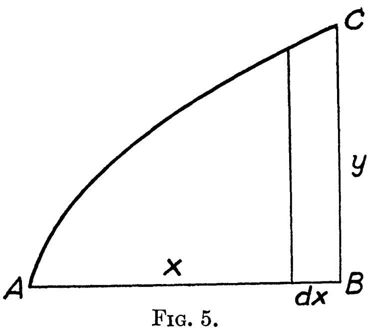


FIG. 5.

If the divisions of the surface are parallel, and if x is the abscissa and y is the ordinate, the differential of the surface will be ydx; namely, the rectangle formed by the ordinate and the differential of the abscissa. If AC is the given curve, then y will have a given ratio to x, so that it is expressed in x alone. Suppose that AC is a parabola, then $ax = y^2$ or $y = \sqrt{(ax)}$. The integral of this, which is $\frac{2}{3}x\sqrt{(ax)}$ or $\frac{2}{3}xy$, is the area desired.[1] See Fig. 5.

This method of Bernoulli's for finding the area is simple, and is satisfactory for elementary work, although the idea of limits involved is taken care of largely by intuition.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

29. LEGENDRE AND CAUCHY SPONSOR ABEL.

In his work on Abel,[2] M. Ch. Lucas de Pesloüan gives, in a supplementary note (p. 144), the beginning of a memoir which, as he says, "probablement fut lu par Fourier en séance publique, le 30 Octobre 1826, à l'Institut, et que l'on ne comprit pas," stating in the text (p. 55) that it is said to be a development of the addition theorem.

In volume 2 of Abel's *Oeuvres Complètes* (1839) the editor, B. Holmboe, has the statement that, much as he had wished to publish in his edition the memoir presented by Abel about the close of the year 1826, "tous les efforts pour obtenir une copie de ce mémoire ont eté infructueux jusqu'à présent."

In the *Mémoires présentés par divers savants à l'Académie Royale des Sciences de l'Institut de France*, published at Paris two years later (1841),[3] however, is the very memoir referred to, with the title: "Mémoire sur une propriété générale d'une classe très étendue de Fonctions Transcendantes, par M. N. H. Abel, Norvégien, présenté à l'Académie le 30 Octobre, 1826." On the last page (264) there is a note by Libri, the historian of mathematics, as follows: "L'Académie m'ayant fait l'honneur de me charger de surveiller l'impression de ce Mémoire, je me suis appliqué à corriger, autant que possible, les fautes d'impression," with apologies for not being able to do as well as could have been done had he been able to see the original manuscript. Indeed, when the second edition of Abel's work was published at Christiania (1881), under the editorship of Sylow and Lie, there was inserted a note on this memoir (volume 2, p. 294) to the effect that Lie

[1] *Ibid.*, pp. 394–395.
[2] N. H. Abel, *Sa vie et son oeuvre*, Paris, 1906.
[3] Vol. 7, pp. 176–264.

had obtained, in 1874, the permission of the Académie des Sciences of Paris to consult the original manuscript, but upon searching the archives it was found that it had never been seen after the first printing.

Among my autographs is an interesting document bearing upon the matter and, so far as I have been able to ascertain, thus far unpublished. It consists of four pages in the handwriting of Legendre and is signed by him and Cauchy. On the first page is a note in another hand, "29 Juin 1829." There is nothing to show what this date means, but it seems to indicate that the Académie was considering the printing of the article which had already been presented. At any rate, the report is upon the question of the value of the memoir. It begins with the words: "Report on a memoir relative to a general property of 'une classe très étendue de functions transcendantes.' The Académie has directed M. Legendre[1] and me to report concerning a memoir of M. Abel relative to a general property of a class 'très étendue,' of transcendent functions." It then goes on to describe the nature of the functions and to mention that they have certain properties which are analogous to those of logarithmic and elliptic functions. It speaks of the fact that Abel had already planned for printing a portion of it in Crelle's journal, of his great promise in science, and of the serious loss to the world in his early death, just as he was working upon several new memoirs. It concludes with the recommendation that the memoir be published in the collection of those of foreign scientists.

The recommendation seems to show that it was written about the date mentioned, which was nearly three months after Abel's death, and that this advice was followed. In the publication of 1841, Libri had attempted the necessary corrections.

That there should have seemed to be necessary such a recommendation with respect to a memoir by a man like Abel only goes to show how little he was generally known at the time and how careful the Académie was as to the papers which bore its seal of approval.[2]

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

When the forces of modernism have reformed our civil year and reduced its irregularities to a minimum, all calendars will be perpetual, and the Dominical or Sunday letter will cease to control our destinies. Until that time, though less mysterious than those lunar influences that combine with it to generate the movements of the movable feasts, the Sunday letter retains its place in our

[1] Although written by Legendre, he phrases the report as if it was due to Cauchy.

[2] I am indebted to Mr. Jekuthial Ginsburg for aid in tracing some of these facts.

almanacs, and continues to be a source of some trouble and confusion to the unlearned. Cheap printing and advertising methods have today so decorated our walls with calendars that the problem of the perpetual calendar has ceased to be immediate and vital. Nevertheless there is a steady interest in these questions, and some practical importance when remote dates have to be considered. The article by Dr. W. H. Vail appearing below contains a rule for rapidly finding the Dominical letter for any year of our era. Readers should have no difficulty in supplying a proof of the rule. Two other articles on the subject have appeared in this department (*1921*, 127, 260).

As companions to the mnemonic lines cited by Dr. Vail, some others may be noted that were obtained through editorial correspondence. Dr. Edgar Dehn of Waterloo, Canada, contributes this "little verse, sufficiently short not to scare those who don't like verses, yet long enough to help in memorizing" the sequence of digits in the *decimal expansion of* π.

"And I wish I could recollect
3 1 4 1 5 9
My number, known and select."
2 6 5 3 6

There is a recurrent interest in mnemonics for π, but Dr. Dehn was under the impression that such mnemonic curiosities existed only in the French and Russian languages. The Russian verse cannot be printed in the Latin alphabet, and will therefore not be given here. The French lines which follow are well known.[1]

"Que j'aime à faire apprendre un nombre utile aux sages!
3 1 4 1 5 9 2 6 5 3 5
Immortel Archimède, sublime ingénieur,
8 9 7 9
Qui de ton jugement peut sonder la valeur?
3 2 3 8 4 6 2 6
Pour moi ton problème eut de pareils avantages!"
4 3 3 8 3 2 7 9

Numerous variant forms have been mentioned, but few of them compare in elegance with that given above.

The *Scientific American* for March 21, 1914, gave the following:

"See I have a rhyme assisting
My feeble brain, its tasks ofttimes resisting!"

Now it is known that the first cipher in the decimal expansion of π occurs in the thirty-third place. Hence it is pointed out by Professor A. A. Bennett that a mnemonic stanza of exactly thirty-two words serves also to give thirty-three figures, if it be recalled that the digit next in order is a cipher. Professor Bennett cites as a "somewhat inelegant" example of such a complete stanza the following lines, composed by him after seeing the verse in the *Scientific American*.[2]

"Now I read a rhyme, expressed in verses clear,
The ratio circular expanding (science inerudite);
And if old Ludolph's very number do appear,
Then 'tis but Egyptian art, or ancient Syracusan might."

[1] These lines and two in German are quoted in this MONTHLY, *1905*, 215.

[2] Two more examples are given in this MONTHLY, *1906*, 50.

The allusion is to Ludolph van Ceulen (d. 1610), who computed π to 35 places, and in whose honor the term "Ludolph's number" is current in Germany. The references to Egypt and Archimedes of Syracuse need not be discussed.

Mr. Haldeman shows that the vertices of a regular undecagon inscribed in a given circle lie on a certain sextic curve. The equation of the sextic involves a parameter by variation of which the undecagon may be rotated into all positions on the circle. The method is along the same line as one given in the MONTHLY by the same author in connection with the regular pentagon and the regular heptagon (*1920*, 257–258, *1919*, 390). Some editorial remarks follow the article.

Dr. J. Rosenbaum investigates a problem suggested by the Pythagorean relation. It may be shown that for a triangle ABC with $a > b > c$ there exists a unique $n \geqq 1$ such that $a^n = b^n + c^n$; but only when A is a right angle or a straight angle does n depend solely on A. In general there is a relation connecting A, b/c, and n, which becomes independent of b/c when A is put equal to $\pi/2$ or π. Dr. Rosenbaum shows that, conversely, when A and n are given, this relation is satisfied by at most one real value of b/c, except when $n = 1$ or 2.

The note by Professor Swift calls attention to a fact in connection with trigonometric tables which is something of a paradox, and illustrates the distinction between a transcendental number and a transcendental function.

The failure at the origin of the formula in polar coördinates for the direction of a curve appears to receive no notice in a number of calculus texts. Professor H. J. Ettlinger supplies this want in the short note which forms Discussion V. His result could also be obtained by transforming dy/dx into polar form.

I. UNCLE ZADOCK'S RULE FOR OBTAINING THE DOMINICAL LETTER FOR ANY YEAR.[1]

By W. H. VAIL, Newark, N. J.

All of our measurements of time are either natural or conventional in character. Thus the solar year, the solar day, and the lunar month are natural measurements of time, being governed by the movements of the sun, the moon, and the earth in their various orbits; while all the other measurements of time, such as the second, the minute, the hour, the week, the civil month, the civil year, and the century, are conventional or arbitrary in character, having been formed for the convenience of man.

All almanacs and diaries which contain the Church Calendar, and the list of eclipses, with the chronological cycles, will cite among other items the Dominical letter for the year which the diary or almanac represents. Thus for the year 1921 they will state that the Dominical letter is B, which is pure Greek to almost every person who consults it. The first problem which the construction of a calendar presents is to find the day of the week corresponding to a given day of the year. The week contains seven days, and the problem would be perfectly

[1] An explanation of the Dominical Letter with tables is given in *The Calendar, its History, Structure and Improvement*, by Alexander Philip, Cambridge University Press, 1921.

simple if the year was composed of a whole number of weeks; but the ordinary year contains fifty-two weeks and one day, while the leap year contains fifty-two weeks and two days.

In order to facilitate the solution of this problem, the days of the week are denoted by the first seven letters of the alphabet, A, B, C, D, E, F, G, which are placed for convenience opposite the days of the year; thus beginning with A standing opposite the first day of January, B stands opposite the second, G opposite the seventh, then A opposite the eighth, and so on throughout the year. Following this rule it is found that A will represent January 1, 8, 15, 22, and 29, which would bring B to represent January 30 and C January 31. Then D will represent February 1, and as February in the ordinary year contains only twenty-eight days, D will also represent March 1, 8, 15, 22, and 29, bringing E for March 30, F for March 31, and G for April 1. In this connection it may be remarked that the letter for each month is fixed. That is, A is always the letter for the first day of January, D for the first day of February and of March, whether the year be an ordinary year or a leap year, G the letter for April, and so on.

To aid the memory in remembeıing which letters represent the different months of the year, the following couplet has been formed: At Dover Dwell George Brown Esquire, Good Carlos Finch, And David Friar. The initial letters of the above twelve words in their order stand for the first days of the twelve months of the year. Or you may take the twelve letters themselves in their regular order, thus: *ADD G, BEG C, FAD F.* A little practice will soon associate the individual letters with the months they represent.

Now proceeding a step further, if one of the days of the week, Sunday for example (as is the case with the year 1921), is represented by the letter B, Monday would be represented by C, Tuesday by D, Wednesday by E, Thursday by F, Friday by G, and Saturday by A, returning for Sunday to B again; every Sunday would be represented by the same letter, B, through the year. If we are able to obtain the letter which represents Sunday for the year, we know in this way the letters which represent the other days of the week, with a special provision for leap years. Having established the letters which shall stand for the first days of the months, it only remains to obtain the Sunday letter for the year. The letter which represents Sunday for the year is called the Dominical or Sunday letter and when this is obtained for any year, the letters which denote the first days of the various months tell us the days of the week upon which these first days will fall. Thus B representing Sunday for the year 1921 (as you will find so stated in any almanac) and B being the letter which represents May 1, it follows that May 1, 1921, occurs on Sunday. C being the letter which represents the first of August, it follows that August 1, 1921, occurs on Monday, and so on through the seven letters which represent the first days of the various months for the year 1921. Thus we are able very readily to state upon what day of the week the first day of any month occurs, and knowing this the days of the week upon which the other days of the month occur follow automatically after a little practice.

This naturally brings us to the consideration of the subject proper of this paper which is Uncle Zadock's rule for obtaining the Dominical or Sunday letter for any year. Some fourscore years ago there appeared an article in the *New York Observer* entitled "A rule for obtaining the Dominical letter for any year." This rule was originated by a school teacher in Pennsylvania whose real name was concealed under the pseudonym of "Uncle Zadock." Therefore the rule came to be known as "Uncle Zadock's Rule." As far as we are aware the real name of the author of this rule was never revealed, and during all these years we have seen it in print only once, when the writer's father had it inserted in a local paper in northern New Jersey, where he resided. This rule was given without proof. At the same time it is absolutely correct in its application to any date in the past, present or future of the Christian Era.[1] It is also most simple in its operation; and if any person is inclined to question its simplicity, let him scan the rule given in the *Encyclopædia Britannica* and wade through the mazes of figures and unknown quantities, as they occupy page after page of that work, before being able to obtain the Dominical letter for any year in question.

In contrast let us now consider Uncle Zadock's rule, and learn how readily the Dominical or Sunday letter for any year is obtained. The rule is as follows:

First, divide the centurial number of the year in question by four.

Second, multiply the remainder, if any, by two.

Third, subtract the product from six.

Fourth, to the remainder add the odd years and also the fourth part of the odd years, rejecting fractions.

Fifth, divide the sum of these numbers by seven.

Sixth, subtract the remainder, if any, from seven.

The last remainder indicates the Dominical or Sunday letter for the year under consideration in the order of the first seven letters of the alphabet. Thus should the last remainder be 1, the Dominical letter would be A. If the remainder were 2, the letter would be B, and so on.

Should the year under consideration prove to be a leap year, the Dominical or Sunday letter obtained by this rule would apply to the last ten months of the year, and the Sunday letter for the months of January and February in that year would be the next letter in the order of the first seven letters of the alphabet. Thus for the year 1920, by this rule, you would obtain the Dominical letter as C and this would be the Dominical letter for the last ten months of the year. The Sunday letter for January and February of this year would be the next letter, which is D.

As an example under the above rule, let us see upon what day of the week the fourth of July, 1776, fell. To obtain the Dominical letter for 1776, divide the centurial number 17 by 4, and you have, as a remainder, 1. Twice this remainder is 2, 2 from 6 leaves 4, and 4 added to the odd number of years, 76, and to one fourth of 76, which is 19, gives 99. Divide 99 by 7 and subtract the

[1] The direct application of the rule holds only for the Gregorian Calendar or "New Style" dating; "Old Style" dates must first be converted into "New Style," as is readily done.

remainder 1 from 7. The final remainder 6 represents the sixth letter of the alphabet which is F; and therefore F is the Dominical or Sunday letter for the last ten months of the year 1776; for 1776 was a leap year, and Uncle Zadock's rule always gives you the Sunday letter for the last ten months of leap years. F being the Sunday letter for the last ten months of the year, and the letter for July always being G, it follows that July 1, 1776, fell on Monday and July 4 on Thursday.

We may note that within a given century the first three steps in the process will always yield the same result. Thus in the nineteenth century there will always be two and in the twentieth century nothing to be added to the odd numbers and the fourth part of them. In applying this rule to any date within the present century, we may simply ignore the centurial number entirely and proceed with the fourth step in the rule.

II. Construction of the Regular Undecagon by a Sextic Curve.

By Cyrus B. Haldeman, Ross, Butler County, Ohio.

Consider the sextic

$$x(a^2 + 3ay^2 + y^4)\sqrt{-(b^2 + a^{11})} = a^5y^6 - by^5 + 6a^6y^4 - 5aby^3 + 9a^7y^2 - 5a^2by + 2a^8, \tag{1}$$

which will be real when $b^2 + a^{11}$ is negative, and the circle

$$x^2 + y^2 = -4a, \tag{2}$$

which will be real when a is negative.

Eliminate x from (1) by means of (2). After expansion, reduction, and separation into factors, the result may be written in the form

$$(y^{11} + 11ay^9 + 44a^2y^7 + 77a^3y^5 + 55a^4y^3 + 11a^5y + 2b)(a^5y - 2b) = 0.$$

By the transformation $y = -2s\sqrt{-a}$, the first of these factors placed equal to zero gives the equation

$$11s - 220s^3 + 1232s^5 - 2816s^7 + 2816s^9 - 1024s^{11} = \frac{b}{a^5\sqrt{-a}};$$

and because

$$11\sin A - 220\sin^3 A + 1232\sin^5 A - 2816\sin^7 A + 2816\sin^9 A - 1024\sin^{11} A = \sin 11A,$$

we may take

$$s = \sin A, \qquad \text{where} \qquad \sin 11A = \frac{b}{a^5\sqrt{-a}},$$

and get

$$y = -2\sqrt{-a}\sin\frac{1}{11}\sin^{-1}\frac{b}{a^5\sqrt{-a}},$$

$$y = -2\sqrt{-a}\sin\frac{1}{11}\left(2\pi + \sin^{-1}\frac{b}{a^5\sqrt{-a}}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$y = 2\sqrt{-a}\sin\frac{1}{11}\left(2\pi - \sin^{-1}\frac{b}{a^5\sqrt{-a}}\right).$$

This solution indicates that the eleven intersections whose ordinates are the eleven real roots of the above-mentioned factor are the vertices of the regular undecagon required.

The above factor of the eleventh degree may be resolved algebraically by the transformation $y = x - a/x$. The result is

$$y = \sqrt[11]{-b + \sqrt{b^2 + a^{11}}} + \sqrt[11]{-b - \sqrt{b^2 + a^{11}}}.$$

REMARKS BY THE EDITOR.

The interest of this problem is not in the mere determination of a sextic through the vertices. It is easy to find a set of six straight lines through the eleven points, and to rotate the set about the origin to obtain the various positions of the undecagon. The problem consists rather in the determination of a curve of the special type $xY_4 = Y_6$, where Y_i denotes a polynomial in y of degree i. Now Y_6/Y_4 is a rational function of y involving eleven constants, so that this may be looked upon as a case of Cauchy's general problem of interpolation by rational functions.[1] The coefficients being determined as symmetric functions of the coördinates of the vertices, we should, after simplification, have the equation (1).

This would, however, be a tedious procedure, and the problem invites a more special treatment. Mr. Haldeman does not give the steps by which he was led to his sextic equation, though it would seem to be through some use of the formula giving sin 11θ in terms of sin θ. It may also be derived from the formulas for 5θ and 6θ, as the following remarks show.

Let a regular polygon of $4n + 3$ sides be inscribed in the circle $x^2 + y^2 = c^2$, and let one of the vertices have vectorial angle α. Then the vectorial angle θ of any vertex satisfies the equation

$$\cos\{(2n+1)\theta - (4n+3)\alpha\} = \cos(2n+2)\theta,$$

or

$$\cos(4n+3)\alpha\cos(2n+1)\theta = \cos(2n+2)\theta - \sin(4n+3)\alpha\sin(2n+1)\theta.$$

Now $\cos(2n+1)\theta/\cos\theta$, $\cos(2n+2)\theta$, and $\sin(2n+1)\theta$ may be expressed[2] as polynomials in $\sin\theta$ of degrees $2n$, $2n+2$, and $2n+1$. Hence, on writing x/c for $\cos\theta$ and y/c for $\sin\theta$, we obtain an equation of the type

$$xY_{2n} = Y_{2n+2}, \qquad (A)$$

which is equivalent to (1) when $n = 2$.

[1] G. Boole, *A Treatise on the Calculus of Finite Differences*, London, 1872, p. 43.

[2] E. W. Hobson, *Plane Trigonometry*, edition 2, Cambridge, 1897, pp. 104–106.

Equation (A) is the general equation of a curve of degree $2n + 2$ having at infinity a point of multiplicity $2n + 1$, whose $2n + 1$ tangents are the line at infinity and $2n$ lines parallel to the x-axis. Such a curve is uniquely determined when c and α are given. The positions of the asymptotes $Y_{2n} = 0$ are independent of α.

It is not difficult to trace the changes in the form of the curve as α varies. It makes $4n + 4$ real intersections with the circle, the extra point having a vectorial angle $-(4n + 3)\alpha$. This extra point always crosses an asymptote at the same time as a vertex crosses the same asymptote on the opposite side of the circle, and the curve then becomes composite, the asymptote being a part of the curve. When no vertex is on an asymptote, the strips into which the asymptotes divide the plane contain two vertices each, one to the right and one to the left, except that the strip containing the extra intersection contains three vertices. This strip also contains (if $\alpha \neq \pm \pi/2$) the only point on the curve where the tangent is parallel to the y-axis. When $\alpha = \pm \pi/2$ (that is, when in equation (1) $b^2 = - a^{11}$), the curve becomes six straight lines.

To obtain similar results for a regular polygon of $4n + 1$ sides (as in Mr. Haldeman's former article on the pentagon) we may start from the equation

$$\sin \{2n\theta - (4n + 1)\alpha\} = - \sin (2n + 1)\theta,$$

and proceed in very much the same way.

An equally simple type of curve (for a polygon of m sides) is $xY_{m-1} = Y_{m-1}$, that is a curve of degree m with $m - 1$ asymptotes parallel to the x-axis and one parallel to the y-axis. To get this, we use, if $m = 4n + 3$, the equation

$$\sin \{(2n + 1)\theta - (4n + 3)\alpha\} = - \sin (2n + 2)\theta;$$

and, if $m = 4n + 1$, the equation

$$\cos \{2n\theta - (4n + 1)\alpha\} = \cos (2n + 1)\theta.$$

III. A Generalization of The Pythagorean Theorem.

By J. Rosenbaum, Bloomfield, Conn.

There is a given angle XOY, and a given number n, and we look for pairs of points, P and Q on OX and OY respectively, such that

$$\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n.$$

When the given angle is a straight angle or a right angle, such a pair of points does not exist unless $n = 1$ or $n = 2$, respectively, and for these values of n any pair of points will satisfy the condition. Conversely, if $n = 1$ or $n = 2$, there is no solution unless the angle is a straight angle or a right angle respectively, and for these angles any pair of points will do. Furthermore, for n such that $0 \leqq n < 1$ there is no solution, because then the condition would require either that PQ be greater than $PO + OQ$, or that 1 be equal to $1 + 1$.

We shall accordingly confine our attention to values of n greater than 1 and different from 2. (Solutions also exist for negative n's but we shall not consider them here.)

It is easy to prove that if P_1, Q_1 is a pair of points satisfying the condition, then all pairs of points P, Q, such that triangles POQ (or QOP) and P_1OQ_1 are similar, will also satisfy the condition. In accordance with the terminology used in algebra, we shall refer to solutions of that kind as dependent. Since we are looking for independent solutions, we can denote PO by 1, and OQ by x.

From the required condition together with the law of cosines, we obtain

$$(x^n + 1)^{2/n} = x^2 + 1 - 2xz, \qquad \text{where} \qquad z = \cos XOY. \tag{1}$$

Solving for z,

$$z = \frac{x^2 + 1 - (x^n + 1)^{2/n}}{2x} = f(x, n). \tag{2}$$

We notice that $f(x, n) = f(1/x, n)$ which means nothing more than this: if in triangle POQ $\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n$, then there is a point Q' on OY such that $\overline{PQ'}^n = \overline{PO}^n + \overline{OQ'}^n$, where $OQ' = (1/OQ)$. The triangles POQ and $Q'OP$ are similar, so that the pairs of points P, Q and P, Q' are not independent.

In what follows we shall endeavor to prove that the converse is also true; i.e. if $f(x_1, n) = f(x_2, n)$, then $x_1 = (1/x_2)$. This will say that if P, Q and P, Q' both satisfy the condition the triangles POQ and $Q'OP$ are similar; i.e. for every given angle excepting for a straight angle or a right angle there is no more than one independent solution. This will be accomplished by proving that for any given $n > 2$, f is a decreasing function of x for all $x > 1$, and that for any given n between 1 and 2, f is an increasing function of x for all $x > 1$.

Differentiating (2) with regard to x,

$$\frac{\partial z}{\partial x} = \frac{(x^2 - 1)(x^n + 1)^{(n-2)/n} - (x^n - 1)}{2x^2(x^n + 1)^{(n-2)/n}}. \tag{3}$$

It is this derivative which we wish to prove is either always positive or always negative for $x > 1$.

Since x is to take on all values greater than 1, and since n is positive, we can make the substitution

$$x = y^{1/n}.$$

The numerator of the right-hand member of (3) now becomes

$$(y^{2/n} - 1)(y + 1)^{(n-2)/n} - (y - 1).$$

This equals $(y + 1)^{-2/n} \cdot \phi(y, n)$, where

$$\phi(y, n) = (y^{2/n} - 1)(y + 1) - (y - 1)(y + 1)^{2/n}. \tag{4}$$

We may disregard the factor $(y + 1)^{-2/n}$ because y is positive, and it is only a question of whether $\partial z/\partial x$ is positive or negative.

Differentiating (4) with regard to n, we have

$$\frac{\partial\phi}{\partial n} = 2n^{-2}(y-1)(y+1)^{2/n}\log(y+1) - 2n^{-2}(y+1)y^{2/n}\log y$$
$$= 2n^{-2}(y+1)(y-1)y[\psi(y+1) - \psi(y)],$$

where $\psi(y) = (y-1)^{-1}y^{(2-n)/n}\log y$.

For $n = 1$, $\psi(y) = (y\log y)/(y-1)$. This is an increasing function of y for all values of $y > 1$, as may be proved by differentiating it and then differentiating the numerator of its derivative.

Therefore $\partial\phi/\partial n > 0$ for $n = 1$.

For $n = 2$, $\psi(y) = (\log y)/(y-1)$; which is found in the same way to be a decreasing function of y for all values of $y > 1$.

Therefore $\partial\phi/\partial n < 0$ for $n = 2$.

If we equate $\partial\phi/\partial n$ to zero, putting $\psi(y+1) = \psi(y)$, and take the logarithms of both sides, we shall obtain a linear equation in n, satisfied by only one value of n for any given value of y. This value of n comes between 1 and 2 and, as $\phi(y, 1) = \phi(y, 2) = 0$, it follows that $\phi(y, n)$ is positive when n lies between 1 and 2, and negative when $n > 2$, and therefore that f is an increasing function of x in the former case and a decreasing function of x in the latter case.

When $x = 1$, $f = 1 - 2^{(2-n)/n}$ and as f represents cos XOY, we have the following results:

(1) *For any given $n > 2$, and for a given angle XOY, there exists one and only one independent pair of points P, Q on OX and OY, respectively, such that $\overline{PQ}^n = \overline{PO}^n + \overline{OQ}^n$, provided the angle XOY satisfies*

$$\text{arc cosine}\,(1 - 2^{-(n-2)/n}) \leqq XOY \leqq 90^\circ;$$

otherwise such a pair of points does not exist.

(2) *For n between 1 and 2, and for a given angle XOY there exists one and only one pair of points satisfying the condition, provided the angle XOY satisfies*

$$\text{arc cosine}\,(1 - 2^{(2-n)/n}) \geqq XOY \geqq 90^\circ;$$

otherwise there is no pair of points to satisfy the condition.

(3) *In either case this range decreases as n approaches 2, condensing to a single angle 90°, but for 90° any pair of points is a solution and there is no limit to the number of independent solutions.*

We can also state now this theorem:

If two triangles have an angle of one equal to an angle of the other, and if in each triangle the n-th power of the side opposite is equal to the sum of the n-th powers of the other two sides, then, for n greater than 1 and different from 2, the triangles are similar.

IV. Note on Trigonometric Functions.

By Elijah Swift, University of Vermont.

Mr. R. S. Underwood's note in this Monthly, *1921*, 374 (see also *1922*, 255, 346), concerning the irrationality of certain trigonometric functions, leads me to supplement his statements in one particular.

We call the sine and cosine transcendental functions and one might assume that their values as given in the usual table were, for the most part, transcendental numbers. So far from that being the case, none of the values of the natural functions listed in the usual trigonometric table are transcendental.

The proof is immediate. The angles whose functions are listed are all commensurable with π, being given in degrees and fractions of degrees. But if $\theta = m\pi/n$, where m and n are integers, then $\sin n\theta = \sin m\pi = 0$. But there is a well-known formula giving $\sin n\theta$ as a rational algebraic function of $\sin \theta$, if n is odd, or of $\sin \theta$ and $\cos \theta$, if n be even. In the latter case the equation may be rationalized in $\sin \theta$, and in either case on substituting zero for $\sin n\theta$ we find that $\sin \theta$ satisfies a rational algebraic equation with rational coefficients and is consequently an algebraic number. The fact that the values of the other trigonometric functions of θ are algebraic follows from the formulas connecting them with the value of $\sin \theta$.[1]

V. Slope of a Curve in Polar Coördinates at the Pole.

By H. J. Ettlinger, University of Texas.

The pole is an exceptional point in polar coördinates. For example to every point in the plane, except the pole, there corresponds one pair of coördinates, (ρ, θ), $\rho > 0$ and $0 \leqq \theta < 2\pi$. Conversely to each pair of such numbers there corresponds one point in the plane. The pole, however, corresponds to the single coördinate $\rho = 0$ and conversely.

At the pole, this fact produces a singular situation with respect to the slope of a curve, $\rho = f(\theta)$ or $\theta = F(\rho)$, which passes through it. At an ordinary point, the slope of the tangent is given as $m = \tan \varphi$, where $\varphi = \theta + \psi$ and ψ is the angle formed by the tangent at P with the radius OP. By the usual method, we find

$$\tan \psi = \rho \frac{d\theta}{d\rho} \cdot \tag{1}$$

At the pole this equation yields $\psi = 0$ and hence

$$\varphi = \theta. \tag{2}$$

To give a meaning to θ at $\rho = 0$, we consider a point P on the curve whose coördinates are $(\Delta\rho, \bar{\theta})$. As $\Delta\rho$ approaches zero, if $\bar{\theta}$ approaches a limit θ_0, this is the direction of the tangent to the curve at the pole. This justifies equation (2). To find this value of θ_0, we solve $f(\theta) = 0$ or $\theta_0 = F(0)$, where $F(\rho)$ is supposed continuous for $\rho = 0$. If the equation of the curve is so defined that to each value of ρ there corresponds only one value of θ, the slope of the tangent at the pole is uniquely determined as

$$m = \tan \theta_0.$$

[1] On the other hand, any angle whose radian measure is a rational number does have transcendental numbers for the values of its trigonometric functions. See F. Klein, *Famous Problems of Elementary Geometry*, translation by Beman and Smith, Boston, 1897, p. 77.

RECENT PUBLICATIONS.

REVIEWS.

First Course in the Theory of Equations. By L. E. Dickson. New York, John Wiley and Sons, 1922. 8vo. 6 + 168 pages. Price, $1.75.

Elementary algebra like elementary physics is less a single subject than an aggregate of distinct sciences loosely related by analogy of material and method. Even when the scope is restricted as here to the theory of equations, there need be no obvious continuity in the range of topics covered. The textbook under review is devoted to the customary questions together with a few important additions. The particular sequence followed is satisfactory but perhaps no better than any one of a dozen other possible arrangements of the same material. There is an unevenness in the book that is inherent in any such aggregation of separate studies. Some chapters are concrete to the extent that details of technique will occupy most of the student's attention, while other portions are so abstract that only the author's careful exposition and fortunate selection of proofs for exercises saves the subject from being entirely beyond the reach of the elementary student. Just what is to be regarded as elementary and what as advanced is perhaps more a question of the manner and style of exposition and the character of the previous preparation of the student than an inherent feature of the subject itself. For example, certain treatises on the calculus are surely beyond the grasp of the college freshman, while other treatments are perhaps ideal material for first-year work at institutions of high grade. Part of the difficulty that many of the less alert students are sure to find in a course based upon this or any similar book lies in the very instability of the problem, if one may venture so harsh a figure. No sooner has one point of view been attained than an entirely new line of investigation demands a change of attitude. The reviewer feels that many plodding students would find it easier to follow a suitable, connected course in even such a subject as the Galois theory, regularly regarded as advanced, than a course such as this which is more traditional but less unified. He would raise the question as to whether the investigation of numerical approximations to the roots of an algebraic equation here featured does not fit more readily into a course on numerical methods (a course which might develop the theory of least squares, interpolation, and numerical integration) than into a discussion devoted to such items as the representation of any symmetric function in terms of the elementary symmetric functions, and the impossibility of trisecting a general angle under the usual restrictions on method. However, traditional reasons will render this particular selection of subject matter acceptable to many teachers who may not be enthusiastic over the new material but who would miss any topic now included had it been omitted from this but not from older works.

The chapter headings[1] give a fair idea of material embraced in this text, but

[1] I: Complex numbers, 1–10; II: Elementary theorems on the roots of an equation, 11–28; III: Constructions with ruler and compasses, 29–44; IV: Cubic and quartic equations; their discriminants, 45–54; V: The graph of an equation, 55–70; VI: Isolation of the real roots of a real equation, 71–85; VII: Solution of numerical equations, 86–100; VIII: Determinants, systems of linear equations, 101–127; IX: Symmetric functions, 128–142; X: Elimination, resultants and discriminants, 143–154; Appendix: The fundamental theorem of algebra, 155–158.

fail to suggest the surprising richness of content and elegance of proof that mark the entire work. It is seldom that an American author succeeds in incorporating so much solid information in an elementary text of this size. This has been accomplished by careful discussions, judicious illustrative examples, and a relegation of all subsidiary material to exercises. The exercises in fact constitute a significant part of the work. Few of them are extremely hard and none are trivial. They may come as a shock to students fed on the sort of drill so common in freshman texts consisting of nothing but substituting numbers in literal formulas. They will require much more class-room discussion but the hints that are given should bring them within the range of well-prepared students.

Instructors who have used the author's *Elementary Theory of Equations* with undergraduates will find much familiar material here. They will note some corrections (for example, the author here defines the minor of a general determinant, while in the other book the definition was only to be inferred from a special case), but the most grateful feature is the revision and simplification of the subject matter and the very greatly enriched discussions. The work is indeed distinctly simplified and without any serious loss in completeness. Most of the interesting technical details that have been omitted for the sake of simplification are referred to in footnotes.

Fledglings are notoriously incapable of long flights. Any discussion which requires the student's consecutive attention for more than a page of distinct logical steps unrelieved by illustration or summary, exhausts and almost eliminates most students and is pedagogically indefensible. The individual items may be clear and simple but a many-linked chain of reasoning may easily drag down a student and can seldom support him. The most serious pedagogical criticism of the book lies in the failure to provide breathing spaces in one or two of the more extended discussions (for example, in paragraph 31, "Cubic equations with a constructible root"). Complicated statements of theorems tend to be meaningless for most students, but this difficulty is usually inherent in the subject, and the manner of exposition is not to be criticized. (Compare, page 81, "If the discriminant Δ of $z^4 + qz^2 + rz + s = 0$ is negative, there are two distinct real roots and two imaginary roots; if $\Delta > 0$, $q < 0$, $L > 0$, four distinct real roots; if $\Delta > 0$ and either $q \geqq 0$ or $L \leqq 0$, no real roots." Here $L = 8qs - 2q^3 - 9r^2$.)

But whatever the objections one may raise, there is no doubt that the book can be used with profit in classes with sophomores. The poorest students cannot fail to acquire much important and interesting information, and good students will have their eyes opened to many fruitful fields of mathematical thought. At all stages the book hints of other unexplored territories and is as suggestive an incitement to the ambitious beginner as could be asked. There is no need to dwell upon the logical neatness with which the author usually succeeds in saying exactly what he intends. All through the subject of algebra there are scattered pitfalls for the unwary and few texts are not guilty of laxity if not of deliberate error in the statement of definitions or theorems. The most delightful element

of this book is that the statements made do not require modification by implicit restrictions to be inferred from the context. The more fundamental the notions used, the more satisfactory is the clearcut language. We need more texts like this in America, books that can be used in every college, but which represent from elementary aspects the finest of mathematical scholarship.

A few minor infelicities may be mentioned: A complex number is discussed under two distinct representations. The less common is called explicitly (page 3) "the trigonometric form," while the other remains unnamed. Immediately after emphasizing the fact that a complex number has many amplitudes differing by integral multiples of 360°, the term, "the amplitude" is used without a hint as to which one is "the amplitude." The terms, "linear factor" and "factored form" are defined in the quadratic case, while the terms are then employed in connection with the general polynomial without further discussion. A product constituting a *polynomial* is called the factored form of an *equation*. In connection with mathematical induction the author speaks of "changing n into $n + 1$." The phrase, "upper limit to the real roots" is defined as a single concept without any inquiry as to the meaning of the word, "limit," or the sense of "upper," but in the exercises, the words "lower limit" are mentioned casually as though then familiar to the reader. "The numbers (10) are known as *Cardan's Formulas*." The discriminant, after being defined in general on page 47, is redefined for the quartic on page 51. "A point on the graph at which the tangent is both horizontal and an ordinary tangent is a bend point. . . ." In the exercises on page 74 the letters must denote only real quantities although no such warning is given. But minor blemishes such as these and rare misprints do not seriously impair the usefulness of a simple but scholarly text.

ALBERT A. BENNETT.

Mathematical Philosophy. A study of fate and freedom. Lectures for educated laymen. By CASSIUS J. KEYSER. New York, E. P. Dutton & Company, 1922. 8vo. 14 + 466 pages. Price $4.70.

In the preface of this book its author expresses the hope that these lectures may not be ungrateful to the following two classes of readers, among others:

"To the growing class of such professional mathematicians as are not without interest in the philosophical aspects of their science. To the growing class of such teachers of mathematics as endeavor to make the spirit of their subject dominate its technique."

The lectures were designed primarily for students whose major interest is in philosophy, but the present review is restricted to a consideration of the merits of the book for the two classes of readers just noted. Such readers will find in this volume much that is inspiring, much that they will enjoy to re-read, much that will be instructive and will lead them to look at subjects from a new point of view. Comparatively few of these readers will probably find here the enduring qualities of real mathematics, but they will find certain views which will extend their horizon as regards the nature and bearing of real mathematics and which will enable them to present their subject in a more popular form.

The mathematical knowledge presupposed on the part of the reader is limited to those facts about algebra, geometry and trigonometry which a capable student can acquire in one collegiate year, but the more mature student of mathematics will evidently read with much more insight the chapters on such subjects as transformation, invariance, the group concept, variables and limits, infinity, hyperspace, and non-euclidean geometries. In fact, certain parts of the chapters just noted cannot be read with much satisfaction by mathematical students whose maturity is below that of the ordinary beginning graduate, but the less mature student is likely to be attracted by the style to read many things that he does not fully comprehend, and such reading is sometimes very profitable.

The first mathematical subject which receives serious attention is the postulates of geometry. The second lecture is explicitly devoted to this subject and in several of those which follow the same subject is more fully elucidated in a masterly manner. This does not imply that the critical mathematical readers will agree with the author as regards all the details. In particular, some of these readers will doubtless disagree with the following definitions found on page 114. "What I wish now to say is that any geometry built upon a postulate system containing Euclid's parallel-postulate, or its equivalent, is called Euclidean, however widely it may differ in other respects from Euclid's Elements; and, correspondingly, any geometry, like that of Lobachevski or that of Riemann, whose postulate system contains a contradictory of Euclid's parallel-postulate, is said to be non-Euclidean, no matter how much it may be like Euclid's Elements in other respects. Such are the specific and more usual senses in which these familiar adjectives are employed in the literature of geometry."

Many mathematicians would doubtless have substituted for this definition of euclidean geometry one based on the group concept as is done, for instance, on page 344 of tome 3, volume 1, of the *Encyclopêdie des Sciences Mathêmatiques.* It is true that in the early period of the development of non-euclidean geometry only one geometry besides the euclidean seemed possible, as is also stated on page 3 of this encyclopedia, and the definitions quoted above have been extensively used. In fact, Professor Keyser would probably not wish that all his readers should agree with him in every case. His object seems to have been a higher one, as he dared to speak of many things in regard to which differences of opinion can normally be expected and in regard to which he did not claim expert knowledge. Some of his observations even on these subjects are valuable as the present writer can testify after re-reading the chapter of 33 pages on the group concept.

The teachers of mathematics who are mainly interested in enlarging their technical knowledge will doubtless find other books more helpful than the one under review, but those who are looking for a lucid exposition of some of the fundamental notions of mathematics, especially as regards their presence in other fields of human interest, will find here a richness which they cannot well afford to miss. Professor Keyser has for a long time stood at the head of American mathematicians as regards a certain type of popularization of mathematics, and

the present volume seems to embody his most significant efforts in this direction thus far. We hope others may follow for our subject needs popularization as may be seen from some of the attacks thereon.

G. A. MILLER.

Stereometrie. By KARL ROHN, with an introductory note by FELIX KLEIN. Leipzig, Robert Noske, 1922. xvi + 188 pages. Price in Germany, $1, American currency.

This work was substantially ready for publication at the time of Professor Rohn's death, in August, 1920,[1] the necessary completion of the manuscript in minor details having been done by his friend and former pupil Dr. Friedrich Wünschmann. Dr. Rohn was himself a pupil of Professor Klein, and the latter, in his appreciative introduction, speaks highly of his skill in the field of geometry.

The work sets forth in succinct form the essential features of modern projective geometry with respect to solids, thus extending the ordinary treatment of the projective properties of figures in a plane to those of three dimensions. It begins with a review of plane geometry (50 pages) and then considers the sphere, cylinder, and cone, proceeding later to the properties of conic sections and other plane figures in space.

The work shows a return to the better type of German bookmaking of prewar days and will be welcomed by students of modern geometry as an aid to their advanced work in this field.

DAVID EUGENE SMITH.

NOTES.

Professor SOLOMON LEFSCHETZ, of the University of Kansas, is now one of the collaborators on the *Bulletin des Sciences Mathématiques*, Paris. In the issue for December, 1922, pages 417–424, the three reviews of recent publications are by him. They are of L. SILBERSTEIN, *The Theory of General Relativity and Gravitation*, University of Toronto Press, 1922; G. C. EVANS, *Functionals and their Applications*, American Mathematical Society, 1918; and O. VEBLEN, *Analysis Situs*, American Mathematical Society, 1922.

The *Bulletin des Sciences Mathématiques* for December, 1920 (series 2, volume 44) devotes forty-one pages (297–337) to a review by E. CARTAN of Sir THOMAS MUIR, *The Theory of Determinants in the Historical Order of Development*, volumes 1–3 (London, 1906, 1911, 1920, see this MONTHLY, *1920*, 419). In this review an account is given chapter by chapter of the contents of the three volumes, bringing the history of determinants down to 1880. The reviewer points out how difficult it is in mathematics and in all branches of science to determine the paternity of any important theorem or discovery, and the inestimable value of the work of Sir Thomas in securing for us this information in the case of determinants.

[1] See this MONTHLY, *1921*, 43.

The *Bulletin of the Calcutta Mathematical Society* has been published continuously since April, 1909, and the third number of volume 12 was issued in December, 1921. The first volume of a new periodical, *Journal of the Department of Science*, was published by the University of Calcutta in 1919. The first part of 70 pages was devoted to papers in Chemistry; the second of 208 pages to papers in Mathematics; then Physics (43 pages) and Botany (5 pages). The 14 mathematical papers dealt almost wholly with topics of applied mathematics. The second volume of the *Journal*, 1920, contained Mathematics (186 pages, 17 papers), Physics (91 pages), and Botany (36 pages); the third volume, 1921, Mathematics (236 pages, 20 papers), Physics (54 pages), and Botany (8 pages). In the later volumes there were several papers on pure mathematics: for example, "Origin of the Indian cyclic method for the solution of $Nx^2 + 1 = y^2$" (volume 2, 69–76), "A note on Whittaker's formula for the solution of algebraic or transcendental equations" (volume 3, pp. 41–44).

Just forty years ago appeared a memoir of AUGUSTUS DE MORGAN (1806–1871) with a selection from his correspondence, and a fairly complete list of his publications; it was prepared by his wife Sophia (Frend). De Morgan was one of seven children and he had seven children himself. His eldest son was William Frend De Morgan (1839–1917), artist, potter, inventor, and finally, at the age of 67, novelist; he was the author of *Joseph Vance* (1906), *Alice-for-Short* (1907), *Somehow Good* (1908), and other notable books. "I paid no heed," he said, "to the wisest and best man I have ever known—my father, of course—and went my own headstrong way. . . I went my own way and wasted an odd forty or fifty years." "If you work hard, Willy," his father had said to him, "you will live to write something worth reading." To her list of delightful biographies, Mrs. Anna M. W. Stirling has now added: *William De Morgan and his Wife* (New York, Holt, 1922). The mathematician will naturally turn with interest to this volume where, among other things, there is new information concerning Augustus De Morgan's Anglo-Indian ancestry.

The third and last part (15 + 945–1509 pages) of *Jahrbuch über die Fortschritte der Mathematik*, volume 45, for 1914–1915, was published in September, 1922; compare *1920*, 268, and *1921*, 315. The price of this part for Germany is (November, 1922) 920 marks; for the United States, $7.30 plus cost of postage and packing. The publishers, Vereinigung wissenschaftlicher Verleger, Berlin, demand $20.95 for the complete volume of 1524 pages. For volume 42, 1211 pages, published before the war and on far better paper, the price was $9.25. In the list of over one hundred "Mitarbeiter" for volume 45 is the name of Professor W. C. GRAUSTEIN of Harvard University.

The notable list of works in Teubner's "Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen" is well known; but except for Pringsheim's *Vorlesungen über Zahlenlehre*, 1916–1921, no works have lately been added to this series. The leading publisher of new mathematical works in Germany now is Julius Springer of Berlin. He has recently started to issue a series of works under the general

heading "Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen." The series is under the general editorship of Professor RICHARD COURANT, Klein's successor at the University of Göttingen, assisted by Professor WILHELM BLASCHKE, of the University of Hamburg, and Professors MAX BORN and CARL RUNGE, of the University of Göttingen. At this date (November, 1922) three volumes have appeared. I: *Vorlesungen über Differential-Geometrie und Geometrische Grundlagen von Einsteins Relativitätstheorie* by W. BLASCHKE, 1921, 10 + 230 pages. This is simply the first of three volumes and deals with the elements of the subject. An English translation will probably be published in England within a year. II: *Theorie und Anwendung der unendlichen Reihen* by K. KNOPP, 1922, 10 + 474 pages. III: *Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen* by ADOLF HURWITZ, *ergänzt durch einen Abschnitt über geometrische Funktionentheorie* by R. COURANT, 1922, 11 + 399 pages. Professor Courant's section occupies pages 245–392. The fourth volume of the series, also announced for 1922, is to be: *Die mathematischen Hilfsmittel des Physikers* by E. MADELUNG. The volumes so far published are very valuable, exceedingly interesting, and issued in attractive form, on good paper, in both bound and unbound editions. It is to be hoped that Springer will not drive away American purchasers by extortionate charges such as he has recently made for *Mathematische Zeitschrift.*

Among recent German publications are the following (the prices are those for Germany, and the dates of publication are 1922 unless otherwise indicated): L. BIEBERBACH, *Differential- und Integralrechnung,* vol. 1 (Teubner's technische Leitfäden, vol. 4). Leipzig, Teubner (6 + 132 pp. 24 marks)—L. BIEBERBACH, *Lehrbuch der Funktionentheorie, Band 1: Elemente.* Leipzig, Teubner, 1921 (6 + 314 pp. 70 marks)—H. DINGLER, *Relativitätstheorie und Ökonomieprinzip,* Leipzig, Hirzel (77 pp. 30 marks)—H. DÖLP, *Grundzüge und Aufgaben der Differential- und Integralrechnung nebst den Resultaten,* neu bearb. von E. Netto, 16 Aufl. Giessen, Topelmann (2 + 214 pp. 24 marks)—A. EINSTEIN, *Vier Vorlesungen über Relativitätstheorie gehalten im Mai, 1921, an d. Universität Princeton.* Braunschweig, Vieweg (3 + 70 pp. 60 marks)—J. E. GERLACH, *Kritik der mathematischen Vernunft.* Bonn, F. Cohen (162 pp. 65 marks, bound) —A. MOSZKOWSKI, *Einstein, Einblicke in seine Gedankenwelt. Gemeinverständliche Betrachtungen über die Relativitätstheorie und ein neues Weltsystem. Entwickelt aus Gesprächen mit Einstein.* 35–40th thousand. Berlin, F. Fontane (240 pp. 45 marks, bound)—G. SALMON- W. FIEDLER, *Analytische Geometrie des Raumes.* Unter Mitwirkung von A. Brill neu herausgegeben von K. Kommerell. Teil 1, Die Elemente u. d. Theorie d. Flächen zweiter Ordnung. 5. Aufl. Lieferung 1, Leipzig, Teubner (10 + 366 pp. 180 marks)—L. SCHRUTKA, *Elemente der höheren Mathematik für Studierende der technischen und Naturwissenschaften.* 2. verb. Aufl. Leipzig and Vienna, F. Deuticke, 1921 (30 + 635 pp. 68 marks).

This year marks the fortieth anniversary of one of the most distinguished of mathematical journals, *Acta Mathematica.* This periodical is of particular

interest to American scholars on account of its consistently international character and high scholarly standing. The recent outbreak of nationalistic jealousies renders of timely interest an account of the circumstances leading to the establishment of this famous quarterly. No small part of the popularity of the *Acta Mathematica* has been the prolific writings of POINCARÉ that have appeared in its pages.

To quote briefly from a prospectus,

"The periodical was founded under the inspiration and with the active encouragement of the greatest mathematicians of that time: CHARLES HERMITE and KARL WEIERSTRASS.

"Feeling deeply that mathematical science has a place which is independent of local and national points of view, these two pioneers in abstract thought were able to disengage themselves from the animosity, which the then recent war had quite naturally excited among the belligerents.

"This animosity had been such as to threaten to break even the scientific ties between two great nations. CHARLES HERMITE and KARL WEIERSTRASS wished to reestablish these connections as soon as possible, and gave as their opinion that mathematicians, whose subject was fundamental in every other science, ought here to take the first step. . . .

"A committee of the most noted mathematicians of the four Scandinavian countries assumed scientific control of the periodical, the first part of which was issued in 1882. The first page is illustrated by a reproduction of the only original portrait in existence of NIELS HENRIK ABEL, the greatest mathematician of Scandinavia and the intellectual teacher—though not, it is true, through personal contact—of HERMITE and WEIERSTRASS. The periodical begins with the memoir:

"'Théorie des groupes fuchsiens' by the young and still unknown pupil of the 'Ecole polytechnique' and 'Ecole nationale des Mines' of France, HENRI POINCARÉ (born in 1854, d. in 1912), he who during his short life was—as PAUL PAINLEVÉ once said—'le cerveau vivant des sciences rationnelles, mathématiques, astronomie, physique, cosmogénie, géodésie, l'inventeur incomparable qui a tout embrassé, tout pénétré, tout approfondi.'

"Nearly every one of the first 35 volumes of the *Acta Mathematica* (vols. 1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 16, 20, 21, 22, 26, 29, 31, 32, 33, 35) contains one or several memoirs by POINCARÉ, all characterized by the penetrating power of thinking, the genius for discovery, and the masterly insight into the greatest problems of science, on which Painlevé had dwelt. The works published by Poincaré in the *Acta Mathematica* occupy in all 1,468 pages."

Since our last report (*1921*, 316) concerning *Publications of the Massachusetts Institute of Technology*, "Contributions from the Department of Mathematics," twenty parts (nos. 23–42, November, 1921–June, 1922) have been published. These parts contain articles by F. L. HITCHCOCK, JOSEPH LIPKA, C. L. E. MOORE (2), H. B. PHILLIPS (3), L. H. RICE (2), GEORGE RUTLEDGE, J. S. TAYLOR, NORBERT WIENER (5), S. D. ZELDIN (2), in addition to joint papers by F. L. HITCHCOCK and N. WIENER, and J. L. WALSH and N. WIENER, reprinted from: *Bulletin of the American Mathematical Society* (see *1922*, 73), *Comptes Rendus du Congrès International des Mathématiques à Strasbourg* (*1920*, 440; *1921*, 378), *Journal of Mathematics and Physics* (*1921*, 380–381), *Proceedings of the London Mathematical Society, Proceedings of the National Academy of Sciences* (*1922*, 75–76). The papers not listed in the above references are as follows:—In the *Journal*, volume 1: "Hyperquaternions" by Moore, 63–77; "Explicit determination of Cotes' coefficients for polynomial area" by Rutledge, 77–84; "A certain type of product and the combinatory analysis involved in its expansion" by Rice, 85–87; "The equivalence of expansions in terms of orthogonal functions" by Walsh and Wiener, 105–122; "Note on the normal planes to a surface in a space of four dimensions" by Moore, 147–156; "Commutativity of contact

transformations of mechanics" by Zeldin, 157–159; "On the expression of the sum of any two determinants as a determinant of more dimensions" by Rice, 160–166; "A new type of integral expansion" by Wiener, 167–176; and "Note on Einstein's theory of gravitation" by Phillips, 177–190. In *Proc. Lond. M. S.*, new series, volume 20: "The group of the linear continuum" by Wiener, 329–346. In *Proc. Nat. A.*, volume 8, 1921: "A solution of the linear matrix equation of double multiplication" by Hitchcock, 78–83.

ARTICLES IN CURRENT PERIODICALS.

American Journal of Mathematics, volume 44, no. 1, January, 1922: "An arithmetical dual of Kummer's quartic surface" by E. T. Bell, 1–11; "Incidences of straight lines and plane algebraic curves and surfaces generated by them" by A. Emch, 12–19; "On the theorems of Gauss and Green" by V. C. Poor, 20–24; "An extension of the Sturm-Liouville expansion" by C. C. Camp, 25–53; "Conformal transformations of period n and groups generated by them" by H. Langman, 54–86.

L'Enseignement Mathématique, volume 22, nos. 3–4, published July, 1922: "Familles additives et fonctions additives d'ensembles abstraits" by M. Fréchet, 113–129; "Sur les foyers rationnels d'une courbe algébrique" by P. Appell, 129–132; "Sur les foyers rationnels des courbes planes" by E. Turrière, 133–135; "Sur les tractrices et les courbes équitangentielles" by C. de Jans, 136–145; "Sur certaines identités géométriques et leur traduction algébrique by P. C. Delens, 146–152; "Sur le déplacement d'un point dans l'espace à n dimensions. Géométrie du n-èdre" by G. Tiercy, 152–167; "Sur les formules de Lorentz" by B. Niewenglowski, 167–169; "Applications géométriques de la crystallographie" (conclusion) by M. Winants, 170–194; "Déduction des dérivées de fonctions circulaires par la méthode géométrique des limites" by B. Petronievics, 195–208; "Déduction géométrique de l'expression pour le rayon de courbure" by J. M. Child and B. Petronievics, 209–214; "Camille Jordan (1838–1922)" by A. Buhl, 214–218; "Einstein au Collège de France" by R. Wavre, 219–222; "Bibliographie" and "Bulletin bibliographique," 228–248.

Isis, volume 4, no. 2, 1922: "The teaching of the history of science" by G. Sarton, 225–249 ["What is the present status of the teaching of the history of science in European and American universities? The lists and announcements of courses which have been published in various journals, may give the reader a very misleading impression. For these lists are many and some are quite long, but most of it is mere bluff. For example, many such courses have been extemporized in America, but, with one exception (Cajori), I do not know of any course delivered by a lecturer having the rank and emoluments of a professor and devoting himself exclusively to it. To be sure, some of these courses offered as a 'side show' by scholars whose main business is to study and to teach something totally different, may be very interesting. . . . We owe some of the best work in every field to the capricious efforts of *dillettanti*, but we can never depend upon them and we must of necessity expect the main advances in knowledge to be made by men whose sole duty is to make them and who give their every thought to it. . . . The *history* of *science* is a historical discipline; it is also a scientific discipline. The historian of science must have a sound knowledge of two sets of facts: historical facts and scientific facts. . . . Historical and scientific errors must be equally avoided, but many historical errors are only venial offences, while the scientific errors are deadly sins. The former, indeed, imply merely a misapprehension of the accessory circumstances, while the latter prove that the very substance of one's investigations has not been understood."]; 'L'enseignement de la mécanique en France au XVII^e siècle" by P. Boutroux, 276–294; "The development of trigonometric methods down to the close of the XVth century. (With a general account of the methods of constructing tables of natural sines, down to our days)" by J. D. Bond, 295–323.

Mathesis, volume 36, July, 1922: "Nouveaux triangles spéciaux" by J. Neuberg, 257–259 ["*L'Index du Répertoire bibliographique des Sciences mathématiques* contient la rubrique: *Triangles spéciaux.* *a.* Triangle isoscèle. *b.* Triangle équilatéral. *c.* Triangle rectangle. Il y manque évidemment la division: *d.* Autres triangles spéciaux. En effet, est nombreuse la liste des triangles particuliers que l'on recontre dans la Géométrie récente du triangle; elle fournit de bons exercices et d'intéressants sujets d'étude."]; "Sur la vie moyenne d'une obligation" by A.

Claeys, 260–261; "Sur l'isopôle d'une droite par rapport à un triangle" by C. Servais, 262–268; "Sur l'equation $x^3 + px + q = 0$" by M. Sterkens, 269–270; "Sur le point de Feuerbach" by P. de Lépiney, 271–274; "Remarques sur l' 'Arithmetique' de SIMON STEVIN" (continued) by H. Bosmans, 275–281; "Sur les points $\varphi, \varphi_a, \varphi_b, \varphi_c$" by R. Deaux, 282–285, "Notes mathématiques," 286–291; Questions and Solutions, 291–304.

THE MONIST, volume 32, no. 1, January, 1922: "The relation of space and geometry to experience" by N. Wiener, 12–60—No. 2, April: "The relation of space and geometry to experience" by N. Wiener (continued), 200–247—No. 3, July: "The relation of space and geometry to experience" by N. Wiener (conclusion), 364–394.

NATURE, volume 110, July, 1922: Review by W. E. H. B. of C. Tweedie, *James Stirling* (Oxford, 1922), 111—August 12: "The elliptic logarithmic spiral" by H. S. Rowell, 214—August 26: Review of H. Malet, *Etude géométrique des transformations birationnelles et des courbes planes* (Paris, 1921), 276—September 2: Review of P. Humbert, *Introduction à l'étude des fonctions elliptiques* (Paris, 1922), 308.

THE OBSERVATORY, volume 45, August, 1922: "Herschel's world-view in the light of modern astronomy" by H. Macpherson, 254–261; "Jacobus Cornelius Kapteyn," 261–265; "From an Oxford note-book," 271–272 [Reminiscences of Kapteyn]—September: "Memorial to Sir Norman Lockyer," 277–280; "Flamsteed's letters to Richard Towneley" by J. L. E. Dreyer, 280–294; Review by H. Jeffreys of H. Weyl, *Space—Time—Matter* (London, 1922), 297–301.

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE U. S. A., volume 7, November, 1921: "On the approximate solutions in integers of a set of linear equations" by H. F. Blichfeldt, 317–319—December: "Group of isomorphisms of a transitive substitution group" by G. A. Miller, 325–328; "Einstein static fields admitting a group G_2 of continuous transformations into themselves" by L. P. Eisenhart, 328–334; "Geometric aspects of the Abelian modular functions of genus four (II)" by A. B. Coble, 334–338—Volume 8, January, 1922: "The Riemann geometry and its generalization" by L. P. Eisenhart and O. Veblen, 19–23; "Ricci's principal directions for a Riemann space and the Einstein theory" by L. P. Eisenhart, 24–26; "Note on the definition of a linear functional" by C. A. Fischer, 26–29—March: "On the relation of a continuous curve to its complementary domains in space of three dimensions" by R. L. Moore, 33–38—April: "A solution of the linear matrix equation by double multiplication" by F. L. Hitchcock, 78–83—June: "On the location of the roots of the derivative of a polynomial" by J. L. Walsh, 139–141—July: "Normal coordinates for the geometry of paths" by O. Veblen, 192–197; "Principal directions in a Riemannian space" by J. L. Synge, 198–203; "Principal directions in the Einstein solar field" by J. L. Synge, 204–207; "Fields of parallel vectors in the geometry of paths" by L. P. Eisenhart, 207–212—August: "Spaces with corresponding paths" by L. P. Eisenhart, 233–238; "Number of substitutions omitting at least one letter in a transitive group" by G. A. Miller, 238–240—September: "The meaning of rotation in the special theory of relativity" by P. Franklin, 265–268.

REVUE GÉNÉRALE DES SCIENCES, volume 33, July 15, 1922: "La vitesse-limite de la lumière et le finitisme" by B. Petronievics, 401–402—August 15–30: "J. C. Kapteyn" by C. H. Hins, 449–450—September 15–30: "Pour servir à l'histoire de la nomographie" by R. Soreau, 518–523.

SCIENCE, new series, volume 56, August 25, 1922: Review by G. A. Miller of C. J. Keyser, *Mathematical Philosophy* (New York, 1922), 229–230—September 1: "The algebraic method of balancing a chemical equation" by H. A. Curtis, 258–260—October 13: "The theory of numbers" by G. H. Hardy, 401–405 [From the address given at the Hull meeting of the British Association for the Advancement of Science, September, 1922]; "An unusual solitaire game" by L. E. Dickson, 418–419; Review by G. A. Miller of W. F. Osgood and W. C. Graustein, *Plane and Solid Analytic Geometry* (New York, 1921), 420–421; "The American Mathematical Society" by R. G. D. Richardson, 423 [Report of the summer meeting held at Rochester, September 7–8, 1922].

SCIENCE PROGRESS, volume 16, April, 1922: "Applied mathematics" by S. Brodetsky, 517–527 [Recent advances in relativity, etc.]; "The Einstein theory of relativity" by D. Laugharne-Thornton, 641–643—Volume 17, July: "Mathematics" by F. P. White, 1–12 [Recent advances]—October: "Mathematics" by F. P. White, 173–180 [Recent advances]; "Applied mathematics" by S. Brodetsky, 180–190 [Recent advances in relativity, etc.]; Review of L. E. Dickson, *First Course in the Theory of Equations* (New York, 1922), 328–329.

Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht, vol. 52, nos. 11-12, November 30, 1921: "Ueber ein Verfahren zur Veranschaulichung der Konvergenz unendlicher Reihen" by E. Dintzl, 249-253; "Direkte Herleitung des relativistischen Dopplerprinzips und der zeitlichen Lorentztransformation aus den nichtrelativistischen Gleichungen Dopplers" by H. Meurer, 254-257; "Beweis des pythagoreischen Lehrsatzes mit Hilfe des Satzes von Menelaus" by J. Salachowski, 257-258; "Ein Modell zu den Sätzen des Ceva und des Menelaos" by E. Lipken, 258-259; "Die Simpsonsche Regel" by A. Witting, 259 [An especially simple derivation of the rule]; "Ueber drei stereometrische Aufgaben" by W. Gaedecke, 260-261; "Ein Nomogramm für die Zinseszinsformel" by P. Hauck, 261-263; "Ueber die mathematischen Bezeichnungen im Unterricht" by W. Lietzmann, 266-267 [Synopsis of the recommendations of the (American) National Committee on Mathematical Requirements, concerning this topic]; "Bücherbesprechungen," 269-282—Volume 53, nos. 1-2, January 20, 1922: "Zur Entwicklung der mathematischen Erfindungsgabe" by P. Maennchen, 2-7; "Der Schenkel-Transversalensatz" by H. Dörrie, 8-14; "Herleitung der Lorentztransformation eines Längenabschnittes durch Vergleich der relativistischen mit den nichtrelativistischen Gleichungen des Dopplerprinzips" by H. Meurer, 15-17; "Winkel an Gleichlaufenden" by C. H. Tietjen, 17-18; "Die Winkelmessung des Artilleristen" by P. Lötzbeyer, 18-20; "Zur Konstruktion des Apollonischen Kreises" by E. Lipken, 20; "Dreikant und Polarkant" by C. Stengel, 21-22; "Zur Berechnung der Kugelfläche" by A. Czwalina, 22-23; "Der Krümmungskreis in einem Punkte einer Ellipse" by G. Diem, 23-25; "Aufgaben-Repertorium," 25-31; "Die Internationale Mathematische Unterrichtskommission" by W. Lietzmann, 31-33; "Die Mathematikerversammlung in Jena 1921" by W. Lietzmann, 34-35; "Bücherbesprechungen," 41-51—Nos. 3-4, March 15: "Die Erbteilungsaufgaben bei Muhammed ibn Musa Alchwarazmi" by H. Wieleitner, 57-67; "Die Spiegelung als primitiver Begriff im Unterricht" by H. Willers, 68-77; "Die Ermittlung der molekularen Grössenordnung im Unterricht" by H. Hermann, 77-81; "Ein Paradoxon der Gravitation" by H. Teege, 81-84; "Eine einfache Wahrscheinlichkeitsaufgabe die auf die Zahl e führt" by W. Simons, 84-85; "Bücherbesprechungen," 87-95—Nos. 5-6, May 12: "Die Spiegelung als primitiver Begriff im Unterricht" (conclusion) by H. Willers, 109-119; "Verallgemeinerung der Cardanischen Formel" by K. Siemon, 120-126; "Die verschiedenen Methoden zur Lösung von Aufgaben der darstellenden Geometrie bei ungünstigen Lageverhältnissen" by A. Baruch, 126-133; "Die Cheopspyramide als Fundgrube mathematischer Schulaufgaben" by M. Zacharias, 133-135; "Aufgaben-Repertorium," 136-139; "Bücherbesprechungen," 146-150—Nos. 7-8, July 20: "Die Stellung der Mechanik zwischen Physik und Mathematik" by W. Lietzmann, 153-161; "Die verschiedenen Methoden zur Lösung von Aufgaben der darstellenden Geometrie bei ungünstigen Lageverhältnissen" (concluded) by A. Baruch, 161-168; "Ableitung der sphärisch trigonometrischen Formeln aus der darstellenden Geometrie" by A. Launer, 168-171; "Bandknoten" by W. Bastiné, 172-174; "Beitrag zur Behandlung der Sätze über die Winkel am Kreis" by W. König, 174-175; "Bücherbesprechungen," 183-189.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. Dodd, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

The Grinnell College Mathematics Club, Grinnell, Ia.

[*1922*, 78.]

The following meetings were held in 1921-1922:

September 22, 1921: Business meeting.

October 4: President's inaugural address by Robert Hannelly '23; "History of mathematical organization" by Professor R. B. McClenon.

October 18: "Trisection of an angle" by Elizabeth Pace '22.

November 1: "Applications of arithmetic" by Burton Jones '23; "Einstein, his life and work" by Margaret Divelbess '23.

November 15: "Report of the Committee on Mathematical Requirements" by Professor O. W. Albert; "Russian peasant method of multiplication" by Ruth Boyce '22.

December 6: "Vibrating strings" by Raymond Weigen '22.

February 14, 1922: "Mathematical requirements for agricultural study" by Louise Pinkerton '22; "Leibnitz" by Robert Hannelly '23.
February 28: "Theorems about circles" by Vera Kreger '24; "Properties of a triangle" by Professor H. Leypolt.
March 14: "Repeating decimals" by Burton Jones '23; "Mathematical recreations" by Ethel Warnick '24.
April 25: "Map projections" by Robert Atwater '23; "Permutations and combinations" by Robert Hannelly '23.
May 9: "Nine-point circle" by Marian Piersol '24; "Fourth dimension" by Ferrell Rodgers '24; "Composition of an atom" by Dr. Lloyd Taylor of the Department of Physics, University of Chicago.
May 23: "Life and work of Descartes" by Evelyn Forthun '24; "Work of Newton" by Harold Armstrong '24.
June 1: Picnic supper at the home of Professor McClenon. The following officers were elected for the year 1922–1923: President, Burton Jones '23; vice-president, Beulah Mott '24; secretary-treasurer, Edwin Kingery '24.

THE MATHEMATICS CLUB OF NORTHWESTERN UNIVERSITY, Evanston, Ill. [*1919*, 363.]

The following officers were elected for the first semester of 1921–1922: Laura Hill, president; James Bronnenberg, vice-president; Mildred Giesecke, secretary; Lawrence Fenner, treasurer; Dr. H. B. Curtis, faculty adviser.

During the first semester the following addresses were delivered: "Descartes' rule of signs" by Professor D. R. Curtiss; "Conformal representation" by Dr. Curtis, instructor; "Least squares" by Laura Hill; "Coördinate systems" by Professor E. J. Moulton; "The measurement of the earth" by Professor J. F. Hyford; "Line coördinates" by Professor C. E. Wilder.

The following officers were elected for the second semester: Elizabeth Bruschke '22, president; Thomas Stitt '23, vice-president; Esther Eisler '22, secretary; Margaret Furrey, assistant in Mathematics, treasurer; Dr. Curtis, faculty adviser.

During the second semester the following addresses were delivered: "Astronomical surveying" by Professor P. Fox; "Computing machines" by Dr. I. Roman, instructor; "Things mathematical" by Dr. A. J. Snow, instructor in Psychology; "Applications of mathematics to chemistry" by Professor W. J. Evans, of the Chemistry Department; "Mathematics in Ancient Central America" by Mr. Sidney Huguenin, assistant; "Mathematics according to scientists and philosophers" by Mr. Theodore Doll, instructor.

The Club gave a Beach Party on May 20, 1922.

(Report by Miss Bruschke.)

WHITE MATHEMATICS CLUB, UNIVERSITY OF KENTUCKY, Lexington, Ky. [*1921*, 389.]

March 9, 1921: "Magic squares" by Dewey Duncan '22.
March 16: "Early mathematical manuscripts of Leibnitz" by Mr. W. E. Payne, instructor.
March 30: "Differential equations from the Lie viewpoint" by Walter Armentrout, Gr.
April 6: "Mechanical devices for integrating differential equations of certain types" by Professor J. M. Davis.
April 13: "Mathematical fallacies" by Nelson Conkwright, Gr.
April 27: "Some applications of vector analysis to the theory of twisted curves" by Professor E. L. Rees.
May 25: "Some applications of mathematics to chemistry" by Eger Murphree, Gr.
October 18: The following officers were elected: President, Professor Flora LeStourgeon; secretary, Professor H. H. Downing.
November 1: "Relating to definitions, axioms, postulates, and assumptions" by Dewey Duncan '22.
November 15: "Class and discrete sequences" by Mr. W. J. Brezler, instructor.
November 29: "Time and its measurement" by Professor Downing.
December 15: "Denumerable and non-denumerable classes" by Mr. R. V. Blair, instructor.

January 22, 1922: "Vector treatment of certain plane concurrences" by Professor Rees.
February 10: "Why is it impossible to trisect an angle?" by Professor J. M. Davis.
February 23: "The real positive number system" by Mr. G. A. Seubert, instructor.
March 9: "How to read secret messages" by Professor P. P. Boyd.
March 24: "Applications of inversion to geometrical constructions" by Nelson Conkwright, Gr.
April 6: "Riemann surfaces for the function $w = z^3 + 3z$" by Vada Nelson, Gr.
April 20: "Van der Waal's and related equations" by Castle Foard, Gr.
May 4: "Flatland" by Helen McGurk '23.
May 18: "Some applications of complex variables to hydrodynamics" by Leland Snoddy, Gr.

(Report by Professor Downing.)

Pi Mu Epsilon, University of Pennsylvania, Philadelphia, Pa. [*1922*, 79.]

In the fall of 1921, a chapter of Pi Mu Epsilon, the honorary mathematical fraternity, was established at the University of Pennsylvania, taking the place of "The Vinculum." The following officers were elected: Director, Professor E. S. Crawley; vice-director, Dr. J. Minnick; secretary, Ella Rosentoor '22; treasurer, Esther Bernstein '23; librarian, Mabel Kessler '22; executive committee, the officers, Thomas Benton '23, and Edward Weyl '23; scholarship committee, Professor F. H. Safford, Professor H. H. Mitchell, Thomas Benton '23, Marion Bromily '22 and Louisa Lotz '22.

The following papers were presented:

November 18, 1921: "The numbers of Bernoulli" by Thomas Benton '23.
December 16: "The area under the equilateral hyperbola" by Professor M. J. Babb.
January 20, 1922: "The solution of the inverse function" by Perry Caris, Gr.; "Problems in the theory of numbers" by Professor Crawley.
February 17: "The circulating decimal" by Enos Witmer, Gr.
March 17: "The new Thorndike tests" by Professor Mitchell.
April 21: New members were initiated. The following officers were elected: Director, Professor Crawley; vice-director, Professor Mitchell; secretary, Helen Lotz '23; treasurer, Thomas Benton '23; librarian, Marion Bromily '22; executive committee, the officers, Esther Bernstein '23, and Enos Witmer, Gr.; scholarship committee, Professor Safford, Mr. H. M. Gehman, instructor, Marie Wilhelmi '22, Georgina Yeatman '23, and Tobias Hahn, Gr.
May 19: "Available standard tests and measurements" by Dr. J. T. Rorer, of the William Penn High School for Girls. Discussion was led by Professor G. G. Chambers. This was an open meeting to which all high-school teachers of mathematics were especially invited.

(Report by Miss Lotz.)

The Pentagram, University of Texas, Austin, Tex. [*1921*, 275.]

The following officers were elected for the year 1921–1922: President, Mary Cook '23; secretary, Lewis Vogelsang '23; faculty adviser, Professor A. A. Bennett. The following papers were presented:

October 20, 1921: "Rules of the game" by Professor Bennett.
November 3: "Science and society" by Professor M. B. Porter; "Magic squares" by Renke Lubben, Gr.
November 17: Social meeting.
December 1: "Logarithms" by Dean H. Y. Benedict; "Euclidean geometry" by Mr. C. M. Cleveland, instructor.
January 18, 1922: "Algebraic equations" by William Whyburn '23; "Mechanics of flying" by Professor H. J. Ettlinger.
February 15: "Squaring a circle" by Carriella Bell '23; "Fallacies of elementary geometry" by Professor R. L. Moore.
March 22: "Vector addition" by Professor J. M. Bryant, of the Department of Electrical Engineering.
April 19: "Life and works of Newton" by Dr. Goldie Horton, instructor.
May 10: "Synthetic methods" by Professor Bennett.
May 18: Sixth annual banquet. Toasts on the advanced courses in mathematics.

(Report by Mr. Vogelsang.)

The Pascal Circle, Trinity College, Washington, D. C. [*1920*, 425, 481.]

The officers for the year 1921–1922 were: Honorary president, Professor Marie Cecilia Mangold; president, Mary Kelley '22; vice-president, Martha Crowley '22; secretary, Helen Ormond '23; treasurer, Marguerite Dwyer '24. The following meetings were held:

October 18, 1921: Business meeting. It was decided that a period of probation of three months be set during which candidates for admission into the circle must maintain high rank in some class in college mathematics.

November 15: "Life and accomplishments of Blaisé Pascal" by Mary Kelley '22; "Number games" by Marie Costello '22.

December 6: Supper party to welcome the new members. An original mathematical entertainment including a one-act comedy "A Trinity Triangle," songs, recitations, toasts, with mathematical terms applicable to each member. Attractive menu cards were designed by Martha Crowley '22.

February 7: "Einstein's laws of gravitation and relativity" by Martha Crowley '22.

March 14: "David Eugene Smith's address on 'Religio Mathematici'" by Elizabeth Herbert '22. Paper folding, and catch problems in algebra.

April 4: "Short methods" by Professor Mangold. Discussion of "The clock problem" and "The age problem."

May 2: "The computation of the calendar, with special reference to Easter as an immovable feast" by Carol O'Brien '22.

May 29: The following officers were elected for the year 1922–1923: President, Margaret Kelly '23; vice-president, Margaret McAuliffe '23; secretary, Agnes Perrot '24; treasurer, Anna Foley '25.

(Reported by Miss Kelley.)

The Mathematics Club of Wellesley College, Wellesley, Mass.

The Mathematics Club of Wellesley College was started in the spring of 1921, with membership limited to the members of the junior and senior classes taking elective mathematics. The following officers were elected for the ensuing year: President, Mary Hankinson '22; vice-president, Margaret Merrell '22; member of the executive committee, Hope Anglemann '22. On October 14, the following officers were elected from the junior class: Secretary-treasurer, Mary Wheeler '23; member of the executive committee, Margaret Ingraham '23. Plans for the year were discussed; and Professor Helen Merrill told the Club about the meetings of the American Mathematical Society and the Mathematical Association of America, held in Wellesley in September.

The programs of the other meetings were as follows:

November 18, 1921: "The angle bisector and the use of two right angles in the solution of the cubic" by Miss Marion Stark, instructor; "Paradromic rings" by Eleanor Johnson '23; "How to draw a straight line" by Margaret Merrell '22; "Mechanical construction of the conics" by Grace Lybrand '23; "Russian peasant method of calculation" by Georgia Gambrill '22.

December 2: "Early days in the Mathematics Department" by Miss Ellen Hayes, formerly head of the Mathematics Department.

January 27, 1922: "The cycloid" by Professor J. L. Coolidge, of Harvard University. Open meeting.

February 24: Library meeting. Old and valuable mathematical books were exhibited.

March 17: "Stephen Leacock" by Mary Wheeler '23; "Lewis Carroll" by Jeannette Gruener '23; "π" by Mary Hankinson '22.

May 8: Social meeting at the home of Professors Smith and Copeland. Fallacies proven and disproven, original songs, refreshments.

May 19: "Paper-folding, with demonstration" by Edith Barrows '22. Officers for the year 1922–1923 were elected as follows: President, Mary Wheeler '23; vice-president, Alice Darnell '23; executive committee, Jeannette Gruener '23, Professor Smith.

(Report by Mary Wheeler.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

2949 (Corrected; see 1922, 29). Proposed by J. B. REYNOLDS, Lehigh University.

Find the lateral area of the cone with vertex at $(0, 0, h)$ and whose base is the epicycloid,

$$2x = a(3 \cos \theta - \cos 3\theta), \quad 2y = a(3 \sin \theta - \sin 3\theta).$$

2992. Proposed by AUGUSTUS BOGARD, Teresian University, Winona, Minn.

A semi-circle rotates at a uniform velocity about its diameter and slides along the line of that diameter at such a uniform rate as just to pass the full length of the diameter while making one revolution about it. Find the equation of the surface thus generated.

2993. Proposed by H. C. BRADLEY, Massachusetts Institute of Technology.

Let ABC be any triangle, and O the center of its circum-circle. Bisect the arcs AB, BC, and CA at F, D, and E. With F, D, and E as centers draw arcs passing in each instance through the adjacent corners of the triangle. Prove that these arcs intersect at the in-center of the triangle ABC.

2994. Proposed by R. M. MATHEWS, Wesleyan University.

Can the following construction be made without the use of a regulus? Construct a line which meets four given skew lines.

2995. Proposed by S. A. COREY, Des Moines, Iowa.

Give a geometric proof of each of the identities:

(a) $\cos (a + 2mx) = \cos a - 2 \sin x[\sin (a + x) + \sin (a + 3x) + \cdots + \sin (a + (2m - 1)x)]$,

and

(b) $\sin (a + 2mx) = \sin a + 2 \sin x[\cos (a + x) + \cos (a + 3x) + \cdots + \cos (a + (2m - 1)x)]$,

where m is a positive integer.

2996. Proposed by E. J. OGLESBY, Flushing, N. Y.

Given $u_1 = .2500$, $u_2 = .4113$, $u_3 = .4785$, $u_4 = .4965$, find x when $u_x = .4311$.

2997. Proposed by M. ZAMETKIN, Jamaica, N. Y.

Given $a = \sin 5°$, $b = \sin 49°$, and $c = \sin 87°$, prove that

$$\sin 73° = \frac{a^2 - b^2 + ac}{4a(a^2 - b^2 + ac) - (a - b + c)}.$$

2998. Proposed by F. M. GARNETT, Augusta, Ga.

A cube has removed from it a right pyramid whose base is a face of the cube and whose altitude is the altitude of the cube. How far from the base of the cube must a plane be passed parallel to the removed face so as to divide the remaining volume of the cube into two equal parts?

SOLUTIONS.

2897 [1921, 228]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Discuss the conditions under which the angles made by two circles on a sphere have the same measures as the distances between their poles.

SOLUTION BY F. L. WILMER, Omaha, Neb.

Write the equations of the circles:

$$\left.\begin{array}{r} x^2 + y^2 + z^2 = 1 \\ x \cos \phi + y \sin \phi = q \end{array}\right\}, \qquad \left.\begin{array}{r} x^2 + y^2 + z^2 = 1 \\ x = p \end{array}\right\},$$

where ϕ is the lesser angular distance between their poles. If (x', y', z') is the point of intersection for which z' is positive, then

$$x' = p, \qquad y' = \frac{q - p \cos \phi}{\sin \phi}, \qquad z' = \frac{1}{\sin \phi} \sqrt{\sin^2 \phi - d^2},$$

where d is the distance between the centers of the two circles. The length of the subnormal at (x', y', z') is $\frac{q - p \cos \phi}{\sin \phi}$ for the circle of radius $\sqrt{1 - p^2}$, and $\frac{p - q \cos \phi}{\sin \phi}$ for the circle of radius $\sqrt{1 - q^2}$.

The direction cosines are then:

$$0, \qquad -\frac{\sqrt{\sin^2 \phi - d^2}}{\sin \phi \sqrt{1 - p^2}}, \qquad \frac{q - p \cos \phi}{\sin \phi \sqrt{1 - p^2}}$$

for the one, and

$$-\frac{\sqrt{\sin^2 \phi - d^2}}{\sqrt{1 - q^2}}, \qquad \frac{\cos \phi \sqrt{\sin^2 \phi - d^2}}{\sin \phi \sqrt{1 - q^2}}, \qquad \frac{p - q \cos \phi}{\sin \phi \sqrt{1 - q^2}}$$

for the other tangent line at (x', y', z').

The proposition requires:

$$-\frac{\cos \phi (\sin^2 \phi - d^2)}{\sin^2 \phi \sqrt{(1 - p^2)(1 - q^2)}} + \frac{(q - p \cos \phi)(p - q \cos \phi)}{\sin^2 \phi \sqrt{(1 - p^2)(1 - q^2)}} = \cos \phi.$$

This reduces to

$$\frac{qp - \cos \phi}{\sqrt{(1 - p^2)(1 - q^2)}} = \cos \phi = \frac{qp}{1 + \sqrt{(1 - p^2)(1 - q^2)}}.$$

NOTE BY THE EDITORS—This equation may also be obtained by considering the stereographic projection of the sphere upon the xy-plane. Circles on the sphere project into circles in the plane and angles are preserved by the projection. The equations of the two circles are

$$x^2 + y^2 - (2/p)x + 1 = 0 \text{ and } x^2 + y^2 - (2/q)(x \cos \varphi + y \sin \varphi) + 1 = 0,$$

and for the cosine of the angle between them we find the same expression as that given in the solution above.

Another way of treating the problem is by spherical trigonometry. The two poles on the sphere and the intersection of the two circles are the vertices of a spherical triangle in which the measure of one angle is equal or supplementary to the measure of the opposite side. These conditions are satisfied when a triangle has two right angles. In other cases the cosine law and the cosine law for the polar triangle, to give consistent results, require that the supplementary relation hold for all three angles and their opposite sides, or only for one. On the other hand, at least two angles of a spherical triangle are in the same quadrants as their opposite sides. Therefore, unless our triangle has two right angles, it must have one angle supplementary and each of the other two equal to their opposite sides. The side whose opposite angle is supplementary is the one which is nearest to 90°.[1] The equation obtained above is merely the cosine law of spherical trigonometry.

2901 [1921, 277]. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Given the length of the base of a variable triangle and the positions of the feet of the altitudes on the other two sides, find (*a*) the locus of the vertex opposite the base; (*b*) the locus of the foot of the altitude on the base.

[1] See Chauvenet, *Plane and Spherical Trigonometry*, edition 9, Philadelphia, 1881, p. 178.

I. SOLUTION BY WILLIAM HOOVER, Columbus, Ohio, and H. P. MANNING, Providence, R. I.

Let P and Q be the two given points, $PQ = 2a$, with axes taken so that their coördinates are $(-a, 0)$ and $(a, 0)$. Let C (x, y) be the vertex whose locus is to be found in (*a*), and let A (x_1, y_1), B (x_2, y_2) be the other two vertices; take P on BC and Q on AC and let the length of AB be denoted by $2b$.

(*a*) If m and m' are the slopes of AC and BC, these various quantities will be connected by the relations

$$\begin{aligned} y &= m(x-a) = m'(x+a) \\ y_1 &= m(x_1 - a) = -\frac{1}{m'}(x_1 + a) \\ y_2 &= -\frac{1}{m}(x_2 - a) = m'(x_2 + a). \end{aligned} \tag{1}$$

Therefore

$$y_1 y_2 = -(x_1 + a)(x_2 + a) = -(x_1 - a)(x_2 - a),$$

or

$$x_1 + x_2 = 0 \quad \text{and} \quad y_1 y_2 = x_1^2 - a^2.$$

The expression for $\overline{AB}^2$ gives the equation

$$4x_1^2 + (y_1 - y_2)^2 = 4b^2.$$

Subtracting $4a^2$, substituting y_1y_2 for $x_1^2 - a^2$, and reducing, we have

$$y_1 + y_2 = \pm 2\sqrt{b^2 - a^2} = 2k, \text{ say.}$$

Now from (1)

$$yy_1 = -(x_1 + a)(x + a), \qquad yy_2 = (x_1 + a)(x - a).$$

Hence, adding and dividing by 2, we have

$$ky = -a(x_1 + a), \tag{2}$$

and then

$$ay_1 = k(x + a), \qquad ay_2 = -k(x - a).$$

The two expressions in (1) for y give

$$mm' = \frac{y^2}{x^2 - a^2},$$

and the two expressions for y_1,

$$mm' = -\frac{x_1 + a}{x_1 - a}.$$

Hence

$$\frac{x^2 + y^2 - a^2}{y^2} = \frac{2a}{x_1 + a} = -\frac{2a^2}{ky},$$

or

$$x^2 + y^2 + \frac{2a^2}{k} y = a^2.$$

This equation represents the locus of C.

(*b*) The slope of the base AB is $(y_1 - y_2)/(x_1 - x_2) = kx/ax_1$, and the foot of the altitude upon this line is the point R, whose coördinates satisfy the equations

$$Y - y_1 = \frac{kx}{ax_1}(X - x_1) \quad \text{and} \quad Y - y = -\frac{ax_1}{kx}(X - x).$$

We may write these

$$\begin{aligned} kxX &= ax_1Y - x_1(ay_1 - kx) = ax_1Y - kax_1, \\ ax_1X &= -kxY + x(ky + ax_1) = -kxY - a^2x, \end{aligned}$$

and if we multiply corresponding members and remove the factor $kaxx_1$, we have

$$X^2 = -Y^2 + \left(\frac{k^2 - a^2}{k}\right) Y + a^2,$$

or

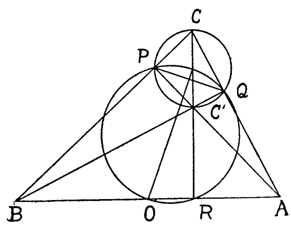
$$X^2 + Y^2 - \frac{b^2 - 2a^2}{k} Y = a^2.$$

This equation represents the locus of R.

II. Solution by Arthur Pelletier, Montreal, Canada.

We have given AB in length only and the fixed points P and Q.

(a) O, the middle point of AB, is determined, for $OP = OQ = AB/2$. In the circle $AQPB$, $\angle ACB$ is measured by $\frac{1}{2}(\text{arc}AB - \text{arc } PQ)$ and is therefore constant. Hence the locus of the vertex C, opposite the base AB, is the arc of the circle PCQ.



(b) Let R be the foot of the altitude on the base AB, and let C' be the intersection of the three altitudes. In the circle $ACPR$, with diameter AC, $\angle CAP = \angle CRP$. In the circle $AQC'R$, with diameter AC', $\angle QAC' = \angle QRC'$. Finally in the fixed circle $AQPB$, already mentioned, $\angle QAP$, measured by $\frac{1}{2}$ arc PQ, is a constant. Therefore $\angle PRQ$, the sum of the other two angles, is a constant and the locus of R is the arc of the circle PRQ.

As a second point, symmetrical to O with respect to the line PQ, may be the mid-point of AB, it follows that two arcs symmetrical to PCQ and PRQ belong also to the required loci.

Note by Otto Dunkel, Washington University—The triangle $AC'B$ also satisfies the conditions of the problem and the entire circle PCQ is included in the locus under (a). Likewise, the entire circle PRQ is the locus of R. If AB rotates about O, C and C' will trace the circle PCQ and change places with the rotation of AB through 180°. It should be noted that the figure takes a somewhat different form when AB intersects PQ.

However, this problem admits of an easier solution by aid of the nine-point circle which is the same for all the triangles and is the locus desired in (b); and then it is easy to show that the locus of C is a circle through P and Q with its center at the extremity of the diameter of the nine-point circle that has its other extremity at O.

2902 [1921, 277]. Proposed by C. N. MILLS, Tiffin, Ohio.

Find the locus of a point the feet of perpendiculars from which, on the sides of a triangle, lie on a straight line.

Note by the Editors: It is well known that the circumscribed circle of the triangle is at least part of the locus, by virtue of the theorem of William Wallace, *Mathematical Repository*, March, 1799. No proof of this result is called for in this problem.

I. Solution by R. M. Mathews, Wesleyan University.

With reference to rectangular axes, let the vertices of the triangle be $O \equiv (0, 0)$, $A \equiv (a, 0)$ and $B \equiv (b, c)$. Let $P \equiv (x', y')$ be any point from which perpendiculars PR, PS, PT are dropped to the sides OA, AB, BO, respectively.

Write the equations of the sides of the triangle, then the equations of the perpendiculars and so find the coördinates of the feet as

$$\begin{aligned} R &\equiv (x', 0), \\ S &\equiv \left(\frac{ac^2 - c(a-b)y' + (a-b)^2x'}{(a-b)^2 + c^2}, \frac{c^2y' - c(a-b)x' + ac(a-b)}{(a-b)^2 + c^2}\right), \\ T &\equiv \left(\frac{b^2x' + bcy'}{b^2 + c^2}, \frac{bcx' + c^2y'}{b^2 + c^2}\right). \end{aligned}$$

Applying the necessary and sufficient condition that R, S, and T be collinear, we have

$$\begin{vmatrix} x', & 0, & 1 \\ b^2x' + bcy', & bcx' + c^2y', & b^2 + c^2 \\ ac^2 - c(a-b)y' + (a-b)^2x', & c^2y' - c(a-b)x' + ac(a-b), & (a-b)^2 + c^2 \end{vmatrix} = 0.$$

This gives the equation of condition

$$x'^2 + y'^2 - ax' + \frac{ab - b^2 - c^2}{c} y' = 0.$$

The coördinates of O, A and B satisfy this equation. Therefore this is the circumscribed circle, and the circumscribed circle is the required locus.

II. Solution by William Hoover, Columbus, Ohio, and Otto Dunkel, Washington University.

Take the given triangle, ABC, as the triangle of reference in trilinear coördinates, the side opposite A being $\alpha = 0$; also $(\alpha', \beta', \gamma')$ the point from which the perpendiculars are drawn. Take any point in $\alpha = 0$, as $(0, \beta_1, \gamma_1)$; then the straight line through $(\alpha', \beta', \gamma')$ and $(0, \beta_1, \gamma_1)$ is given by

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ 0 & \beta_1 & \gamma_1 \end{vmatrix} = 0$$

or,

$$\alpha(\beta'\gamma_1 - \gamma'\beta_1) - \beta\alpha'\gamma_1 + \gamma\alpha'\beta_1 = 0.$$

This is perpendicular to $\alpha = 0$ if

$$\beta'\gamma_1 - \gamma'\beta_1 - \alpha'\beta_1 \cos B + \alpha'\gamma_1 \cos C = 0$$

or

$$-(\gamma' + \alpha' \cos B)\beta_1 + (\beta' + \alpha' \cos C)\gamma_1 = 0;$$

so that we can write for the coördinates of the foot of the perpendicular from $(\alpha', \beta', \gamma')$ upon $\alpha = 0$,

$$0, \qquad \beta' + \alpha' \cos C, \qquad \gamma' + \alpha' \cos B.$$

In the same way we get the feet of the other two perpendiculars.

These three points will be on a straight line if

$$\begin{vmatrix} 0 & \beta' + \alpha' \cos C & \gamma' + \alpha' \cos B \\ \alpha' + \beta' \cos C & 0 & \gamma' + \beta' \cos A \\ \alpha' + \gamma' \cos B & \beta' + \gamma' \cos A & 0 \end{vmatrix} = 0.$$

This reduces to [1]

$$(\alpha' \sin A + \beta' \sin B + \gamma' \sin C)(\beta'\gamma' \sin A + \gamma'\alpha' \sin B + \alpha'\beta' \sin C) = 0.$$

The first factor is equal to the area of the triangle divided by the radius of the circumscribed circle and hence cannot be zero. The second factor put equal to zero gives the equation of the circumscribed circle. Hence the circumscribed circle comprises the entire locus.

A simple proof may be obtained by reversing the reasoning given on page 118 of Salmon's *Conic Sections*, edition 6 (London, 1879).

If we assume Wallace's theorem the above may be regarded as a derivation of the equation of the circumscribed circle; for it is the equation of a conic which includes all the points of this circle.

Also solved by T. L. Bennett, A. M. Harding, Arthur Pelletier, and H. L. Olson.

2906 [1921, 277]. Proposed by ELIJAH SWIFT, University of Vermont.

Given any number of five digits, reverse the order and subtract the smaller of the two numbers thus formed from the larger. Show that if told the last three digits of this difference, we can find the first two, and give a simple rule for determining them.

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[1] If we multiply the three rows by $\beta'\gamma' \sin A$, $\gamma'\alpha' \sin B$ and $\alpha'\beta' \sin C$, respectively, and add, we shall get a new row whose elements are α', β', and γ' times the expression

$$\beta'\gamma' \sin A + \gamma'\alpha' \sin B + \alpha'\beta' \sin C,$$

which is, therefore, a factor.

SOLUTION BY A. V. RICHARDSON, Bishop's College, Lennoxville, Quebec, Canada.

We may indicate the subtraction as follows:

$$\begin{array}{ccccc} a & b & c & d & e \\ e & d & c & b & a \\ \hline x & y & m & n & p \end{array}$$

where the letters denote the digits of the numbers. Since the first number is the larger, either $a > e$ or $a = e$ and $b > d$. The digits of the difference will be as in the following table.

	x	y	m	n	p
$a > e,\ b > d$	$a - e$	$b - d - 1$	9	$9 + d - b$	$10 + e - a$
$b = d$	$a - e - 1$	9	9	9	$10 + e - a$
$b < d$	$a - e - 1$	$10 + b - d$	0	$d - b - 1$	$10 + e - a$
$a = e,\ b > d$	0	$b - d - 1$	9	$10 + d - b$	0

From the table we may formulate the following rule:

$$x = 10 - p \quad \text{and} \quad y = 8 - n$$

except

(1) when $p = 0$, and then $x = 0$ and $y = 9 - n$;
(2) when $m = 0$, and then $x = 9 - p$, and $y = 9 - n$;
(3) when $m = n = 9$, $p \neq 0$, and then $x = 9 - p$ and $y = 9$.

Also solved by T. M. BLAKSLEE, H. N. CARLTON, A. M. HARDING, L. C. MATHEWSON, C. J. STOWELL, F. L. WILMER.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

In the temporary assignment of Mr. E. B. MODE to duties connected with University administration, the following have been appointed temporary instructors in the College of Liberal Arts, Boston University: C. E. HAIGLER of the Rehabilitation Division, Franklin Union; J. A. MARSH of the English High School, Boston; and R. W. GARDNER of the Eastern Nazarene College, Wollaston.

At the University of Michigan, Professor W. H. BUTTS has retired after serving twenty-four years as instructor and professor of mathematics and fourteen years as assistant dean of the Colleges of Engineering and Architecture. He has been appointed professor emeritus.

Mr. L. J. COMRIE, M.A., F.R.A.S., of New Zealand and St. John's College, University of Cambridge, Cambridge, England, is in residence at the Sproul Observatory of Swarthmore College as Research Assistant for the coming year. Mr. Comrie is the holder of the Isaac Newton Studentship awarded in April 1921 at Cambridge University for research in Astronomy. He has been working under Professor A. S. Eddington for three years and has made a special study of occultations of stars by planets. He will continue this study and in particular the publications of predictions of these occultations. He was successful in organizing a Computing Section of the British Astronomical Association, and also edited the first edition of that Association's *Observer's Handbook*, published in 1922. He will participate in the researches upon the parallaxes of stars which is the prime work of the Sproul Observatory.

Robert Wheeler Willson, professor of astronomy, emeritus, at Harvard University since 1919, died November 1, 1922. He was born at Roxbury, Mass., July 30, 1853. He received the degree of A.B. from Harvard in 1873 and of Ph.D. from the University of Würzburg in 1886. He was an assistant in the Argentine National Observatory in 1873 and in the Harvard Observatory in 1874; tutor in physics at Harvard 1875–1881; assistant astronomer at the Winchester Observatory, Yale, 1881–1884; then instructor in astronomy and physics 1891–1899, assistant professor of astronomy 1899, and professor of astronomy 1903, at Harvard. He was the author of *Laboratory Astronomy,* Boston, 1901 (enlarged edition, 1905), and *Times of Sunrise and Sunset in the United States,* Cambridge, 1908.

Thomas Marcus Blakslee, born at Harpersville, N. Y., December 12, 1854, died at Ames, Iowa, January 30, 1923. He received the degree of Ph.B. from Colgate University in 1874 and of Ph.D. from Yale University in 1880. He pursued his mathematical studies further in the Johns Hopkins and Chicago Universities, and for a short time in the University of Göttingen. For the greater part of his life he held the chair of mathematics in Des Moines College, Des Moines, Iowa. He has been a member of the Association since September, 1919. During the latter years of his life he was an interested and frequent contributor to the Problems and Solutions Department of this Monthly; also his article on "The solution of an equation by a frame" appeared in *1911,* 159–162. He was the author of *Direction as a Quantity. Directed Quantity and Kindred Subjects,* Des Moines, 1887 (19 pages); and of *Academic Trigonometry. Plane and Spherical,* Boston, Ginn, 1888 (30 pages).

The following resolutions upon the death of Professor Halsted were adopted at the April meeting of the Rocky Mountain Section (see *1922,* 198):

By the death of our associate and fellow worker, George Bruce Halsted, March 19, 1922, in New York City, the Rocky Mountain Section of The Mathematical Association of America publicly recognizes that Colorado has lost a distinguished citizen; and the whole country, a mathematician and philosopher and logician, of international standing. Dr. Halsted was a member of the leading scientific societies, a contributor to scientific journals, and the author of well-known mathematical books; and was on terms of personal friendship with mathematicians in various countries. We, in Colorado, knew him well, and gave him freely of our love and admiration. We know that his was a life of service to learning, and particularly to mathematics, the purest of all the sciences. Here, in Greeley, Colorado, his later home, we, the members of The Association, attending the session, pay to him our sincere respects for his scholarship, his productive labors, and his sterling qualities.

The United States has withdrawn its invitation for the International Mathematical Congress to be held "in New York or its vicinity" in 1924. The invitation from Canada will probably be accepted. It is now expected that most of its meetings will be held in Montreal before it adjourns to Toronto for the sessions of the British Association for the Advancement of Science.

Published April 9, 1923.

Important Notice

The Mathematical Association of America, like all other organizations of an educational character, gives manifold more than it receives from its constituents. This discrepancy is accounted for by the gratuitous and arduous work given to the Association by its devoted servants.

Since it is impossible to raise the dues above a certain maximum without going beyond the reach of very many of those to whom the Association means most, it seems clear that an endowment fund is the best solution of the difficulty. Now that the Association is incorporated it is legally qualified to administer such a fund.

An endowment is needed not only to prevent a reduction of the number of pages in the MONTHLY, but also to enable the Association to make just compensation to its servants, and to go forward with its important projects such, for example, as the preparation and publication of a Mathematical Dictionary which is so greatly needed in the English language.

It is believed that, when these conditions are widely known among the friends of mathematics, financial support of this kind will be forthcoming.

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of

America the sum of .... Dollars,

to be known as the..

....Fund, and to be used

for $\begin{cases}\text{Endowment—the income only of which may be expended.}\\ \text{Special Projects—for which both principal and income may be expended.}\end{cases}$

Witness: Signature..

....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.

[2] Indicate which one of the two purposes is desired, and omit the other.

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EDITORIAL CORRESPONDENCE AND BOOKS FOR REVIEW should be addressed to the EDITOR-IN-CHIEF for 1923, W. B. FORD, 204 Mason Hall, Ann Arbor, Mich.
BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

Eighth Summer Meeting of the Association, Vassar College, September 5–6, 1922
Seventh Annual Meeting, Harvard University, December 28, 29, 1922
The following are dates of Section meetings of the Association in 1922 (unless otherwise specified):

ILLINOIS, Rockford, Ill., April 28–29
IOWA, Des Moines, November 3; Cornell College, Mount Vernon, April 27-28, 1923
KANSAS, Topeka, January 21; January 20, 1923
KENTUCKY, Georgetown College, April 8; University of Kentucky, April, 1923
MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA, Washington, December 9; Baltimore, May, 1923
MINNESOTA, St. Paul, June 4, 1921; St. Paul, May 27
MISSOURI, Kansas City Junior College, November 18; University of Missouri, Columbia, November 30–December 1, 1923
OHIO, Columbus, Apr. 14–15; Mar. 30–31, 1923
ROCKY MOUNTAIN, Greeley, Colo., April 14–15; University of Colorado, April, 1923
SOUTHEASTERN, Atlanta, Ga., April 29; Agnes Scott College, Decatur, Ga., March 10, 1923
TEXAS, Dallas, November 25, 1921; Houston, December 1–2